The maximal subsemigroups of semigroups of transformations preserving or reversing the orientation on a finite chain

Ilinka Dimitrova\(^1\), Vítor H. Fernandes\(^2\) and Jörg Koppitz\(^1\)

Faculty of Mathematics and Natural Science
South-West University "Neofit Rilski"
2700 Blagoevgrad, Bulgaria
ilinka.dimitrova@yahoo.com

Departamento de Matemática
Faculdade de Ciências e Tecnologia
Universidade Nova de Lisboa
2829-516 Caparica, Portugal
also: Centro de Álgebra da Universidade de Lisboa, Av. Prof. Gama Pinto 2, 1649-003 Lisboa, Portugal
vhf@fct.unl.pt

Institute of Mathematics
University of Potsdam
14469 Potsdam, Germany
koppitz@rz.uni-potsdam.de

Abstract

The study of the semigroups \(\mathcal{OP}_n\) and \(\mathcal{OR}_n\) respectively of all orientation-preserving transformations and of all orientation-preserving or orientation-reversing transformations on an \(n\)-element chain has began in [10] and [4]. In order to bring more insight into the subsemigroup structure of \(\mathcal{OP}_n\) and \(\mathcal{OR}_n\), we characterize their maximal subsemigroups.

Keywords: finite transformation semigroup, orientation-preserving and orientation-reversing transformations, maximal subsemigroups.

2000 Mathematics Subject Classification: 20M20

Introduction and Preliminaries

For \(n \in \mathbb{N}\), let \(X_n = \{1 < 2 < \cdots < n\}\) be a finite chain with \(n\) elements. As usual, we denote by \(T_n\) the monoid (under composition) of all full transformations of \(X_n\). For every transformation \(\alpha \in T_n\), we denote by \(\ker \alpha\) and \(\text{Im} \alpha\) the kernel and the image of \(\alpha\), respectively. The number \(\text{rank} \alpha = |\ker \alpha| = |\text{Im} \alpha|\) is called the rank of \(\alpha\).

Given a subset \(U\) of \(T_n\), we denote by \(E(U)\) its set of idempotents. The weight of an equivalence relation \(\pi\) on \(X_n\) is the number \(|X_n/\pi|\). Let \(A \subseteq X_n\) and let \(\pi\) be an equivalence relation on \(X_n\) of weight \(|A|\). We say that \(A\) is a transversal of \(\pi\) (denoted by \(A \# \pi\)) if \(|A \cap \bar{x}| = 1\) for every equivalence class \(\bar{x}\) of \(\pi\). A subset \(C\) of the chain \(X_n\) is said to be convex if \(x, y \in C\) and \(x \leq z \leq y\) together imply that \(z \in C\). An equivalence relation \(\pi\) on \(X_n\) is convex if its classes are convex.

We say that a transformation \(\alpha \in T_n\) is order-preserving (respectively, order-reversing) if \(x \leq y\) implies that \(x\alpha \leq y\alpha\) (respectively, \(x\alpha \geq y\alpha\)), for all \(x, y \in X_n\). As usual, \(O_n\) denotes the submonoid of \(T_n\) of all order-preserving transformations of \(X_n\). This monoid has been largely studied, for instance in [1, 5, 8, 9, 12].

Let \(a = (a_1, a_2, \ldots, a_t)\) be a sequence of \(t\) (\(t \geq 1\)) elements from the chain \(X_n\). We say that \(a\) is cyclic (respectively, anti-cyclic) if there exists no more than one index \(i \in \{1, \ldots, t\}\) such that \(a_i > a_{i+1}\) (respectively,

\(^1\)The authors gratefully acknowledges support of FCT, within the project ISFL-1-143 of CAUL.

\(^2\)The author gratefully acknowledges support of FCT, within the projects ISFL-1-143 and PTDC/MAT/69514/2006 of CAUL.
Lemma 1.1. Let \( a_1 < a_{i+1} \), where \( a_{i+1} \) denotes \( a_1 \). We say that a transformation \( \alpha \in T_n \) is orientation-preserving (respectively, orientation-reversing) if the sequence of its images is cyclic (respectively, anti-cyclic). The notion of an orientation-preserving transformation was introduced by McAlister in [10] and, independently, by Catarino and Higgins in [4]. It is easy to show that the product of two orientation-preserving or of two orientation-reversing transformations is orientation-preserving, and the product of an orientation-preserving transformation by an orientation-reversing transformation is orientation-reversing (see [4]). We denote by \( \mathcal{OP}_n \) (respectively, \( \mathcal{OR}_n \)) the monoid of all orientation-preserving (respectively, orientation-preserving or orientation-reversing) full transformations. It is clear that \( \mathcal{OP}_n \) is a submonoid of \( \mathcal{OR}_n \).

Since \( \mathcal{O}_n, \mathcal{OP}_n \) and \( \mathcal{OR}_n \) are regular submonoids of \( T_n \), the definition of the Green’s relations \( \mathcal{L}, \mathcal{R} \) and \( \mathcal{H} \) on \( \mathcal{O}_n, \mathcal{OP}_n \) and \( \mathcal{OR}_n \) follow immediately from well known results on regular semigroups and from their descriptions on \( T_n \). We have \( \alpha \mathcal{L} \beta \iff \text{im} \alpha = \text{im} \beta \) and \( \alpha \mathcal{R} \beta \iff \ker \alpha = \ker \beta \), for every transformations \( \alpha \) and \( \beta \). Recall also that, for the Green’s relation \( \mathcal{J} \), we have (on \( \mathcal{O}_n, \mathcal{OP}_n \) and \( \mathcal{OR}_n \)) \( \alpha \mathcal{J} \beta \iff \text{rank} \alpha = \text{rank} \beta \), for every transformations \( \alpha \) and \( \beta \).

Regarding the monoids \( \mathcal{OP}_n \) and \( \mathcal{OR}_n \), presentations for them were exhibited by Catarino in [3] and by Arthur and Ruškuc in [2], the Green’s relations, their sizes and ranks, among other properties, were determined by Catarino and Higgins in [4] and a description of their congruences were given in [6] by Fernandes, Gomes and Jesus. In [13] Zhao, Bo and Mei characterized the locally maximal idempotent-generated subsemigroups of \( \mathcal{OP}_n \) (excluding the permutations).

In this paper, we aim to give more insight into the subsemigroup structure of the monoids \( \mathcal{OP}_n \) and \( \mathcal{OR}_n \) by characterizing the maximal subsemigroups of these monoids and of their ideals. In Section 1, we study the monoid \( \mathcal{OP}_n \) and its ideals. First, we describe all maximal subsemigroups of \( \mathcal{OP}_n \) (some of them are associated with the maximal subgroups of the additive group \( \mathbb{Z}_n \)). The main result of this section is the characterization of the maximal subsemigroups of the ideals of \( \mathcal{OP}_n \). In Section 2, we study the monoid \( \mathcal{OR}_n \) and its ideals. Again, first we describe all maximal subsemigroups of \( \mathcal{OR}_n \) (some of them are associated with the maximal subgroups of the dihedral group \( D_n \) of order \( 2n \)). The main result of this section is the characterization of the maximal subsemigroups of the ideals of \( \mathcal{OR}_n \), which are associated with the maximal subsemigroups of the ideals of \( \mathcal{OP}_n \).

