A geometrical view of the Nehari manifold

José Maria Gomes∗

Abstract
We study the Nehari manifold \( \mathcal{N} \) associated to the boundary value problem
\[
-\Delta u = f(u), \quad u \in H_0^1(\Omega),
\]
where \( \Omega \) is a bounded regular domain in \( \mathbb{R}^n \). Using elementary tools from Differential Geometry, we provide a local description of \( \mathcal{N} \) as an hypersurface of the Sobolev space \( H_0^1(\Omega) \). We prove that, at any point \( u \in \mathcal{N} \), there exists an exterior tangent sphere whose curvature is the limit of the increasing sequence of principal curvatures of \( \mathcal{N} \). Also, the \( H_1 \)-norm of \( u \in \mathcal{N} \) depends on the number of principal negative curvatures. Finally, we study basic properties of an angle decreasing flow on the Nehari manifold associated to homogeneous non–linearities.


1 Introduction

The variational method introduced by Nehari in [9]–[10] was a significant outcome of his research on the non–oscillating nature of solutions to certain classes of second order equations. For instance, concerning the linear problem
\[
y'' + p(x)y = 0, \quad y(a) = y'(b) = 0,
\]
where \( p \) is a continuous positive function, [Theorem 1, [8]] sets the equivalence between the existence of a positive solution in \( [a, +\infty] \) and the fact that the lowest eigenvalue
\[
\lambda := \min \frac{\int_a^b y'^2 \, dx}{\int_a^b py^2 \, dx}
\]

∗This work is supported by F.B. 2008 ISFL-1-297 from FCT/MCTES/PT.
satisfies $\lambda > 1$ for all $b > a$. In [7], a solution to the non-linear equation

$$y'' + p(x)y^{2n+1} = 0, \quad y(a) = y(b) = 0$$

with a prescribed number $m$ of intermediate zeros $a < a_1 < \ldots < a_m < b$ is obtained by minimizing the functional

$$\tilde{J}(u; a_1, \ldots, a_m) := \sum_{\nu=1}^{m+1} [\tilde{J}_\nu]^{\frac{1}{n}},$$

where $u \in C_0^{1,1}[a, b]$ satisfies $u(a_1) = \ldots = u(a_m) = 0$ and

$$\tilde{J}_\nu(w) = \left( \frac{\int_{a_\nu}^{a_{\nu+1}} w'^2 \, dx}{\int_{a_\nu}^{a_{\nu+1}} pw^{2n+2} \, dx} \right)^{\frac{n+1}{n}},$$

is the Rayleigh coefficient on $C_0^{1,1}([a_\nu, a_{\nu+1}])$. Similar ideas were later exploited in [14] and [15] although these works don’t mention Nehari’s early contribution. In fact, as it was defined in [9], the “normalization condition” (known a posteriori as the Nehari constrain)

$$\int_a^b y'^2 \, dx = \int_a^b y^2 F(y^2, x) \, dx \quad (u \neq 0),$$

was the basis of a more comprehensive method allowing the proof of the existence of solutions to a second order non-linear equation of type

$$y'' + yF(y^2, x) = 0,$$

where the non-homogeneous linear term prevented the method of minimizing a Rayleigh coefficient.

In the past few decades, the Nehari method has been extensively used on the study of existence of ground-state, nodal, multi-spike or multi-bump solutions, in what can be considered as a natural enlargement of Nehari’s concerns about oscillatory aspects of second order non-linear differential equations (see for instance [4],[5] and [12]). For the interested reader on an abstract treatment of the Nehari method (or on further references about the subject) we recommend the survey [13]. Our purpose to bring out a clearer picture of a variational framework known since 1960 was, in some sense, stimulated by the study of [2].

In section 1 we obtain classical estimates of the energy of a function satisfying the Nehari constrain and recall basic facts about the Nehari manifold. In section 2 we use the notion of curvature to provide a local description of the Nehari manifold $\mathcal{N}$. Some regularity assumptions will be required both
on the nonlinear term of the Nehari constrain as well as on the function $u \in \mathcal{N}$. In the last section, we propose an alternative flow on the Nehari manifold (assuming an homogeneous nonlinearity) whose stable stationnary points are, under appropriate conditions, solutions of the second order equation

$$-\Delta u = f(u), \quad u \in H^1_0(\Omega).$$

This work is a personal tribute to Nehari’s pioneering works [9]–[10] fifty years after their publication. I thank Luis Sanchez and Pedro Girão for their interest and support.

2 Preliminary results

Along this article we consider the space $H^1_0(\Omega)$, where $\Omega$ is a bounded and regular domain of $\mathbb{R}^N$. We assume $H^1_0(\Omega)$ is endowed with the norm

$$\|u\| = \langle u, u \rangle := \int_\Omega |\nabla u|^2(x) \, dx.$$

As usual, we denote $2^* = \frac{2N}{N-2}$ and $2^* = +\infty$ if $N = 2$, so that the embedding

$$H^1_0(\Omega) \subset L^q(\Omega)$$

is compact for $1 \leq q < 2^*$. We introduce the classical Euler-Lagrange functional

$$J(u) := \frac{1}{2} \int_\Omega |\nabla u|^2(x) \, dx - \int_\Omega F(u)(x) \, dx,$$

(2.1)

defined over $H^1_0(\Omega)$ where $F(u) = \int_0^u f(s) \, ds$. Critical points of $J$ in $H^1_0(\Omega)$ are classical solutions of the elliptic equation

$$-\Delta u = f(u) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega,$$

(2.2)

provided well known assumptions on the non-linear term $f$ are verified (see, for instance [11]). In our case, we require

(f1) $f \in C^2(\mathbb{R}, \mathbb{R})$.

(f2) $f(u)u \leq \beta f'(u)u^2$ where $0 < \beta < 1$.

(f3) There exist positive constants $\xi_1 \leq \xi_2$ such that

$$\xi_1|u|^{p-2} \leq f'(u) \leq \xi_2|u|^{p-2},$$
where \(2 < p < 2^*\).

Note that condition (f2) implies that \(f(0) = 0\) as well as
\[
\zeta F(u) \leq f(u)u,
\]
for some \(\zeta > 2\), which is the classical Ambrosetti-Rabinowitz condition. Further, we will require

\[(f'3')\] There exist positive constants \(\xi_1 \leq \xi_2\) such that
\[
\xi_1 |u|^{p-2} \leq f''(u)u \leq \xi_2 |u|^{p-2}.
\]

Condition \((f'3')\) implies \((f3)\) (adapting, if necessary, the constants \(\xi_1\) and \(\xi_2\)).

We define a sequence \((e_n)\) in \(H^1_0(\Omega)\) in the following way. Let \(e_1\) be such that
\[
\|e_1\|^2 = \min \left\{ \|u\|^2 : \int_\Omega F(u)(x) \, dx = 1 \right\},
\]
and for \(n > 1\)
\[
\|e_n\|^2 = \min \left\{ \|u\|^2 : \int_\Omega F(u)(x) \, dx = 1, \quad u \in \text{span}\{e_1, ..., e_{n-1}\}^\perp \right\}.
\]

We have the following fact whose proof we postpone to the Appendix.