1 Maximal subsemigroups of the ideals of \( \mathcal{OP}_n \)

Let \( n \in \mathbb{N} \). The semigroup \( \mathcal{OP}_n \) is the union of its \( \mathcal{J} \)-classes \( J_1, J_2, \ldots, J_n \), where

\[
J_k = \{ \alpha \in \mathcal{OP}_n \mid \text{rank} \alpha = k \},
\]

for \( k = 1, \ldots, n \). It is well known that the ideals of the semigroup \( \mathcal{OP}_n \) are the unions of the \( \mathcal{J} \)-classes \( J_1, J_2, \ldots, J_k \), i.e. the sets

\[
\mathcal{OP}(n,k) = \{ \alpha \in \mathcal{OP}_n \mid \text{rank} \alpha \leq k \},
\]

with \( k = 1, \ldots, n \). Every principal factor on \( \mathcal{OP}_n \) is a Rees quotient \( \mathcal{OP}(n,k)/\mathcal{OP}(n,k-1) \) \((2 \leq k \leq n)\) of which we may think as \( J_k \cup \{0\} \), where the product of two elements of \( J_k \) is taken to be zero if it falls into \( \mathcal{OP}(n,k-1) \).

Denote by \( L_\alpha, R_\alpha \) and \( H_\alpha \) the \( \mathcal{L} \)-class, \( \mathcal{R} \)-class and \( \mathcal{H} \)-class, respectively, of an element \( \alpha \in \mathcal{OP}_n \). Since the product \( \alpha \beta \), for \( \alpha, \beta \in J_k \), belongs to the class \( J_k \) (if and only if \( \alpha \beta \in R_\alpha \cap L_\beta \)) if and only if \( \text{im} \alpha \# \ker \beta \), it is easy to show:

**Lemma 1.1** Let \( \alpha, \beta \in J_k \), with \( k = 1, 2, \ldots, n \). Then

\[
\begin{align*}
\alpha R_\beta &= \begin{cases} 
R_{\alpha \beta} = R_\alpha & \text{if } \text{im} \alpha \# \ker \beta, \\
0 & \text{otherwise}
\end{cases} \\
L_\alpha \beta &= \begin{cases} 
L_{\alpha \beta} = L_\beta & \text{if } \text{im} \alpha \# \ker \beta, \\
0 & \text{otherwise}
\end{cases}
\end{align*}
\]

\[
\begin{align*}
L_\alpha R_\beta &= \begin{cases} 
J_k & \text{if } \text{im} \alpha \# \ker \beta, \\
0 & \text{otherwise}
\end{cases}
\end{align*}
\]

and

\[
\begin{align*}
\alpha H_\beta &= H_{\alpha \beta} = \begin{cases} 
H_{\alpha \beta} & \text{if } \text{im} \alpha \# \ker \beta, \\
0 & \text{otherwise}
\end{cases}
\end{align*}
\]
Next, recall that Catarino and Higgins proved:

**Proposition 1.2 ([4])** Let $\alpha \in J_k$, with $k = 1, 2, \ldots, n$. Then $|H_\alpha| = k$. Moreover, if $\alpha$ is an idempotent, then $H_\alpha$ is a cyclic group of order $k$.

Let $G$ be a cyclic group of order $k$, with $k \in \mathbb{N}$. It is well known that there exists an one-to-one correspondence between the subgroups of $G$ and the (positive) divisors of $k$. Moreover, if $r$ is a divisor of $k$ then there exits a (cyclic) subgroup $G_r$ of $G$ such that $|G_r| = r$. On the other hand, being $x \in G$, there exists a (positive) divisor $r$ of $k$ such that $x^r$ is the identity of $G$.

Let us consider the following elements:

$$ g = \left( \begin{array}{ccc} 1 & 2 & \cdots & n-1 & n \\ 2 & 3 & \cdots & n & 1 \end{array} \right) \in J_n \quad \text{and} \quad u_i = \left( \begin{array}{cc} 1 & 2 \\ 1 & 2 \end{array} \right) \begin{array}{c} i-1 \\ i-1 \end{array} = \left( \begin{array}{cc} i & 1 & \cdots & n \\ i+1 & \cdots & n \end{array} \right) \in J_{n-1}, $$

for $i = 1, \ldots, n$ (with $i = n$ we take $i + 1 = 1$).

Notice that $J_n = H_g$, whence $J_n$ is a cyclic group of order $n$.

We will use the following well known result (see [3, 10]).

**Proposition 1.3** $\mathcal{OP}_n = \langle u_1, g \rangle$.

Next, we present alternative generating sets of the monoid $\mathcal{OP}_n$.

**Proposition 1.4** Let $\alpha \in J_{n-1}$ and let $\gamma \in J_n$ be a permutation of order $n$. Then $\mathcal{OP}_n = \langle \alpha, \gamma \rangle$.

**Proof.** Since $\gamma \in J_n$ has order $n$, we have $\langle \gamma \rangle = J_n$ and so $g \in \langle \gamma \rangle$. From $\alpha \in J_{n-1}$, it follows that there exist $1 \leq i, j \leq n$ such that $\im \alpha = X_n \setminus \{j\}$ and $(i, i+1) \in \ker \alpha$ (by taking $i + 1 = 1$, if $i = n$). Put $s = i - j$, if $j < i$, and $s = n + i - j$, otherwise. Then, it is easy to show that $\beta = \alpha g^s \in H_{u_i}$. Now, as $u_i$ is an idempotent of $\mathcal{OP}_n$, by Proposition 1.2, it follows that $u_i$ is a power of $\beta$. On the other hand, it is a routine matter to show that $u_1 = g^{n+i-1}u_ig^{n-i+1}$. Thus, by Proposition 1.3, we deduce that $\mathcal{OP}_n = \langle \alpha, \gamma \rangle$.

For a prime divisor $p$ of $n$, we put $W_p = \langle g^p \rangle = \{1, g^p, g^{2p}, \ldots, g^{n-p}\}$, which is, clearly, a cyclic group of order $\frac{n}{p}$. Furthermore, from well known results regarding finite cyclic groups, we have:

**Lemma 1.5** The groups $W_p$, with $p$ a prime divisor of $n$, are the maximal subgroups of $J_n$.

Now, we can describe the maximal subsemigroups of $\mathcal{OP}_n$.

**Theorem 1.6** A subsemigroup $S$ of the semigroup $\mathcal{OP}_n$ is maximal if and only if $S = \mathcal{OP}(n, n - 2) \cup J_n$ or $S = \mathcal{OP}(n, n - 1) \cup W_p$, for a prime divisor $p$ of $n$.

**Proof.** Let $S$ be a maximal subsemigroup of $\mathcal{OP}_n$. Then, it is clear that $\mathcal{OP}(n, n - 2) \subseteq S$ and thus $S = \mathcal{OP}(n, n - 2) \cup T$, for some subset $T$ of $J_{n-1} \cup J_n$. By Proposition 1.4, we have $T \cap J_{n-1} = \emptyset$ or $T$ does not contain any element of $J_n$ of order $n$. In this latter case, we must have $J_{n-1} \subseteq T$, by the maximality of $S$. This shows that $S = \mathcal{OP}(n, n - 1) \cup T'$, for some subset $T'$ of $J_n$, whence $T'$ must be a maximal subgroup of $J_n$. Thus, by Lemma 1.5, we have $T' = W_p$, for some prime divisor $p$ of $n$. On the other hand, if $T \cap J_{n-1} = \emptyset$ then $S \subseteq \mathcal{OP}(n, n - 2) \cup J_n$, whence $S = \mathcal{OP}(n, n - 2) \cup J_n$, by the maximality of $S$.

The converse part follows immediately from Proposition 1.4 and Lemma 1.5.
Let $n \geq 3$ and $1 \leq k \leq n - 1$. In the remaining of this section, we consider the ideal $OP(n, k)$ of $OP_n$.

Clearly, the maximal subsemigroups of $OP(n, 1)$ are the sets of the form $OP(n, 1) \setminus \{\alpha\}$, for $\alpha \in OP(n, 1)$. Therefore, in what follows, we consider $k \geq 2$.

Notice that, as every element $\alpha \in \mathcal{O}_n$ of rank $r - 1$, for $2 \leq r \leq n - 1$, is expressible as a product of elements of $\mathcal{O}_n$ of rank $r$ (see [7]) and every element $\beta \in OP_n$ admits a decomposition $\beta = g^t\alpha$, for some $1 \leq t \leq n$ and $\alpha \in \mathcal{O}_n$ (see [4]), we deduce that every element of $J_{r-1}$ is a product of elements of $J_r$, for $2 \leq r \leq n - 1$. Thus, we have:

**Lemma 1.7** $OP(n, k) = \langle J_k \rangle$.

Let us denote by $\Lambda_k$ the collection of all subsets of $X_n$ of cardinality $k$. Since two elements of $J_k$ are $\mathcal{L}$-related if and only if they have the same image, an $\mathcal{L}$-class of $J_k$ (which coincides with an $\mathcal{L}$-class of $OP_n$, as $J_k$ is regular) is completely determined by the image set of its transformations. Therefore a typical $\mathcal{L}$-class of $J_k$ has the form

$$L_A = \{\alpha \in J_k \mid \text{im} \alpha = A\},$$

with $A \in \Lambda_k$.

Let $\Omega_k$ be the collection of all equivalence relations $\pi$ on $X_n$ of weight $k$ such that, for all $\bar{x} \in X_n/\pi$, either $\bar{x}$ or $X_n \setminus \bar{x}$ is a convex subset of $X_n$. Since two transformations of $J_k$ are $\mathcal{R}$-related if and only if they have the same kernel, an $\mathcal{R}$-class of $J_k$ (which coincides with an $\mathcal{R}$-class of $OP_n$, as $J_k$ is regular) is completely determined by the kernel of any of its elements. A typical $\mathcal{R}$-class of $J_k$ has then the form

$$R_{\pi} = \{\alpha \in J_k \mid \ker \alpha = \pi\},$$

with $\pi \in \Omega_k$.

Finally, it follows that a typical $\mathcal{H}$-class of $J_k$ has the form

$$H_{(\pi, A)} = R_{\pi} \cap L_A,$$

with $\pi \in \Omega_k$ and $A \in \Lambda_k$.

Notice that, for any $\pi \in \Omega_k$ and for any $\alpha \in R_{\pi}$, it is easy to show that $H_{\alpha} \cap \mathcal{O}_n = \emptyset$ if and only if $(1, n) \in \pi$ (i.e. $\pi$ contains a non-convex class). Observe also that, being $O(n, k) = OP(n, k) \cap \mathcal{O}_n$ (the ideal of $\mathcal{O}_n$ of all elements of rank less than or equal to $k$) and $J'_k = J_k \cap \mathcal{O}_n$ (the $\mathcal{J}$-class of $\mathcal{O}_n$ of all elements of rank equal to $k$), a typical $\mathcal{L}$-class of $J'_k$ has the form $L_A \cap \mathcal{O}_n$, with $A \in \Lambda_k$, and a typical $\mathcal{R}$-class of $J'_k$ has the form $R_{\pi} \cap \mathcal{O}_n$, with $\pi \in \Omega'_k = \{\pi \in \Omega_k \mid (1, n) \not\in \pi\}$.

**Proposition 1.8** Let $C$ be any subset of $J_k$ containing $J_k \cap \mathcal{O}_n$ and at least one element from each $\mathcal{R}$-class of $J_k$. Then $OP(n, k) = \langle C \rangle$.

**Proof.** First, let $\alpha$ be an element of $C$ with kernel $\{\{1, k+1, \ldots, n\}, \{2\}, \ldots, \{k\}\}$. Let $\beta$ be any order-preserving transformation with image $\{1, \ldots, k\}$ such that $\text{im} \alpha \# \ker \beta$. Then, $\ker(\alpha \beta) = \ker \alpha$ and $\text{im} (\alpha \beta) = \text{im} \beta$, from which it follows that the idempotent power of $\alpha \beta$ is the transformation $\begin{pmatrix} 1 & \cdots & k & k+1 & \cdots & n \\ \vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\ 2 & \cdots & k+1 & k+1 & \cdots & k+1 \end{pmatrix}$. Therefore

$$\gamma = \begin{pmatrix} 1 & \cdots & k & k+1 & \cdots & n \\ \vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\ 2 & \cdots & k+1 & k+1 & \cdots & k+1 \end{pmatrix} \begin{pmatrix} 1 & \cdots & k & k+1 & \cdots & n \\ \vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\ 1 & \cdots & k & 1 & \cdots & 1 \end{pmatrix} \in \langle C \rangle.$$

Furthermore, as $\gamma$ generates a cycle group of order $k$, we have $H_\gamma \subseteq \langle C \rangle$. 

4
Now, let \( \varepsilon = \gamma^k = \begin{pmatrix} 1 & \cdots & k & k+1 & \cdots & n \\ 1 & \cdots & k & k & \cdots & n \end{pmatrix} \) be the idempotent of \( H_\gamma \) and let \( H \) be any \( \mathcal{H} \)-class contained in \( R_\varepsilon = R_\gamma \). Since the elements of \( H \) have the same kernel that \( \varepsilon \in \mathcal{O}_n \), then \( H \) has an order-preserving element \( \tau \). From \( \varepsilon R \tau \) it follows that \( \varepsilon \tau = \tau \), whence \( \mathrm{im} \varepsilon \# \ker \tau \) and so, by Lemma 1.1, we have \( H \tau = H_\tau \). As \( \tau \in \mathcal{C} \) and \( H_\tau \subseteq \langle C \rangle \), we also have \( H = H_\tau \subseteq \langle C \rangle \). Hence \( R_\varepsilon \subseteq \langle C \rangle \).