**Lemma 1** The sequence \((e_n)\) is an orthogonal basis of \(H^1_0(\Omega)\). Also \((\|e_n\|)\) is non-decreasing and
\[
\lim_{n \to \infty} \|e_n\| = \infty.
\]

**Remark 1** Each \(e_n\) satisfies the relation
\[
-\Delta e_n = \lambda_{nn} f(e_n) + \sum_{i=1}^{n-1} \lambda_{ni} (-\Delta e_i)
\]
for some Lagrange multipliers \(\lambda_{ni}\). In particular, \(e_n \in C^{3,\alpha}(\Omega) \cap C_0(\Omega)\). Multiplying \((2.5)\) by \(e_n\), and integrating by parts we conclude
\[
\lambda_{nn} = \frac{\|e_n\|^2}{\int_\Omega f(e_n)e_n(x) \, dx} > 0.
\]

A similar argument yields, for all \(m > n\),
\[
0 = \int_\Omega \nabla e_n \nabla e_m(x) \, dx = \lambda_{mn} \int_\Omega f(e_n)e_m(x) \, dx.
\]

Then \((2.6)\) implies
\[
\text{for all } m > n \quad \langle \nabla J(e_n), e_m \rangle = 0.
\]
The Nehari manifold is defined as
\[ \mathcal{N} := \{ u \in H_0^1(\Omega) : u \neq 0 \quad \text{and} \quad \langle \nabla J(u), u \rangle = 0 \}. \]  
(2.7)
Condition \( \langle \nabla J(u), u \rangle = 0 \) writes
\[ \int_{\Omega} |\nabla u|^2(x) \, dx - \int_{\Omega} f(u)u(x) \, dx = 0. \]  
(2.8)

In the next Proposition we obtain estimates on a function \( u \in \mathcal{N} \) based on the dimension of a space where the second derivative of \( J \) at \( u \) is negative definite.

**Proposition 1** Assume \( f \in C^1(\mathbb{R}, \mathbb{R}) \) satisfies \((f2)–(f3)\). Let \( u \in \mathcal{N} \) and \( V_j \) be a \( j \)-dimensional subspace of \( H_0^1(\Omega) \) such that
\[ D^2J_{vv}(u) \leq 0 \quad \text{for all} \quad v \in V_j. \]  
(2.9)
Then
\[ J(u) \geq \max\{C_1\|e_j\|^2, C_2\}, \]
where \( e_j \) was defined in (2.4) and \( C_1, C_2 \) are positive constants independent of \( u \).

**Proof.** By (2.8), our assumptions on \( f \) and Sobolev’s Embedding Theorem we have, for some constant \( c_p \),
\[ \|u\|^2 \leq \frac{\xi_2}{p - 1} \int_{\Omega} |u|^p(x) \, dx \leq \frac{\xi_2 c_p}{p - 1} \|u\|^p \]  
(2.10)
Then, for \( C = \left( \frac{p - 1}{\xi_2 c_p} \right)^{\frac{1}{p - 2}} \), we conclude
\[ \|u\| \geq C. \]  
(2.11)
By (2.1), (2.3) and (2.8),
\[ J(u) \geq \left( \frac{1}{2} - \frac{1}{\xi} \right) \|u\|^2. \]  
(2.12)
The previous estimates prove that \( J(u) \geq C_2 \) with \( C_2 = (1/2 - 1/\xi)C^2 \).

Let
\[ S = \{ v \in V_j : \|v\| = 1 \}. \]
We have \( \gamma(S) = j \) where \( \gamma \) is the the genus of a closed symmetric set (see [11]). Let
\[ E_j = (\operatorname{span}\{e_1, ..., e_{j-1}\})^\perp. \]
Since $\gamma(S) > \text{codimension } E_j$, we conclude by [Proposition 7.8, [11]] that $S \cap E_j \neq \emptyset$.

We may therefore choose $v \in V_j \cap E_j$ and, multiplying if necessary by an appropriate constant, assume $\int_{\Omega} F(v(x)) \, dx = 1$. We have

$$D^2 J_{uw}(u) = \int_{\Omega} |\nabla v|^2(x) \, dx - \int_{\Omega} f'(u)v^2(x) \, dx \leq 0.$$  \hspace{1cm} (2.13)

By (2.13), Holder inequality and (f3),

$$\int_{\Omega} |\nabla v|^2(x) \leq \left( \int_{\Omega} |f(u)|^{\frac{p}{p-2}}(x) \, dx \right)^{\frac{p-2}{p}} \left( \int_{\Omega} |v|^p(x) \, dx \right)^{\frac{2}{p}} \leq C \left( \int_{\Omega} |u|^p(x) \, dx \right)^{\frac{p-2}{p}} \left( \int_{\Omega} F(v)(x) \, dx \right)^{\frac{2}{p}} \left( \int_{\Omega} |v|^p(x) \, dx \right)^{\frac{2}{p}}.$$  \hspace{1cm} (2.14)

where $C = \xi_2 \frac{p-2}{\xi_1} \left( \frac{p-1}{p} \right)^{\frac{p-2}{p}}$.

By the definition of $(e_n)$ and our assumptions on $v$ we have,

$$\int_{\Omega} |\nabla v|^2(x) \, dx \geq \int_{\Omega} |\nabla e_j|^2(x) \, dx.$$  \hspace{1cm} (2.15)

We conclude, by (2.12), (2.14) and (2.15)

$$J(u) \geq \left( \frac{1}{2} - \frac{1}{\zeta} \right) \int_{\Omega} |\nabla u|^2(x) \, dx - \left( \frac{1}{2} - \frac{1}{\zeta} \right) \int_{\Omega} f(u)u(x) \, dx \geq \left( \frac{1}{2} - \frac{1}{\zeta} \right) \frac{\xi_1}{p-1} \int_{\Omega} |u|^p(x) \, dx \geq C_1 \left\| e_j \right\|^{\frac{2p}{p-2}}$$  \hspace{1cm} (2.16)

where $C_1 = C^{-p/(p-2)} \left( \frac{1}{2} - \frac{1}{\zeta} \right)^{\frac{\xi_1}{p-1}}$.

**Remark 2** We conclude from Proposition 1 and Lemma 1 that if $(u_j)$ is a sequence in $\mathcal{N}$ such that, for each $u_j$, there exists a $j$-dimensional space $V_j$ verifying (2.9) then

$$\left\| u_j \right\| \to \infty.$$  \hspace{1cm}

Given $u \in \mathcal{N}$ the tangent space $\mathcal{T}_u$ to $\mathcal{N}$ at $u$ consists on the functions $v \in H_0^1(\Omega)$ such that

$$2 \int_{\Omega} \nabla u \nabla v(x) \, dx - \int_{\Omega} f'(u)uv(x) \, dx - \int_{\Omega} f(u)v(x) \, dx = 0.$$  \hspace{1cm} (2.17)

The next proposition sets some well–known facts.
Proposition 2 Assume $f$ satisfies (f1)--(f3). There exists $C' > 0$ such that
\[ u \in \mathcal{N} \Rightarrow \|u\| \geq C'. \tag{2.18} \]
Moreover, $\mathcal{N}$ is locally diffeomorphic to
\[ S := \{ u \in H^1_0(\Omega), \|u\| = 1 \}. \]
Given $u \in \mathcal{N}$,
\[ \nabla J(u) = 0 \iff \Pi_u(\nabla J(u)) = 0, \tag{2.19} \]
where $\Pi_u$ is the orthogonal projection on $\mathfrak{T}_u$.

**Proof.** Condition (2.18) was already proved in Proposition 1. Given $u \in H^1_0(\Omega)\setminus\{0\}$, consider the function
\[ g(t) := \langle \nabla J(tu), tu \rangle = t^2 \int_{\Omega} |\nabla u(x)|^2 \, dx - \int_{\Omega} t f(tu) u(x) \, dx. \]
By (f2)--(f3), we have $g(t) > 0$ if $0 < t < \epsilon$ for $\epsilon$ sufficiently small. Also
\[ \lim_{t \to +\infty} g(t) = -\infty. \]
Therefore there exists $t_0 > 0$ such that $g(t_0) = 0$. By (2.8) and (f2),
\[ g'(t_0) = 2t_0 \int_{\Omega} |\nabla u(x)|^2 \, dx - \int_{\Omega} f(t_0 u) + f'(t_0 u) u^2 \, dx < 0. \]
Consequently, $t_0 > 0$ is uniquely determined. Also, by the Implicit Function Theorem,
\[ t_0(u) \in C^2(H^1_0(\Omega)\setminus\{0\}), \mathbb{R}\setminus\{0\}). \]
Consider the $C^2$--application
\[ P_N : H^1_0(\Omega)\setminus\{0\} \hookrightarrow \mathcal{N} \quad u \mapsto t_0(u) u. \]
Clearly, the restriction
\[ P_N|_S \hookrightarrow \mathcal{N} \]
is a local diffeomorphism.