Next, let \( \pi \in \Omega_k \) be such that \( (1, n) \not\in \pi \). Then, there exists an order-preserving transformation \( \tau \in L_\varepsilon \cap R_\pi \). Since \( \varepsilon \in L_\varepsilon \cap R_\varepsilon = L_\tau \cap R_\varepsilon \), we have \( \varepsilon \tau = \tau \), whence \( \mathrm{im} \varepsilon \# \ker \tau \) and so, by Lemma 1.1, we obtain \( \tau R_\varepsilon = R_\tau = R_\varepsilon \). As \( \tau \in \mathcal{C} \) and \( R_\varepsilon \subseteq \langle C \rangle \), it follows that \( R_\pi \subseteq \langle C \rangle \).

Finally, let \( \pi \in \Omega_k \) be such that \( (1, n) \in \pi \) and let \( \tau \in \mathcal{C} \cap R_\pi \). Take an order-preserving idempotent \( \varepsilon' \) such that \( \mathrm{im} \varepsilon' = \mathrm{im} \tau \). Then, \( \varepsilon' \in L_{\varepsilon'} \cap R_{\varepsilon'} = L_\tau \cap R_{\varepsilon'} \), whence \( \varepsilon \varepsilon' = \tau \) and so \( \mathrm{im} \varepsilon \# \ker \varepsilon' \). Thus, by Lemma 1.1, we have \( \tau R_{\varepsilon'} = R_\tau = R_{\varepsilon'} \). As \( \tau \in \mathcal{C} \) and \( R_{\varepsilon'} \subseteq \langle C \rangle \) (by the previous case), it follows that \( R_\pi \subseteq \langle C \rangle \).

Hence, we proved that \( J_k \subseteq \langle C \rangle \) and so, by Lemma 1.7, we obtain \( \mathcal{O} P(n, k) = \langle C \rangle \), as required.

Since \( \mathcal{O} n(k) = \langle E(J_k \cap \mathcal{O}_n) \rangle \) (see [7]) and each \( \mathcal{R} \)-class of \( J_k \) contains at least one idempotent, we have:

**Corollary 1.9** \( \mathcal{O} P(n, k) = \langle E(J_k) \rangle \).

Notice that, it is easy to show that, in fact, each \( \mathcal{R} \)-class of \( J_k \) contains at least two idempotents. Moreover, as \( 2 \leq k \leq n - 1 \), it also is easy to show that each \( \mathcal{L} \)-class of \( J_k \) contains at least two idempotents.

Let \( \Lambda \) be a non-empty proper subset of \( \Lambda_k \) and let \( \Omega \) be a non-empty proper subset of \( \Omega_k \) (respectively, of \( \Omega'_k \)). The pair \( (\Lambda, \Omega) \) is called a **coupler** of \( (\Lambda_k, \Omega_k) \) (respectively, of \( (\Lambda_k, \Omega'_k) \)) if the following three conditions are satisfied (see [11]):

1. For every \( A \in \Lambda \) and \( \pi \in \Omega \), \( A \) is not a transversal of \( \pi \);
2. For every \( B \in \Lambda_k \setminus \Lambda \), there exists \( \pi \in \Omega \) such that \( B \# \pi \);
3. For every \( \rho \in \Omega_k \setminus \Omega \) (respectively, \( \rho \in \Omega'_k \setminus \Omega \)), there exists \( A \in \Lambda \) such that \( A \# \rho \).

Next, we consider the following subsets of \( \mathcal{O} P(n, k) \):

1. \( S_A = \mathcal{O} P(n, k - 1) \cup (J_k \setminus L_A) \), for each \( A \in \Lambda_k \);
2. \( S_\pi = \mathcal{O} P(n, k - 1) \cup (J_k \setminus R_\pi) \), for each \( \pi \in \Omega_k \);
3. \( S_{(\Lambda, \Omega)} = \mathcal{O} P(n, k - 1) \cup (\bigcup \{L_A \mid A \in \Lambda\}) \cup (\bigcup \{R_\pi \mid \pi \in \Omega\}) \), for each coupler \( (\Lambda, \Omega) \) of \( (\Lambda_k, \Omega_k) \).

It is routine matter to prove that each of these subsets is a (proper) subsemigroup of \( \mathcal{O} P(n, k) \).

Before we give the description of the maximal subsemigroups of the ideals of the semigroup \( \mathcal{O} P_n \), we recall the following result presented by the first and third author in [5] (see also [12]).

**Theorem 1.10** ([5]) **Let** \( n \geq 3 \) and \( 2 \leq k \leq n - 1 \). **Then** a subsemigroup of \( \mathcal{O} n(k) \) **is maximal if and only if** it belongs to one of the following types:

1. \( S_A \cap \mathcal{O}_n \), with \( A \in \Lambda_k \);
2. \( S_\pi \cap \mathcal{O}_n \), with \( \pi \in \Omega'_k \) such that \( \pi \) does not admit an interval of \( \Lambda \) as a transversal;
3. \( S'_{(\Lambda, \Omega)} = \mathcal{O} n(k - 1) \cup (\bigcup \{L_A \cap \mathcal{O}_n \mid A \in \Lambda\}) \cup (\bigcup \{R_\pi \cap \mathcal{O}_n \mid \pi \in \Omega\}) \), with \( (\Lambda, \Omega) \) a coupler of \( (\Lambda_k, \Omega'_k) \).

**Lemma 1.11** Let \( S \) be a maximal subsemigroup of \( \mathcal{O} P(n, k) \). Then \( S = \bigcup \{H_\alpha \mid \alpha \in S\} \).
Proof. Let $T = \bigcup \{H_\alpha \mid \alpha \in S\}$. First, notice that, by Corollary 1.9, there exists $\varepsilon \in E(J_k)$ such that $\varepsilon \notin S$. Hence, $H_\varepsilon \cap S = \emptyset$ and so $S \subseteq T \subseteq OP(n,k)$. The result follows by proving that $T$ is a subsemigroup of $OP(n,k)$. Clearly, by the maximality of $S$ (and Lemma 1.7), we have $OP(n,k-1) \subseteq S$. So, if suffices to show that, for all $\alpha, \beta \in T \cap J_k$ such that $\alpha \beta \in J_k$, we get $\alpha \beta \in T$. Therefore, let $\alpha, \beta \in T \cap J_k$ be such that $\alpha \beta \in J_k$. Take $\alpha', \beta' \in S$ such that $\alpha \in H_{\alpha'}$ and $\beta \in H_{\beta'}$. Then $\im \alpha' = \im \alpha \# \ker \beta = \ker \beta'$ and $\alpha \in R_\alpha \cap L_\beta$, whence $\alpha' \beta' \in R_{\alpha'} \cap L_{\beta'} = R_{\alpha} \cap L_{\beta} = H_{\alpha \beta}$, and so, as $\alpha' \beta' \in S$, we obtain $\alpha \beta \in H_{\alpha' \beta'} \subseteq T$, as required. 

Now, we have:

**Theorem 1.12** Let $n \geq 3$ and $2 \leq k \leq n - 1$. Then a subsemigroup of $OP(n,k)$ is maximal if and only if it belongs to one of the following types:

1. $S_A$, with $A \in \Lambda_k$;
2. $S_\pi$, with $\pi \in \Omega_k$;
3. $S_{(\Lambda,\Omega)}$, with $(\Lambda, \Omega)$ a coupler of $(\Lambda_k, \Omega_k)$.

**Proof.** We begin by showing that each of these subsemigroups of $OP(n,k)$ is maximal.

First, let $A \in \Lambda_k$ and let $\alpha \in L_A$. Take an idempotent $\varepsilon \in (J_k \setminus L_A) \cap R_\alpha$. As $L_\varepsilon \subseteq S_A$ and, by Lemma 1.1, $L_\varepsilon \alpha = L_A$, we have $\langle S_A, \alpha \rangle = OP(n,k)$. Thus, $S_A$ is maximal.

Similarly, being $\pi \in \Omega_k$ and $\alpha \in R_\pi$, the $L$-class $L_\alpha$ contains at least an idempotent $\varepsilon \in J_k \setminus R_\pi$ and so $R_\varepsilon \subseteq S_\pi$ and, by Lemma 1.1, $\alpha R_\varepsilon = R_\pi$, whence $\langle S_\pi, \alpha \rangle = OP(n,k)$. Thus, $S_\pi$ is maximal.

Finally, regarding the subsemigroups of type 3, let $(\Lambda, \Omega)$ be a coupler of $(\Lambda_k, \Omega_k)$. As $\Lambda_k \setminus \Lambda \neq \emptyset$ and $\Omega_k \setminus \Omega \neq \emptyset$, we may take $\alpha \in R_\rho \cap L_B$, for some $\rho \in \Omega_k \setminus \Omega$ and $B \in \Lambda_k \setminus \Lambda$. Then, there exist $\pi \in \Omega$ and $A \in \Lambda$ such that $B \# \pi$ and $A \# \rho$. Now, by Lemma 1.1, we have $\alpha R_\pi = R_\alpha = R_\rho$. As $R_\pi \subseteq S_{(\Lambda, \Omega)}$, we obtain $R_\rho \subseteq \langle S_{(\Lambda, \Omega)}, \alpha \rangle$. On the other hand, by Lemma 1.1, we also have $L_A R_\rho = J_k$. Since $L_A \subseteq S_{(\Lambda, \Omega)}$, we deduce that $\langle S_{(\Lambda, \Omega)}, \alpha \rangle = OP(n,k)$. Thus, $S_{(\Lambda, \Omega)}$ is maximal.

For the converse part, let $S$ be a maximal subsemigroup of the ideal $OP(n,k)$.

If $S \cap R_\pi = \emptyset$, for some $\pi \in \Omega_k$, then $S = S_\pi$, by the maximality of $S$. Similarly, if $S \cap L_A = \emptyset$, for some $A \in \Lambda_k$, then $S = S_A$. Thus, admit that $S$ contains at least one element from each $R$-class and each $L$-class of $J_k$. If $S \cap \Omega_n = O(n,k)$ then $S = OP(n,k)$, by Proposition 1.8. Therefore $S \cap \Omega_n \subseteq O(n,k)$. Let $S$ be any maximal subsemigroup of $O(n,k)$ such that $S \cap \Omega_n \subseteq S$. Now, by Theorem 1.10, we have three possible cases for $S$.

First, suppose that $\tilde{S} = S_\pi \cap \Omega_n$, for some $\pi \in \Omega_k$. Then, as $S \cap R_\pi \neq \emptyset$, we may take $\alpha \in S \cap R_\pi$. Since $\pi \in \Omega_k$, we have $H_\alpha \cap \Omega_n \neq \emptyset$. Now, as $H_\alpha \subseteq S$ (by Lemma 1.11), we have $(S \cap \Omega_n) \cap R_\pi \neq \emptyset$, whence $\tilde{S} \cap R_\pi \neq \emptyset$, which is a contradiction. Thus, $\tilde{S}$ cannot be of this type.

Secondly, we suppose that $\tilde{S} = S_{A_1} \cap \Omega_n$, for some $A_1 \in \Lambda_k$. Let $A_1, \ldots, A_r$ be the $r \geq 1$ distinct elements of $\Lambda_k$ such that, for all $A \in \Lambda_k$, $L_A \cap \Omega_n \cap S = \emptyset$ if and only if $A \in \{A_1, \ldots, A_r\}$. Notice that, for $i \in \{1, \ldots, r\}$, we have $L_{A_i} \cap S \neq \emptyset$ and, as a consequence of Lemma 1.11, if $\alpha \in L_{A_i} \cap S$ then $(1,n) \in \ker \alpha$. Now, let

$$\Omega = \{\pi \in \Omega_k \mid R_\pi \cap L_{A_i} \cap S \neq \emptyset, \text{ for some } i \in \{1, \ldots, r\}\}.$$ 

Notice that, clearly, $\Omega \neq \emptyset$. Also, let

$$\Lambda = \{A \in \Lambda_k \mid A \text{ is not a transversal of } \pi, \text{ for all } \pi \in \Omega\}.$$ 

Observe that, as $(1,n) \in \pi$, for all $\pi \in \Omega$, then $\{A \in \Lambda_k \mid 1,n \in A\} \subseteq \Lambda$ and so, in particular, $\Lambda \neq \emptyset$. Furthermore, it is a routine matter to check that the pair $(\Lambda, \Omega)$ is a coupler of $(\Lambda_k, \Omega_k)$. Next, we show that $S \cap J_k \subseteq S_{(\Lambda, \Omega)}$. Take $\alpha \in S \cap J_k$. If $\im \alpha \in \Lambda$, then $\alpha \in \bigcup \{L_A \mid A \in \Lambda\}$. Thus, let us suppose that $\im \alpha \notin \Lambda$. Then we have to consider two cases. If $\alpha \in L_{A_i}$, for some $i \in \{1, \ldots, r\}$, then $\ker \alpha \in \Omega$, whence $\alpha \in \bigcup \{R_\pi \mid \pi \in \Omega\}$. Now, let $\alpha \notin L_{A_i}$, for all $i \in \{1, \ldots, r\}$. Then, there exists $\pi \in \Omega$ such that $\im \alpha \# \pi$. As
π ∈ Ω, there exists i ∈ {1, . . . , r} such that \( R_π \cap L_A \cap S \neq \emptyset \). Take β ∈ \( R_π \cap L_A \cap S \neq \emptyset \). Hence \( \text{im} α \# \pi = \ker β \)
and so αβ ∈ \( R_α \cap L_β = R_α \cap L_A \). Moreover, αβ ∈ S, whence αβ ∈ \( R_αβ \cap L_A \cap S \). Then \( \ker α = \ker (αβ) ∈ Ω \)
from which it follows that α ∈ \( \bigcup \{R_π \ | \ π ∈ Ω \} \). So, we have proved that α ∈ \( S(Λ, Ω) \). Therefore \( S ⊆ S(Λ, Ω) \) and thus \( S = S(Λ, Ω) \), by the maximality of S.