We now turn to (2.19). The first implication is trivial. Consider the constraint $\phi(u) := \langle \nabla J(u), u \rangle = 0$. By (f2), for any $u \in \mathcal{N}$,
\[ \langle \nabla \phi(u), u \rangle = \int_{\Omega} |\nabla u|^2(x) \, dx - \int_{\Omega} f'(u) u^2(x) \, dx = \int_{\Omega} f(u) u - f'(u) u^2(x) \, dx < 0, \]
i.e., $u \notin \mathfrak{T}_u$. Then, $\Pi_u(\nabla J(u)) = 0$ and $\langle \nabla J(u), u \rangle = 0$ imply $\nabla J(u) = 0$. ■
3 Local geometry of the Nehari manifold

We use basic notions of Differential Geometry to describe the Nehari manifold as an hypersurface of $H^1_0(\Omega)$ (see for instance, [1] and [6]). In the sequence, we will assume that assumptions (f1), (f2) and (f3') are verified. By the Riesz representation of a linear functional in $H^1_0(\Omega)$ and (2.17), the tangent space can also be characterized as

$$\mathcal{T}_u := \{ v \in H^1_0(\Omega) : \langle N(u), v \rangle = 0 \},$$

with $N(u) = 2u + \Delta^{-1}(h(u))$ and

$$h(u) = f'(u)u + f(u).$$

(3.1)

Prescribe

$$n(u) = \frac{N(u)}{\|N(u)\|},$$

as unitary normal to $\mathcal{T}_u$. By (f2),

$$\langle n(u), u \rangle < 0$$

(3.2)

for all $u \in \mathcal{N}$. Our assumptions on $f$ imply that the map $u \rightarrow n(u)$ is of class $C^1$ in $H^1_0(\Omega) \setminus \{0\}$. Given $u \in \mathcal{N}$, we formally define a Weingarten map

$$L_u : \mathcal{T}_u \mapsto \mathcal{T}_u \quad L_u(v) = Dn(u)[v].$$

In fact, given $u \in \mathcal{N}$, $v \in \mathcal{T}_u$ and a regular path $\gamma$ such that

$$\gamma : ]-1,1[ \rightarrow \mathcal{N}, \quad \gamma(0) = u, \quad \gamma'(0) = v,$$

we have

$$\langle n(\gamma(t)), n(\gamma(t)) \rangle = 1 \quad \forall t \in ]-1,1[.$$

In particular

$$\langle Dn(\gamma(0))[\gamma'(0)], n(\gamma(0)) \rangle = 0,$$

i.e.

$$Dn(u)[v] \in \mathcal{T}_u$$

for all $v \in \mathcal{T}_u$. We also recall the classical formula

$$Dn(u)[v] = -D\Pi_u(v, n(u)).$$

(3.3)

Computing,

$$Dn(u)[v] = \frac{1}{\|N(u)\|} \left( 2v + \Delta^{-1}(h'(u)v) - n(u) \langle 2v + \Delta^{-1}(h'(u)v) , n(u) \rangle \right).$$

(3.4)
If we assume $u \in W^{1,\infty}_0(\Omega)$ the operator

$$L_u(v) := Dn(u)[v] = \frac{1}{\|N(u)\|}(2I + T_u)$$

where

$$T_u(v) = \Delta^{-1}(h'(u)v) - n(u) \left\langle 2v + \Delta^{-1}(h'(u)v), n(u) \right\rangle$$

is well-defined for all $v \in H^1_0(\Omega)$. Moreover the operator

$$T_u : \mathcal{S}_u \mapsto \mathcal{S}_u$$

is self-adjoint and compact (note that the term $\left\langle 2v + \Delta^{-1}(h'(u)v), n(u) \right\rangle$ maps into $\mathbb{R}$). We may therefore provide an orthogonal basis for $\mathcal{S}_u$ of eigenvectors of $T_u$. To an eigenvector $v$ of $T_u$ with associated eigenvalue $\lambda$ corresponds the same eigenvector $v$ of $L_u$ with associated eigenvalue $k = \frac{2 + \lambda}{\|N(u)\|}$.

Remark 3 Of course, the assumption that $u \in W^{1,\infty}_0(\Omega)$ may be weakened. For instance, if $\Omega$ is a bounded regular subset of $\mathbb{R}^2$, as $H^1_0(\Omega) \subset L^q(\Omega)$ for any $q \in [1, +\infty[$ with compact embedding, the principal curvatures are defined for all $u \in H^1_0(\Omega) \cap \mathcal{N}$. However, the class of functions in $W^{1,\infty}_0(\Omega)$ is of special interest regarding its invariance property for a significant class of energy decreasing flows associated to Euler-Lagrange functionals.

We have the following property of the non-zero eigenvalues of the compact operator $T_u$.

Lemma 2 Given $u \in \mathcal{N} \cap W^{1,\infty}_0(\Omega)$, the distinct non-zero eigenvalues of $T_u$ form an increasing sequence $(\lambda_n(u))$ converging to zero.

Proof. As usual, we determine the sequence of the non-zero eigenvalues and corresponding eigenvectors of $T_u$ by means of a recurrent sequence of minimization problems:

$$\lambda_n := \inf \left\{ \left\langle T_u(v), v \right\rangle : v \in \mathcal{S}_u, \|v\| = 1, v \in (\text{span}\{v_1, ..., v_{n-1}\})^\perp \right\}$$

and a corresponding eigenvector $v_n$ is a function where the infimum is attained. Necessarily, $(\lambda_n)$ is an increasing sequence. In case $\lambda_{n+1} = \lambda_n$ the eigenvalue $\lambda_n$ has multiplicity greater than 1. Since $\langle n(u), v \rangle = 0$, we have

$$\langle T_n(v), v \rangle = \langle \Delta^{-1}(h'(u)v), v \rangle = -\int_{\Omega} h'(u)v^2(x) \, dx ,$$
and conclude $\lambda_n \leq 0$ for all $n \in \mathbb{N}$.

Assume, for some $n$ that $\lambda_n = 0$ and $\lambda_{n-1} < 0$. Then for any $k \geq n$, we have $\lambda_k = 0$ and the corresponding eigenfunction $v_k$ satisfies

$$
\int h'(u)v_k^2(x) \, dx = 0.
$$

Then, by (f3),

$$
v_k \equiv 0 \quad \text{in} \quad \text{supp}(u) \quad \forall k \geq n.
$$

As any $w$ such that

$$
\text{support}(w) \subset \text{support}(u)
$$

is orthogonal to $v_k$ with $k \geq n$, $w$ necessarily belongs to $\text{span}\{v_1, \ldots, v_{n-1}\}$. This would imply, for any bounded regular domain $\omega$ such that $\omega \subset \text{supp}(u)$,

$$
\left( H_0^1(\omega) \cap \Sigma_u \right) \subset \text{span}\{v_1, \ldots, v_{n-1}\}
$$

which is absurd since the first subspace is infinite dimensional.

If

$$
\int_{\Omega} h'(u)v^2(x) \, dx > 0, \quad \forall v \in \Sigma_u \setminus \{0\},
$$

the sequence $(v_i)$ of eigenvectors associated to the sequence of non-zero eigenvalues $(\lambda_i)$ provides an Hilbert basis of $\Sigma_u$. This is the case if $u(x) \neq 0$ a.e. in $\Omega$. In general, we may write

$$
\Sigma_u = \text{Ker}(T_u) \oplus R(T_u),
$$

where $R(T_u)$ is the closure of the subspace generated by the family $\{v_i\}$.