Finally, suppose that \( \bar{S} = S'(Λ', Ω') \), for some coupler \((Λ', Ω')\) of \((Λ_k, Ω'_k)\). Let

\[ Λ = \{ A ∈ Λ' \ | \ L_A \cap S \cap \left( \bigcup \{ R_π \ | \ π ∈ Ω'_k \setminus Ω' \} \right) \neq \emptyset \}, \]

which is a nonempty subset of Λk (as \( S \cap R_π \neq \emptyset \), for all \( π ∈ Ω_k \)). Also, let

\[ Ω = \{ π ∈ Ω_k \ | \ A \text{ is not a transversal of } π, \text{ for all } A ∈ Λ \}. \]

Clearly, \( Ω' ⊆ Ω \), whence \( Ω \neq \emptyset \). Furthermore, it is a routine matter to check that the pair \((Λ, Ω)\) is a coupler of \((Λ_k, Ω_k)\). Next, we aim to prove that \( S = S(Λ, Ω) \). First, observe that, from the definition of Λ and from \( S \cap Ω_n ⊆ S'(Λ', Ω') \) in view of Lemma 1.11, we deduce that \( R_π \cap L_A \cap S = \emptyset \), for all \( π ∈ Ω'_k \setminus Ω' \) and \( A ∈ Λ \). Now, take α ∈ \( J_κ \cap S \) and suppose that α /∈ \( S(Λ, Ω) \). Then, \( \text{im} α = \Lambda_κ \setminus Λ \) and \( \ker α ∈ Ω_k \setminus Ω \). Hence, there exists A ∈ Λ such that A\# ker α. Thus, by the definition of Λ, we may take β ∈ \( L_A \setminus \bigcup \{R_π \ | \ π ∈ Ω'_k \setminus Ω' \} \) and so, as \( \text{im} β = A \# \ker α \), we have βα ∈ \( R_β \cap L_α \cap S \), i.e. βα ∈ S, \( \text{im}(βα) = \text{im} β = \Lambda_k \setminus Λ \) and \( \ker β = \ker (βα) = \ker β ∈ Ω'_k \setminus Ω' \), which contradicts the above deduction. Therefore \( α ∈ S(Λ, Ω) \). It follows that \( S ⊆ S(Λ, Ω) \) and then \( S = S(Λ, Ω) \), by the maximality of S, as required.

2 Maximal subsemigroups of the ideals of \( OR_n \)

Let \( n ∈ N \). As for \( OP_n \), the semigroup \( OR_n \) is the union of its \( J \)-classes \( J_1, J_2, \ldots, J_n \), where

\[ J_k = \{ α ∈ OR_n \ | \ \text{rank} α = k \} \]

for \( k = 1, \ldots, n \). Notice that \( J_k \cap OP_n \) is the \( J \)-class \( J_k \) of \( OP_n \), for \( k = 1, \ldots, n \), and \( J_1 = J_1 \) and \( J_2 = J_2 \). Analogously to \( OP_n \), the ideals of the semigroup \( OR_n \) are the unions of the \( J \)-classes \( J_1, J_2, \ldots, J_n \), i.e. the sets

\[ OR(n,k) = \{ α ∈ OR_n \ | \ \text{rank} α ≤ k \}, \]

with \( k = 1, \ldots, n \). To avoid ambiguity, we denote by \( L_α, R_α \) and \( H_α \) the \( L \)-class, \( R \)-class and \( H \)-class, respectively, of an element \( α ∈ OR_n \). Observe that, for \( α ∈ OP_n \), the sets \( L_α \cap OP_n, R_α \cap OP_n \) and \( H_α \cap OP_n \) are respectively the \( L \)-class \( L_α \), the \( R \)-class \( R_α \) and the \( H \)-class \( H_α \) of \( OP_n \).

Taking the product of two elements of \( J_k \), for \( k = 2, \ldots, n \), as being zero if it falls into \( OR(n,k−1) \), a result similar to Lemma 1.11 holds for elements of \( OR_n \).

**Lemma 2.1** Let \( α, β ∈ J_k \), with \( k = 1, 2, \ldots, n \). Then

\[
α R β = \begin{cases} R_α β = R_α & \text{if } \text{im} α \# \ker β, \\ 0 & \text{otherwise}. \end{cases} \quad \text{and} \quad L_α β = \begin{cases} L_α β = L_β & \text{if } \text{im} α \# \ker β, \\ 0 & \text{otherwise}. \end{cases} \]

\[
\bar{L}_α R β = \begin{cases} \bar{L}_k & \text{if } \text{im} α \# \ker β, \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad α H β = \begin{cases} H_α β = H_β & \text{if } \text{im} α \# \ker β, \\ 0 & \text{otherwise}. \end{cases} \]

As \( OR_1 = OP_1 \) and \( OR_2 = OP_2 \), in what follows, we consider \( n ≥ 3 \).

Next, recall that a dihedral group \( D_n \) of order \( 2n \) can abstractly be defined by the group presentation

\[ \langle x, y \ | \ x^n = y^2 = 1, xy = yx^{-1} \rangle. \]
Let
\[ h = \begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ n & n-1 & \cdots & 2 & 1 \end{pmatrix} \in \bar{\mathcal{J}}_n. \]

Hence, we have \( \bar{J}_n = \langle g, h \rangle \) and, as \( g^n = h^2 = (gh)^2 = 1 \), it is easy to see that \( \bar{J}_n \) is a dihedral group of order \( 2n \). Furthermore, Catarino and Higgins proved:

**Proposition 2.2** ([4]) Let \( \alpha \in \bar{J}_k \), with \( k = 3, \ldots, n \). Then \( |\bar{H}_\alpha| = 2k \). Moreover, if \( \alpha \) is an idempotent, then \( \bar{H}_\alpha \) is a dihedral group of order \( 2k \).

Thus, each \( \mathcal{H} \)-class of rank \( k \) of \( \mathcal{O} \mathcal{R}_n \) has \( k \) orientation-preserving transformations and \( k \) orientation-reversing transformations, for \( k \in \{3, \ldots, n\} \).

Notice that, since \( \bar{J}_1 = J_1 \) and \( \bar{J}_2 = J_2 \), for \( \alpha \in \bar{J}_k \) with \( k = 1, 2 \), we have \( |\bar{H}_\alpha| = k \).

Let us consider again the dihedral group \( \mathcal{D}_n = \langle x, y \mid x^n = y^2 = 1, xy = yx^{-1} \rangle \) of order \( 2n \). Observe that \( \mathcal{D}_n = \{1 = x^0, x, x^2, \ldots, x^{n-1}\} \cup \{y, xy, x^2y, \ldots, x^{n-1}y\} \). It is easy to show that the subgroups of \( \mathcal{D}_n \) are of the form \( \langle x^d \rangle \) (a cyclic group of order \( n/d \)) and of the form \( \langle x^d, x^i \rangle \) (a dihedral group of order \( 2n/d \)), for each positive divisor \( d \) of \( n \) and each \( 0 \leq i < d \). It follows that \( \langle x \rangle \) and \( \langle x^p, x^i \rangle \), with \( p \) a prime divisor of \( n \) and \( 0 \leq i < p \), are the maximal subgroups of \( \mathcal{D}_n \).