In view of (3.5), we will refer an eigenvalue $k_i$ of $L_u$ as a (signed) principal curvature of $\mathcal{N}$ at $u$ if the corresponding eigenvalue $\lambda_i$ of $T_u$ satisfies $\lambda_i < 0$. The sequence $(k_i)$ is increasing and converges to $2/\|N(u)\|$. We denote by $\mathcal{K}_u$ the set of all eigenvalues of $L_u$. We have

$$
\mathcal{K}_u \subseteq \{k_i\}_{i \in \mathbb{N}} \cup \{2/\|N(u)\|\}, \quad (3.6)
$$

with equality of sets in the degenerate case $\text{Ker}(T_u) \neq \{0\}$. In particular, at any point $u \in \mathcal{N}$, the principal curvatures are positive, except at most for a finite number.

Let $P$ be a plane containing the inward normal $n(u)$ and a direction $v(u)$ associated to a positive curvature. Using the reference frame of center $u$ and vectors $v(u)$ and $n(u)$, if $w \in P \cap \mathcal{N} \setminus \{u\}$ is sufficiently close to $u$, then

$$
w = x\, v(u) + y\, n(u) \quad \text{with} \quad (x, y) \in \mathbb{R}^2, \quad y < 0.
$$
Remark 4  We may describe the above mentioned property saying that, at any point \( u \in \mathcal{N} \cap W^{1,\infty}_0(\Omega) \), there exists an “exterior” tangent sphere to the Nehari manifold, with center
\[
C(u) = u - \frac{\|N(u)\|}{2} \cdot n(u) = -\frac{1}{2} \Delta^{-1}(h(u)) ,
\]
and radius \( \|N(u)\|/2 \), whose curvature is approximated by the sequence of principal curvatures of the Nehari manifold.

We have the following estimates on the curvatures of the Nehari manifold.

**Lemma 3** There exists \( C > 0 \) such that, for every \( u \in W^{1,\infty}_0(\Omega) \cap \mathcal{N} \) and \( i \in \mathbb{N} \)
\[
- \frac{C(2 + \|u\|^{2(p-2)/p})}{\|u\|} \leq k_i(u) \leq \frac{C}{\|u\|} \tag{3.7}
\]

**Proof.** As
\[
\langle N(u), u \rangle = 2\|u\|^2 - \int f(u)u\,dx - \int f'(u)u^2\,dx
\]
by (2.8) and (f2)
\[
|\langle N(u), u \rangle| \geq \frac{1-\beta}{\beta} \cdot \|u\|^2 ,
\]
and, by Schwarz inequality,
\[
\|N(u)\| \geq \frac{1-\beta}{\beta} \|u\| . \tag{3.8}
\]
In view of (3.5), we conclude from Lemma 2 and (3.8) the right hand-side of (3.7). In order to prove the complete estimate it suffices to set the inequality to \( k_1 \). Assume \( \|v\| = 1 \). Necessarily
\[
\lambda_1 \geq \lambda := \min_{\|v\|=1} - \int_{\Omega} h'(u)v^2(x)\,dx .
\]
By (f3’) and (3.1),
\[
h'(u) \leq C_1 |u|^{p-2}
\]
for \( C_1 = \xi_2/(p-1) \). Then, by Holder inequality, (2.8) and Sobolev Imbedding Theorem, for some constant \( C_2 > 0 \)
\[
\int_{\Omega} h'(u)v^2(x)\,dx \leq C_1 \left( \int_{\Omega} |u|^p(x)\,dx \right)^{p-2/p} \left( \int_{\Omega} |v|^p(x)\,dx \right)^2
\]
\[
\leq C_2 \left( \int_{\Omega} f(u)u(x)\,dx \right)^{2(p-2)/p} = C_2 \|u\|^{2(p-2)/p} , \tag{3.9}
\]
thereby proving inequality (3.7). \( \blacksquare \)
Remark 5 Note that, if \( p \leq 4 \), the curvatures are uniformly bounded below on the Nehari manifold by a negative constant. In particular, there exists \( K > 0 \) such that, for all \( u \in \mathcal{N} \),
\[
|k_i(u)| \leq K \quad \forall i \in \mathbb{N}.
\]

Analogously to Proposition 1, we obtain lower bounds on the energy of \( u \in \mathcal{N} \) based on the number of negative principal curvatures of the Weingarten map \( L_u \).

**Proposition 3** Assume (f1)-(f2)-(f3'). Let \( u \in \mathcal{N} \cap W^{1,\infty}_0(\Omega) \) be such that
\[
k_i(u) \leq 0, \quad i = 1, \ldots, j.
\]
Then, there exist positive constants \( C_1 \) and \( C_2 \) independent of \( u \) such that
\[
J(u) \geq \max\{C_1\|e_j\|^{\frac{2p}{p-2}}, C_2\},
\]
where \( e_j \) was defined in (2.4).

**Proof.** The proof is similar to the proof of Proposition 1 so we omit the details. Consider the subspaces
\[
V_j := \text{span}\{v_1, \ldots, v_j\} \quad \text{and} \quad E_j = (\text{span}\{e_1, \ldots, e_{j-1}\})^\perp
\]
where the \( v_i \)'s are eigenvectors associated to \( k_1, \ldots, k_j \) (necessarily, \( \bar{j} \geq j \)). For any \( v \in V_j \),
\[
\langle Dn_u(v), v \rangle = \|N(u)\|^{-2} \left( 2 \|v\|^2 - \int_\Omega h'(u)v^2(x) \, dx \right) \leq 0,
\]
(3.10)
or
\[
\|v\|^2 - \frac{1}{2} \int_\Omega h'(u)v^2(x) \, dx \leq 0.
\]
As in Lemma 1, we may choose \( v \in V_j \cap E_j \) such that \( \int_\Omega H(v(x)) \, dx = 1 \) for \( H(v) = \int_0^v h(s) \, ds \). Recalling that, by (3.1), \( h'(u) = 2f'(u) + f''(u)u \), we may estimate as in (2.14)–(2.16) and conclude the proof.

**Remark 6** We may assert the existence of points on the Nehari manifold with an arbitrarily large number of negative principal curvatures. In fact, let us consider a multi-bump function
\[
u := \sum_{k=1}^n v_k
\]

12
where, for \( i \neq j \),
\[
support(v_i) \cap support(v_j) = \emptyset
\]
and
\[
v_k \in \mathcal{N} \cap W^{1,\infty}_0(\Omega)
\]
for all \( k = 1, \ldots, n \). Since
\[
2\|v_i\|^2 - \int_\Omega h'(u)v_i^2(x) \, dx = 2\|v_i\|^2 - \int_\Omega h'(u)v_i^2(x) \, dx
\]
and the set of functions \( \{v_i\}_{i=1,\ldots,n} \) is orthogonal, we conclude that
\[
k_1 < \ldots < k_{n-1} < 0,
\]
where \( k_i \) is the sequence of eigenvalues of \( L_u \).

4 An angle-decreasing flow.

In the next section, we assume
\[
f(u) = \begin{cases} c_1|u|^{p-2}u & \text{if } u \leq 0 \\ c_2|u|^{p-2}u, & \text{if } u > 0 \end{cases}
\]
(4.1)

where \( c_1, c_2 > 0 \). In case where the non-linearity \( f \) is as in (4.1), then
\[
J(u) = \left( \frac{1}{2} - \frac{1}{p} \right) \|u\|^2 \quad \forall u \in \mathcal{N}.
\]

In particular, critical points of the distance functional \( u \mapsto \|u\| \) constrained to \( \mathcal{N} \) are solutions of (2.2).