Now, for a prime divisor \( p \) of \( n \) and \( 0 \leq i < p \), consider the dihedral group \( V_{p,i} = \langle g^p, g^i h \rangle \) of order \( 2n/p \). Then, the above observation can be rephrased as:

**Lemma 2.3** The group \( J_n = \langle g \rangle \) and the groups \( V_{p,i} \), with \( p \) a prime divisor of \( n \) and \( 0 \leq i < p \), are the maximal subgroups of \( \bar{J}_n \).

Next, we recall the following well known result (see [3, 10]).

**Proposition 2.4** \( \mathcal{O} \mathcal{R}_n = \langle u_1, g, h \rangle \).

In fact, more generally, we have:

**Proposition 2.5** Let \( \alpha \in \bar{J}_{n-1} \), \( \gamma \) an element of \( J_n \) of order \( n \) and \( \beta \in \bar{J}_n \setminus J_n \). Then \( \mathcal{O} \mathcal{R}_n = \langle \alpha, \gamma, \beta \rangle \).

**Proof.** If \( \alpha \in \bar{J}_{n-1} \cap \mathcal{O} \mathcal{P}_n \) then, by Proposition 1.4, we have \( \mathcal{O} \mathcal{P}_n = \langle \alpha, \gamma \rangle \). If \( \alpha \in \bar{J}_{n-1} \setminus \mathcal{O} \mathcal{P}_n \) then \( \alpha \beta \in \bar{J}_{n-1} \cap \mathcal{O} \mathcal{P}_n \) and, again by Proposition 1.4, we obtain \( \mathcal{O} \mathcal{P}_n = \langle \alpha \beta, \gamma \rangle \). Therefore, \( u_1, g \in \langle \alpha, \gamma, \beta \rangle \).

As \( \beta \in \bar{J}_n \setminus \mathcal{O} \mathcal{P}_n \), there exists \( i \in \{1, \ldots, n\} \) such that \( \beta = \begin{pmatrix} 1 & 2 & \cdots & i-1 & i \\ i-1 & i-2 & \cdots & 1 & n \\ 1 & 2 & \cdots & i-1 & i \\ \vdots & \vdots & \ddots & \ddots & \ddots \\ n & n-i+2 & \cdots & n-i & n-i+1 \end{pmatrix} \). On the other hand, the transformation \( \delta = \begin{pmatrix} 1 & 2 & \cdots & i-1 & i \\ n-i+2 & n-i+3 & \cdots & n-i & n-i+1 \end{pmatrix} \) is an element of \( \mathcal{O} \mathcal{P}_n \) and \( h = \beta \delta \in \langle \alpha, \gamma, \beta \rangle \). Therefore, by Proposition 2.4, we deduce that \( \mathcal{O} \mathcal{R}_n = \langle \alpha, \gamma, \beta \rangle \).

We have now all the ingredients to describe the maximal subsemigroups of \( \mathcal{O} \mathcal{R}_n \).

**Theorem 2.6** A subsemigroup \( S \) of the semigroup \( \mathcal{O} \mathcal{R}_n \) is maximal if and only if \( S = \mathcal{O} \mathcal{R}(n, n-2) \cup \bar{J}_n \) or \( S = \mathcal{O} \mathcal{R}(n, n-1) \cup \bar{J}_n \) or \( S = \mathcal{O} \mathcal{R}(n, n-1) \cup V_{p,i} \), for some prime divisor \( p \) of \( n \) and \( 0 \leq i < p \).

**Proof.** Let \( S \) be a maximal subsemigroup of \( \mathcal{O} \mathcal{R}_n \). Then, by Proposition 2.5, we have \( S = \mathcal{O} \mathcal{R}(n, n-2) \cup T \), for some \( T \subset (\bar{J}_n \cup \bar{J}_i) \) such that \( T \cap \bar{J}_n = \emptyset \) or \( T \) does not contain any element of \( J_n \) of order \( n \) or \( T \cap (\bar{J}_n \setminus \bar{J}_i) = \emptyset \). In the latter two cases, we must have \( \bar{J}_n \subseteq T \), by the maximality of \( S \). Thus, \( S = \mathcal{O} \mathcal{R}(n, n-1) \cup T' \), for some \( T' \subset \bar{J}_n \). Clearly, \( T' \) must be a maximal subgroup of \( \bar{J}_n \), whence \( S = \mathcal{O} \mathcal{R}(n, n-1) \cup \bar{J}_n \) or \( S = \mathcal{O} \mathcal{R}(n, n-1) \cup V_{p,i} \), for some prime divisor \( p \) of \( n \) and \( 0 \leq i < p \), accordingly with Lemma 2.3. On the other hand, if \( T \cap \bar{J}_n = \emptyset \) then \( S \subseteq \mathcal{O} \mathcal{R}(n, n-2) \cup J_n \) and so \( S = \mathcal{O} \mathcal{R}(n, n-2) \cup J_n \), by the maximality of \( S \).

The converse part follows immediately from Proposition 2.5 and Lemma 2.3.

\[ \square \]
From now on we consider the ideals $OR(n, k)$ of $\mathcal{OR}_n$, for $k \in \{1, \ldots, n - 1\}$. Since $OR(n, 1) = OP(n, 1)$ and $OR(n, 2) = OP(n, 2)$, in what follows, we take $k \geq 3$.

Notice that, as $\alpha h \in OP(n, k)$, for all $\alpha \in OR(n, k) \setminus OP(n, k)$, by using Lemma 1.7, it is easy to conclude:

**Lemma 2.7** $OR(n, k) = \langle \bar{J}_k \rangle$.

In fact, moreover, we have:

**Proposition 2.8** $OR(n, k) = \langle J_k, \alpha \rangle$, for all $\alpha \in \bar{J}_k \setminus J_k$.

**Proof.** Let $\alpha \in \bar{J}_k \setminus J_k$ and take an idempotent $\varepsilon \in \bar{L}_\alpha$. Since $im\alpha = im\varepsilon \# ker\varepsilon$, we have $\alpha \bar{R}_\varepsilon = \bar{R}_\alpha$, by Lemma 2.1. Hence, $\alpha(\bar{R}_\varepsilon \cap J_k) = \bar{R}_\alpha \cap J_k$ and so $\bar{R}_\alpha = (\bar{R}_\varepsilon \cap J_k) \cup (\bar{R}_\alpha \cap J_k) = (\bar{R}_\varepsilon \cap J_k) \cup \alpha(\bar{R}_\varepsilon \cap J_k) \subseteq \langle J_k, \alpha \rangle$.

Now, let $\varepsilon'$ be an idempotent of $\bar{R}_\alpha$ and take $\alpha' \in H_{\varepsilon'} \setminus J_k$. Notice that $\alpha' \in \langle J_k, \alpha \rangle$. As $im\varepsilon' \# ker\varepsilon' = ker\alpha'$, we have $\bar{L}_{\varepsilon'} \alpha' = \bar{L}_{\varepsilon'} = \bar{L}_{\varepsilon'}$, by Lemma 2.1. Thus $(\bar{L}_{\varepsilon'} \cap J_k)\alpha' = \bar{L}_{\varepsilon'} \setminus J_k$, whence $\bar{L}_{\varepsilon'} = (\bar{L}_{\varepsilon'} \cap J_k) \cup (\bar{L}_{\varepsilon'} \setminus J_k) = (\bar{L}_{\varepsilon'} \cap J_k) \cup (\bar{L}_{\varepsilon'} \cap J_k)\alpha' \subseteq \langle J_k, \alpha \rangle$.

Finally, as $im\varepsilon' \# ker\varepsilon' = ker\alpha'$, we have $\bar{L}_{\varepsilon'} \bar{R}_\alpha = \bar{J}_k$, again by Lemma 2.1. Therefore, $\bar{J}_k \subseteq \langle J_k, \alpha \rangle$ and so, by Lemma 2.7, $OR(n, k) = \langle J_k, \alpha \rangle$, as required. $\blacksquare$

As an immediate consequence of Proposition 2.8, we have:

**Corollary 2.9** $OR(n, k - 1) \cup J_k$ is a maximal subsemigroup of $OR(n, k)$.

Also, combining Proposition 2.8 with Corollary 1.9, we have:

**Corollary 2.10** $OR(n, k) = \langle E(J_k), \alpha \rangle$, for all $\alpha \in \bar{J}_k \setminus J_k$.

Before we present our description of the maximal subsemigroups of the ideals of $OR_n$, we prove the following result:

**Proposition 2.11** Let $S$ be a maximal subsemigroup of $OR(n, k)$ containing at least one orientation-reversing transformation of rank $k$. Then $S = \bigcup \{\bar{H}_\alpha \mid \alpha \in S \cap OP_n\}$.

**Proof.** Let $\alpha \in S$. As clearly $OR(n, k - 1) \subseteq S$, it suffices to consider $\alpha \in \bar{J}_k$. Take $\beta \in \bar{H}_\alpha$ and suppose that $\beta \notin S$. Hence, by the maximality of $S$, we have $OR(n, k) = \langle S, \beta \rangle$. Let $\tau \in \bar{J}_k \setminus S$. Then, there exist $t \geq 0$, $r_0, r_1, \ldots, r_t \geq 0$ and $\alpha_1, \ldots, \alpha_t \in S$ such that $\tau = \beta^{r_0}\alpha_1^\beta \alpha_2^\beta \cdots \beta^{r_t-1}\alpha_t^\beta$. As $\alpha H \beta$, it follows that $\tau' = \alpha_0^\beta \alpha_1^\beta \alpha_2^\beta \cdots \alpha_t^\beta$. Furthermore, $\tau' \in S$. Thus, for all $\tau \in \bar{J}_k$, $H_\tau \cap S \neq \emptyset$, from which it follows that $E(J_k) \subseteq S$. Since $S$ also contains an orientation-reversing transformation of rank $k$, by Corollary 2.10, we have $S = OR(n, k)$, a contradiction. Therefore $\bar{H}_\alpha \subseteq S$. This shows that $\bar{H}_\alpha \subseteq S$ for all $\alpha \in S$, i.e. $\bigcup \{\bar{H}_\alpha \mid \alpha \in S\} \subseteq S$ and thus $\bigcup \{\bar{H}_\alpha \mid \alpha \in S\} = S$. Since each $H$-class of $OR_n$ contains an orientation-preserving transformation, we obtain $S = \bigcup \{\bar{H}_\alpha \mid \alpha \in S \cap OP_n\}$, as required. $\blacksquare$

In general, if $S'$ is a subsemigroup of $OP(n, k)$ containing $OP(n, k - 1)$, then (using an argument similar to that considered in the proof of Lemma 1.11) it is easy to show that $S = \bigcup \{\bar{H}_\alpha \mid \alpha \in S'\}$ is a subsemigroup of $OR(n, k)$. Furthermore, if $S' \subseteq OP(n, k)$ then $S$ is also a proper subsemigroup of $OR(n, k)$. In fact, in this case, by Corollary 1.9, there exists $\varepsilon \in E(J_k)$ such that $\varepsilon \notin S'$. It follows that $\bar{H}_\varepsilon \cap S' = \emptyset$ and so also $\varepsilon \notin S$.

Finally, we have:

**Theorem 2.12** Let $n \geq 4$ and $3 \leq k \leq n - 1$. Then, a subsemigroup $S$ of $OR(n, k)$ is maximal if and only if $S = OR(n, k - 1) \cup J_k$ or $S = \bigcup \{\bar{H}_\alpha \mid \alpha \in S'\}$, for some maximal subsemigroup $S'$ of $OP(n, k)$.
Proof. First, let \( S \) be a maximal subsemigroup of \( OR(n, k) \) and admit that \( S \neq OR(n, k-1) \cup J_k \). Then \( S \) must contain an orientation-reversing transformation of rank \( k \) and so \( S = \bigcup \{ \tilde{H}_\alpha \mid \alpha \in S \cap \mathcal{OP}_n \} \), by Proposition 2.11. Clearly, \( S \cap \mathcal{OP}_n \) is a proper subsemigroup of \( \mathcal{OP}(n, k) \), whence there exists a maximal subsemigroup \( S' \) of \( \mathcal{OP}(n, k) \) such that \( S \cap \mathcal{OP}_n \subseteq S' \). Then, by the above observation, \( \bigcup \{ \tilde{H}_\alpha \mid \alpha \in S' \} \) is a proper subsemigroup of \( OR(n, k) \) and, as it contains \( S \), it follows that \( S = \bigcup \{ \tilde{H}_\alpha \mid \alpha \in S' \} \), by the maximality of \( S \).

Conversely, if \( S = OR(n, k-1) \cup J_k \), then \( S \) is a maximal subsemigroup of \( OR(n, k) \), by Corollary 2.9. Hence, let us admit that \( S = \bigcup \{ \tilde{H}_\alpha \mid \alpha \in S' \} \), for some maximal subsemigroup \( S' \) of \( \mathcal{OP}(n, k) \). Then, by the above observation, \( S \) is a proper subsemigroup of \( OR(n, k) \). Moreover, \( S \) must contain an orientation-reversing transformation of rank \( k \). Let \( \hat{S} \) be a maximal subsemigroup of \( OR(n, k) \) such that \( S \subseteq \hat{S} \). Then \( \hat{S} \) also contains an orientation-reversing transformation of rank \( k \) and so, by Proposition 2.11, \( \hat{S} = \bigcup \{ \tilde{H}_\alpha \mid \alpha \in \hat{S} \cap \mathcal{OP}_n \} \). On the other hand, \( S' \subseteq S \cap \mathcal{OP}_n \subseteq \hat{S} \cap \mathcal{OP}_n \subseteq \mathcal{OP}(n, k) \), whence \( S' = S \cap \mathcal{OP}_n \), by the maximality of \( S' \). It follows that \( S = \hat{S} \) and thus \( S \) is a maximal subsemigroup of \( OR(n, k) \), as required.  

References