We introduce an auxiliary functional on the Nehari manifold:
\[
\theta_u \equiv \theta(u) = : \left\langle n(u), \frac{u}{\|u\|} \right\rangle.
\]

The functional \( \theta \) is the restriction to \( \mathcal{N} \) of a functional of class \( C^1(H_0^1(\Omega)\setminus\{0\}, \mathbb{R}) \) that we will denote by \( \theta \). Note that, by (3.2) and Schwarz inequality
\[
\theta(\mathcal{N}) \subset [-1,0[.
\]

Also, \( \arccos(\theta_u) \) corresponds to the angle between the vectors \( u \) and \( n(u) \).

Assuming \( u \in W^{1,\infty}_0(\Omega) \), we use our previous decomposition of the tangent space \( \mathcal{T}_u \) to calculate
\[
\Pi_u(\nabla \theta_u).
\]
For any \( v \in T_u \),
\[
\langle \nabla \theta_u, v \rangle = D\theta_u(v) = \left\langle Dn(u)[v], \frac{u}{\|u\|} \right\rangle - \langle n(u), u \rangle \frac{\langle u, v \rangle}{\|u\|^3} \quad (4.2)
\]
Choosing \( v \) an eigenvector with corresponding eigenvalue \( k \), as \( \langle n, v \rangle = 0 \) we obtain by (3.4),
\[
\langle \nabla \theta_u, v \rangle = \left( k - \frac{\theta_u}{\|u\|} \right) \langle v, \frac{u}{\|u\|} \rangle. \quad (4.3)
\]
We may write, in the non-degenerate case \( \text{Ker}(T_u) = \{0\} \),
\[
\Pi_u(\nabla \theta_u) = \frac{1}{\|u\|} \sum_{i=1}^{\infty} \left( k_i - \frac{\theta_u}{\|u\|} \right) \langle v_i, u \rangle \cdot v_i. \quad (4.4)
\]
More generally, denoting by \( \Pi_0^u \) the projection on \( \text{Ker}(T_u) \subset T_u \),
\[
\Pi_u(\nabla \theta_u) = \frac{1}{\|u\|} \sum_{i=1}^{\infty} \left( k_i - \frac{\theta_u}{\|u\|} \right) \langle v_i, u \rangle \cdot v_i + 2 - \frac{\theta_u}{\|u\|^2} \cdot \Pi_0^u(u) \quad (4.5)
\]
**Remark 7** Using (4.4)–(4.5) and Lemma 3, a simple estimate shows that, for some \( C > 0 \),
\[
\|\Pi_u(\nabla \theta_u)\| \leq C\|u\|^{(p-4)/p} \leq C\|u\|, \quad \forall u \in \mathcal{N}.
\]
In case \( \nabla J(u) = 0 \) then \( \nabla \theta_u = 0 \) but the inverse is not true. However, in case \( \theta_u/\|u\| \notin \mathcal{R}_u \),
\[
\nabla J(u) = 0 \iff \nabla \theta_u = 0.
\]
Note that eventually unstable stationary points of the \( H^1 \)-distance decreasing flow on the Nehari–manifold are minimizers of the angle functional. We have the following

**Proposition 4** Let
\[
\Phi : W^{1,\infty}_0(\Omega) \cap \mathcal{N} \mapsto W^{1,\infty}_0(\Omega), \quad \Phi(u) = \Pi_u(\nabla \theta_u).
\]
Given \( u_0 \in \mathcal{N} \cap W^{1,\infty}_0(\Omega) \), the initial value problem
\[
\eta(0, u_0) = u_0, \quad \frac{d\eta}{dt}(t) = -\Phi(\eta(t, u_0)). \quad (4.6)
\]
has a unique solution
\[
\eta : \mathcal{N} \times [0, \tau_0] \mapsto \mathcal{N},
\]
for some \( \tau_0 > 0 \). In case \( \Omega \) is a bounded regular domain of \( \mathbb{R}^2 \) then \( \tau_0 = +\infty \) for all \( u_0 \in \mathcal{N} \). Moreover, for \( 0 < t_1 < t_2 \)
\[
\theta(\eta(t_1, u_0)) \geq \theta(\eta(t_2, u_0)). \quad (4.7)
\]
The proof of Proposition 4 will follow from the next lemmas.

**Lemma 4** Let \( f : \mathbb{R} \to \mathbb{R} \) be a locally Lipschitz function. Define \( \Psi : W^{1,\infty}_0(\Omega) \to W^{1,\infty}_0(\Omega) \), \( u \to \Delta^{-1}(f(u)) \).

Then \( \Psi \) is locally Lipschitz continuous.

**Proof.** Trivially, \( W^{1,\infty}_0(\Omega) \subset C_0(\Omega) \) with continuous injection. By standard regularity theory (see [3], theorems 8.33-8.34) we have \( \Delta^{-1}(f(u)) \in C^{1,\alpha}_0(\Omega) \) so \( \Psi \) is well-defined.

Let \( B_\epsilon(u) \) be the ball of radius \( \epsilon \) and center \( u \) in \( C_0(\Omega) \). By our assumptions on \( f \), for any \( v \in B_\epsilon(u) \), we have

\[
|f(u) - f(v)| \leq K \epsilon |u - v|,
\]
for some \( K \epsilon > 0 \). We conclude that the functional

\[
\psi : W^{1,\infty}_0(\Omega) \to C(\overline{\Omega}) \, , \quad u \to f(u)
\]

is locally Lipschitz continuous. Since \( \Delta^{-1} : C(\overline{\Omega}) \to C^{1,\alpha}_0(\Omega) \) is Lipschitz continuous, we conclude that \( \Psi = \Delta^{-1} \circ \psi \) is locally Lipschitz continuous. The proof is complete.

**Remark 8** With similar arguments, we may prove that, for locally Lipschitz functions \( f, g : \mathbb{R} \mapsto \mathbb{R} \),

\[
u \mapsto \Delta^{-1}[\Delta^{-1}(f(u))g(u)]
\]
is locally Lipschitz continuous in \( W^{1,\infty}_0(\Omega) \).

**Lemma 5** Let

\[
\Phi : W^{1,\infty}_0(\Omega) \cap \mathcal{N} \mapsto W^{1,\infty}_0(\Omega) \, , \quad \Phi(u) = \Pi_u(\nabla \theta_u).
\]

For any \( u_1 \in W^{1,\infty}_0(\Omega) \cap \mathcal{N} \), there exists a \( W^{1,\infty} \)-ball \( B_1 \) centered at \( u_1 \) and a Lipschitz continuous function

\[
F : B_1 \cap W^{1,\infty}_0(\Omega) \mapsto W^{1,\infty}_0(\Omega)
\]
such that

\[
F(u) = \Phi(u) , \quad \forall u \in B_1 \cap W^{1,\infty}_0(\Omega) \cap \mathcal{N}.
\]
Proof. Let $B_1$ be a $W^{1,\infty}$-ball centered at $u_1$ such that $\|u\|$ and $\|N(u)\|$ are uniformly bounded below by a positive constant in $B_1$. We consider the following extensions $N, n : B_1 \mapsto W_0^{1,\infty}(\Omega)$ and $\theta : B_1 \mapsto \mathbb{R}$,

$$N(u) = 2u + \Delta^{-1}(h(u)), n(u) = \frac{N(u)}{\|N(u)\|}, \theta(u) = \left\langle n(u), \frac{u}{\|u\|} \right\rangle.$$ 

In the homogeneous case, $h(u) = f(u) + f'(u)u = pf(u)$. Then,

$$\theta(u) = \frac{1}{\|N(u)\| \cdot \|u\|} \left(2\|u\|^2 - p\int_{\Omega} f(u)u \, dx\right).$$

Define

$$J_1(u) = \|N(u)\|^2 = 4\|u\|^2 + 4p\langle u, \Delta^{-1}(f(u)) \rangle + p^2\|\Delta^{-1}(f(u))\|^2,$$

and

$$J_2(u) = \left(2\|u\| - \frac{p}{\|u\|} \int_{\Omega} f(u)u \, dx\right),$$

so that

$$\theta(u) = \frac{J_2(u)}{\sqrt{J_1(u)}}. \quad (4.8)$$

By Lemma 4 $J_1 : W_0^{1,\infty}(\Omega) \to \mathbb{R}$ is locally Lipschitz continuous. Moreover

$$\langle \nabla J_1(u), v \rangle = 8\langle u, v \rangle + 4p\langle v, \Delta^{-1}(f(u)) \rangle + 4p\langle u, \Delta^{-1}(f'(u)v) \rangle + 2p^2\langle \Delta^{-1}(f'(u)v), \Delta^{-1}(f(u)) \rangle. \quad (4.9)$$

As

$$\langle u, \Delta^{-1}(f'(u)v) \rangle = -\int_{\Omega} f'(u)uv \, dx = (p - 1)\langle v, \Delta^{-1}(f(u)) \rangle,$$

and

$$\langle \Delta^{-1}(f'(u)v), \Delta^{-1}(f(u)) \rangle = (p - 1)\langle \Delta^{-1}(f(u))f'(u) \rangle,$$

we conclude that

$$\nabla J_1(u) = 8u + 4p^2\Delta^{-1}(f(u)) + 2p^2(p - 1)\Delta^{-1}[\Delta^{-1}(f(u))f'(u)]. \quad (4.10)$$

Then, by Remark 8, we conclude that $\nabla J_1 : W_0^{1,\infty}(\Omega) \to W_0^{1,\infty}(\Omega)$ is locally Lipschitz continuous. Similarly, we may prove that $J_2 : W_0^{1,\infty}(\Omega) \to \mathbb{R}$ and $\nabla J_2 : W_0^{1,\infty}(\Omega) \to W_0^{1,\infty}(\Omega)$ are locally Lipschitz continuous. We conclude
from (4.8) that $\nabla \theta : W_0^{1,\infty}(\Omega) \cap B_1 \to W_0^{1,\infty}(\Omega)$ is Lipschitz continuous. Finally, writing

$$F(u) := \nabla \theta(u) - \langle \nabla \theta(u), n(u) \rangle n(u)$$

we conclude that

$$F : W_0^{1,\infty}(\Omega) \cap B_1 \mapsto W_0^{1,\infty}(\Omega),$$

is Lipschitz continuous and

$$F(u) = \Pi_u(\nabla \theta_u), \quad \forall u \in \mathcal{N}.$$
As \( \eta([0, \tau_0]) \subset \mathcal{N} \cap W^{1,\infty}_0(\Omega) \), for any \( u \in \eta([0, \tau_0]) \) we may provide an orthonormal basis of \( \Sigma_u \) consisting of eigenvectors of \( L_u \). Let us study how the norm of the projection \( \Pi_\eta(\eta) \) and of the normal component \( \langle \eta, n \rangle \cdot n \) evolve along the flow defined in (4.6). For simplicity of notation, we assume \( \text{Ker}(T_u) = \{0\} \) although minor changes provide the more general case.

\[
\frac{d}{dt} \left( \frac{1}{2} \|\Pi_\eta(\eta(t))\|^2 \right) = \\
\left\langle D \Pi_\eta(\eta'(t)), \Pi_\eta(\eta(t)) \right\rangle + \left\langle \Pi_\eta(\eta'(t)), \Pi_\eta(\eta(t)) \right\rangle
\]  
(4.11)

Denoting \( \eta(t) = u \) and \( n(u) = n \), we have, by (4.4),

\[
\left\langle \Pi_\eta(\eta'(t)), \Pi_\eta(\eta(t)) \right\rangle = \left\langle -\Pi_u(\nabla \theta_u), u \right\rangle = \frac{1}{\|u\|} \sum_{i=1}^{\infty} \left( -k_i + \frac{\theta_u}{\|u\|} \right) \langle v_i, u \rangle^2.
\]  
(4.12)

Also

\[
\left\langle D \Pi_\eta(\eta'(t)), \Pi_\eta(\eta(t)) \right\rangle = \left\langle D \Pi_u(-\Pi_u(\nabla \theta_u), u), \Pi_u(u) \right\rangle.
\]  
(4.13)

We decompose

\[
D \Pi_u(-\Pi_u(\nabla \theta_u), u) = D \Pi_u(-\Pi_u(\nabla \theta_u), \Pi_u(u) + \langle u, n \rangle n)
\]

and observe that

\[
D \Pi_u(-\Pi_u(\nabla \theta_u), \Pi_u(u)) \in \Sigma_u^\perp
\]

(since it is the second fundamental form of \( \mathcal{N} \) at \( u \)). Then, by (3.3), we may re-write (4.13)

\[
\left\langle D \Pi_u(-\Pi_u(\nabla \theta_u), u), \Pi_u(u) \right\rangle = \langle u, n \rangle \left\langle D n(u)[\Pi_u(\nabla \theta_u)], u \right\rangle = \sum_{i=1}^{\infty} \theta_u k_i \left( k_i - \frac{\theta_u}{\|u\|} \right) \langle u, v_i \rangle^2.
\]  
(4.14)

Combining (4.11), (4.12) and (4.14) we obtain, for \( u = \eta(t) \),

\[
\frac{d}{dt} \left( \frac{1}{2} \|\eta(t)\|^2 \right) = \sum_{i=1}^{\infty} \left( k_i(\eta) - \frac{\theta_u}{\|\eta\|} \right) \left( \theta_u k_i(\eta) - \frac{1}{\|\eta\|} \right) \langle \eta, v_i(\eta) \rangle^2.
\]  
(4.15)

Let us turn to the study of the normal component \( \langle \eta, n \rangle n \). Differentiating in \( t \), assuming \( \eta(t) = u \), we obtain

\[
\frac{d}{dt} \left( \frac{1}{2} \langle \eta(t), n(\eta(t)) \rangle^2 \right) = \langle u, n \rangle \left( \langle u, D n_u(-\Pi_u(\nabla \theta_u)) \rangle + \langle -\Pi_u(\nabla \theta_u), n \rangle \right).
\]

18
Noting that \(-\Pi u(\nabla \theta u), n) = 0\), we may write, for \(u = \eta(t)\),
\[
\frac{d}{dt} \left( \frac{1}{2} \langle \eta(t), n(\eta(t)) \rangle^2 \right) = \sum_{i=1}^{\infty} \theta_{\eta} k_i(\eta) \left( -k_i(\eta) + \frac{\theta_{\eta}}{\|\eta\|} \right) \langle \eta, v_i(\eta) \rangle^2.
\]
(4.16)

We have the following monotone property of the angle decreasing flow:

**Proposition 5** Let \(u \in \mathcal{N} \cap W^{1,\infty}(\Omega)\). Consider the solution \(\eta(t, u)\) of (4.6) and denote \(\eta^\top = \Pi \eta \eta\) and \(\eta^\perp = \eta - \eta^\top\).

If \(K_u \cap \frac{\theta_{\eta}}{\|\eta\|} - 1 \neq \emptyset\), then \(\frac{d}{dt} \|\eta^\top\|_{\eta = u} \leq 0\). In case
\[
K_u \cap \frac{\theta_{\eta}}{\|\eta\|}, 0 = \emptyset
\]
then \(\frac{d}{dt} \|\eta^\perp\|_{\eta = u} \geq 0\).

**Proof.** In the non-degenerate case, the proof follows from (4.16), (4.15), recalling that \(\theta_{\eta} < 0\) for all \(\eta \in \mathcal{N}\). In the general case, we obtain
\[
\frac{d}{dt} \left( \frac{1}{2} \|\Pi_\eta \eta\|^2 \right) = \sum_{i=1}^{\infty} \left( k_i(\eta) - \frac{\theta_{\eta}}{\|\eta\|} \right) \left( \theta_{\eta} k_i(\eta) - \frac{1}{\|\eta\|} \right) \langle \eta, v_i(\eta) \rangle^2 + K_1 \|\Pi_\eta \eta\|^2.
\]
(4.17)

where \(K_1 = (2/\|\eta\| - \theta_{\eta}/\|\eta\|^{-1})(2\theta_{\eta}/\|N(\eta)\| - \|\eta\|^{-1}) < 0\) and
\[
\frac{d}{dt} \left( \frac{1}{2} \langle \eta(t), n(\eta(t)) \rangle^2 \right) = \sum_{i=1}^{\infty} \theta_{\eta} k_i(\eta) \left( -k_i(\eta) + \frac{\theta_{\eta}}{\|\eta\|} \right) \langle \eta, v_i(\eta) \rangle^2 + K_2 \|\Pi_\eta \eta\|^2.
\]
(4.18)

where \(K_2 = 2\theta_{\eta}/\|N(\eta)\|(-2/\|N(\eta)\| + \theta_{\eta}/\|\eta\|) > 0\) and the proof follows from similar estimates.

**An Example.** We will now study an example of convergence of the angle decreasing flow \(\eta(t, u_0)\) to a critical point of the distance functional on \(\mathcal{N}\) —i.e. a solution of (2.2). We assume \(\Omega\) is a bounded regular domain of \(\mathbb{R}^2\) so that, by Proposition 4, \(\eta(\cdot, u_0)\) is defined in \([0, +\infty[\). Moreover we assume that, for all \(t > 0\), \(\text{Ker}(T_{\eta(t, u_0)}) = \{0\}\). This last hypotheses may be removed provided minor changes are added to the following assumptions. We shall denote \(\theta(u_0) = c\) (recall \(-1\leq c < 0\)) and suppose the following:
There exists a positive sequence \((\alpha_n)_{n \in \mathbb{N}}\) such that
\[
\sum_{n=1}^{\infty} \alpha_n < \infty \quad \text{and} \quad u \in \eta([0,+\infty[, u_0) \Rightarrow \langle u, v_n(u) \rangle^2 \leq \alpha_n \langle u, v_1(u) \rangle^2.
\]
(As usual, \((v_n(u))\) is the basis of \(\Xi_u\) composed by the eigenvectors of the Weingarten map \(L_u\).)

There exists \(K, \rho > 0\) and \(\bar{n}\) such that, for all \(u \in \eta([0,+\infty[, u_0)\),
\[
-K \leq k_{\bar{n}}(u) \leq -\left(\rho + \frac{1}{|c|}\right) \frac{1}{\|u\|},
\]

For \((\alpha_n)\), \(\bar{n}\) and \(\rho\) defined in (N1)-(N2), for some \(C_1 > 0\),
\[
-c\rho^2 + \left(\frac{1}{2c} - \frac{c}{2}\right)^2 \sum_{i=\bar{n}+1}^{\infty} \alpha_i \leq -C_1 \cdot \sum_{n=1}^{\infty} \alpha_n
\]
\[
-c\rho^2 - \frac{1}{4} \sum_{i=\bar{n}+1}^{\infty} \alpha_i \geq C_1  \tag{4.19}
\]

Intuitively, assumptions (N1)–(N3) impose that, all along the flow, \(\Pi_{\eta}(\eta)(t)\) mainly concentrates on directions of the tangent space associated to certain negative eigenvalues of \(L_u\). We have the following convergence property.

Suppose conditions (N1)–(N3) are verified. Then, as \(t \to \infty\), \(\eta(t, u_0)\) converges in \(H^1\)-norm to a critical point \(u^*\) of \(J\). Moreover, \(\|\eta^{-}(t, u_0)\|\) is a decreasing function of \(t\) whereas \(\|\eta^{+}(t, u_0)\|\) is an increasing function of \(t\).

The proof of the convergence will follow after a number of steps. For simplicity, denote \(\eta(t) := \eta(t, u_0)\). We have \(\theta_\eta \leq c\) so that, by (N1)
\[
|\langle \eta(t), v_n(\eta(t)) \rangle| \leq \alpha_n |\langle \eta(t), v_1(\eta(t)) \rangle| \quad \forall n \in \mathbb{N}, \forall t \in [0,+\infty[.
\]

**Step 1: Increasing of \(\|\eta^{+}(t)\|\)**

We prove that the norm of the normal component is an increasing function of \(t\). As usual, we shall denote \(k_i := k_i(\eta)\) and \(v_i := v_i(\eta)\). We have, by (4.16),
\[
\frac{d}{dt} \frac{1}{2} \|\eta^{+}\|^2 = \sum_{i=1}^{\bar{n}} \theta_\eta k_i \left(-k_i + \frac{\theta_\eta}{\|\eta\|}\right) \langle \eta, v_i \rangle^2 + \sum_{i=\bar{n}+1}^{\infty} \theta_\eta k_i \left(-k_i + \frac{\theta_\eta}{\|\eta\|}\right) \langle \eta, v_i \rangle^2. \tag{4.21}
\]
Note that, by (N1),
\[ \sum_{i=\bar{n}+1}^{\infty} \theta_{\eta} k_i \left( -k_i + \frac{\theta_{\eta}}{\|\eta\|} \right) \langle \eta, v_i \rangle^2 \geq \sum_{i=\bar{n}+1}^{\infty} \frac{\theta_{\eta}^3}{4\|\eta\|^2} \langle \eta, v_i \rangle^2 \]
\[ \geq -\frac{1}{4\|\eta\|^2} \langle \eta, v_1 \rangle^2 \sum_{i=\bar{n}+1}^{\infty} \alpha_i. \quad (4.22) \]

By (N2), (4.20), (4.21) and (4.22)
\[ \frac{d}{dt} \frac{1}{2} \|\eta\|^2 \geq \left( \frac{|c| \rho^2}{\|\eta\|^2} - \frac{1}{4\|\eta\|^2} \sum_{i=\bar{n}+1}^{\infty} \alpha_i \right) \langle u, v_1 \rangle^2 \geq \frac{C_1}{\|\eta\|^2} \langle u, v_1 \rangle^2 \geq 0. \quad (4.23) \]

**Step 2: Decreasing of \( \|\eta^\top(t)\| \)**

By (4.15), (N1)–(N3) we write
\[ \frac{d}{dt} \frac{1}{2} \|\eta^\top(t, u_1)\|^2 = \]
\[ \sum_{i=1}^{\bar{n}} \left( k_i - \frac{\theta_{\eta}}{\|\eta\|} \right) \left( \theta_{\eta} k_i - \frac{1}{\|\eta\|} \right) \langle \eta, v_i \rangle^2 + \sum_{i=\bar{n}+1}^{\infty} \left( k_i - \frac{\theta_{\eta}}{\|\eta\|} \right) \left( \theta_{\eta} k_i - \frac{1}{\|\eta\|} \right) \langle \eta, v_i \rangle^2 \]
\[ \leq -\frac{|c| \rho^2}{\|\eta\|^2} \langle \eta, v_1 \rangle^2 + \frac{\gamma}{\|\eta\|^2} \langle \eta, v_1 \rangle^2 \cdot \sum_{i=\bar{n}+1}^{\infty} \alpha_i, \]

where
\[ \gamma = \left( \frac{c}{2} - \frac{1}{2c} \right)^2. \]

Therefore, by (4.19),
\[ \frac{d}{dt} \frac{1}{2} \|\eta^\top(t)\|^2 \leq -\frac{C_1}{\|\eta(t)\|^2} \langle \eta(t), v_1 \rangle^2. \]

As \( \langle \eta, v_1 \rangle^2 \geq (\sum_{n=1}^{\infty} \alpha_n)^{-1} \|\eta^\top\|^2 \), we conclude
\[ \frac{d}{dt} \frac{1}{2} \|\eta^\top(t)\|^2 \leq -\frac{C_1}{\|\eta(t)\|^2} \|\eta^\top(t)\|^2. \quad (4.24) \]

**Step 3: Convergence of \( \eta(t) \) to a global minimum of \( \theta \)**

By the previous steps, we have
\[
\frac{d}{dt}\|\eta\|^2 = \frac{d}{dt}\|\eta^\perp\|^2 + \frac{d}{dt}\|\eta^\top\|^2 \leq \frac{d}{dt}\|\eta^\perp\|^2, \tag{4.25}
\]

and, by (N2) and (4.21), for some \(K\) such that \(|k_i| \leq K\),
\[
\frac{d}{dt}\|\eta^\perp\|^2 \leq K^2\|\eta^\top\|^2. \tag{4.26}
\]

We conclude from (4.25)–(4.26)
\[
\frac{d}{dt}\|\eta\|^2 \leq K^2\|\eta^\top\|^2.
\]
or
\[
\|\eta\|^2(t) \leq \|\eta\|^2(0) + K^2\int_0^t \|\eta^\top\|^2(s) \, ds.
\]

Then, by (4.24),
\[
\frac{d}{dt}\|\eta^\top(t)\|^2 \leq -\frac{2C_1\|\eta^\top(t)\|^2}{\|\eta(0)\|^2 + K^2\int_0^t \|\eta^\top(s)\|^2 \, ds}.
\]

By Lemma 7 (Appendix), we conclude that
\[
\int_0^{+\infty} \|\eta^\top(t)\| \, dt \leq C(C_1, \|\eta(0)\|, K).
\]

In particular, by (2.18) and (4.4),
\[
\|\eta'(t)\| = \|\Pi_\eta(\nabla \theta_\eta)\| \leq \frac{K}{\|\eta\|} \|\eta^\top\| \leq M\|\eta^\top\| \tag{4.27}
\]
for an adequate constant \(M\) independent of \(\eta\). Then
\[
\int_0^{+\infty} \|\eta'(t)\| \, dt \leq C(C_1, \|\eta(0)\|, K),
\]

and the flow \(\eta(t)\) necessarily converges in \(H^1\)-norm to \(u^*\). By (N1) and (4.23), using a simple approximation argument, one concludes that \(\Pi_{u^*}(u^*) = 0\). Then \(\theta(u^*) = -1\), \(u^*\) is a critical point of the distance functional on the Nehari Manifold and a solution to (2.2).

\begin{remark}
Note that, in view of Remark 7 and estimate (4.27), the \(H^1\)-convergence of \(\eta(t)\) and \(\eta^\perp(t)\) are equivalent.
\end{remark}
5 Appendix

5.1 A suitable basis of $H^1_0(\Omega)$.

Let $F \in C(\mathbb{R}, \mathbb{R})$ be such that $F(0) = 0$, $F(u) > 0$ if $u \neq 0$. Moreover, assume

$$\lim_{u \to \pm\infty} F(u) = +\infty,$$

and

$$\lim_{u \to \pm\infty} \frac{F(u)}{|u|^q} = 0,$$

for some $1 \leq q < 2^*$.

We define by recurrence a family of orthogonal vectors. Consider the following minimization problem:

$$\min \left\{ \int_{\Omega} |\nabla u|^2(x) \, dx : u \in H^1_0(\Omega), \int_{\Omega} F(u)(x) \, dx = 1 \right\}.$$

(5.3)

By (5.1)–(5.2), a minimizer exists, that we shall denote by $e_1$. More generally, we define $e_n$ to be a minimizer of the Dirichlet integral $\int_{\Omega} |\nabla u|^2(x) \, dx$ over the weakly closed set

$$\left\{ u \in H^1_0(\Omega) : \int_{\Omega} F(u)(x) \, dx = 1 \text{ and } u \in \langle e_1, \ldots, e_{n-1} \rangle_{\perp} \right\}.$$

**Lemma 6** The sequence $(e_n)$ is an orthogonal basis of $H^1_0(\Omega)$. Also $(\|e_n\|)$ is non-decreasing and 

$$\lim_{n \to \infty} \|e_n\| = \infty.$$

**Proof.**

Trivially, the sequence $(\|e_n\|)$ is non-decreasing. We assert that

$$\lim_{n \to \infty} \|e_n\| = \infty.$$

Suppose, in view of a contradiction, the existence of $C > 0$ such that $\|e_n\| \leq C$ for all $n \in \mathbb{N}$. Passing to a weakly convergent subsequence, denoted by $(e_{n_j})$, we have

$$e_{n_j} \rightharpoonup v \quad \text{and} \quad \int_{\Omega} F(v)(x) \, dx = 1.$$

(5.4)

Let $n_j \in \mathbb{N}$ be fixed. We have

$$\langle v, e_{n_j} \rangle = \lim_{k \to \infty} \langle e_{n_k}, e_{n_j} \rangle = 0.$$
Now letting \( n_j \to \infty \) we conclude \( \|v\| = 0 \) and contradict (5.4). The assertion is proved.

Let \( w \in H^1_0(\Omega) \) be such that
\[
\langle w, e_i \rangle = 0 \quad \text{for all } i \in \mathbb{N}.
\] (5.5)
If \( w \neq 0 \) assume (without loss of generality)
\[
\int_\Omega F(w)(x) \, dx = 1.
\]
The previous assertion, together with (5.5), imply that there exists \( n \in \mathbb{N} \) such that \( \|e_n-1\| \leq \|w\| < \|e_n\| \). This, contradicts the definition of the function \( e_n \). Then \( w = 0 \) and the proof is complete. \( \square \)

## 5.2 A Gronwall type estimate

**Lemma 7** Let \( f \in C^1([0, +\infty[, \mathbb{R}^+) \) be such that
\[
f'(t) \leq -\frac{f(t)}{a + b \int_0^t f(u) \, du}
\] (5.6)
for some \( a, b > 0 \). Then
\[
\int_0^\infty \sqrt{f(u)} \, du \leq C(a, b, f(0)).
\] (5.7)

**Proof.** Integrating equation (5.6),
\[
f(t) - f(0) \leq -\frac{1}{b} \left[ \ln \left( a + b \int_0^t f(u) \, du \right) \right]_0^t,
\]
or
\[
f(t) + \frac{1}{b} \ln \left( a + b \int_0^t f(u) \, du \right) \leq f(0) + \frac{\ln(a)}{b}
\]
and, as \( f(t) \geq 0 \), we conclude, by passing to the limit in \( t \),
\[
\ln \left( a + b \int_0^{+\infty} f(u) \, du \right) \leq bf(0) + \ln(a)
\]
or
\[
\int_0^{+\infty} f(u) \, du \leq C_1
\] (5.8)
where \( C_1 = (ae^{bf(0)} - a)/b \). Writing \( f(t) = h^2(t) \) with \( h(t) > 0 \), inequality (5.6) becomes
\[
2h(t)h'(t) \leq -\frac{h^2(t)}{a + b \int_0^t f(u) \, du}.
\]
By (5.6)–(5.8), we conclude

\[ h'(t) \leq -\frac{h(t)}{2(a + bC_1)} \]

or

\[ h(t) \leq \sqrt{f(0)e^{-C_2 t}}, \]

where \( C_2 = (2(a + bC_1))^{-1} \). This proves the lemma.

**References**


José Maria Gomes
Centro de Matemática e Aplicações,
Faculdade de Ciências e Tecnologia de Universidade Nova de Lisboa
2829-516 Caparica
Portugal.

e-mail: jm.gomes@fct.unl.pt