Mathematical Problems in Mechanics

Waves in a thin and periodically oscillating medium

Rita Ferreira [a], M. Luísa Mascarenhas [b]

[a] *C.M.A.-U.N.L., Quinta da Torre, 2829-516 Caparica, Portugal*

[b] *Departamento de Matemática da F.C.T.-U.N.L. e C.M.A.-U.N.L., Quinta da Torre, 2829-516 Caparica, Portugal*

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Abstract

We study the asymptotic behavior of the spectrum of an elliptic operator with periodically oscillating coefficients, in a thin domain, with vanishing Dirichlet conditions. Two cases are treated: the case where the periodicity of the oscillations has the same order of thickness as the domain and the case where the oscillations have a frequency much greater than the thickness of the domain. A physical motivation can be to understand the behavior of the probability density associated to the wave function of a particle confined to a very thin domain, with periodically varying characteristics. *To cite this article: R. Ferreira, M.L. Mascarenhas, C. R. Acad. Sci. Paris, Ser. I ... (2007).*

Résumé

Ondes dans un milieu fin à caractéristiques périodiquement oscillantes. On étudie le comportement asymptotique du spectre d'un problème elliptique à coefficients périodiques dans un domaine mince, à condition de Dirichlet nulle. On analyse deux cas : le cas où la périodicité des oscillations est du même ordre que l'épaisseur du domaine et le cas où la fréquence des oscillations est trés grande devant l'épaisseur. Une motivation physique est de comprendre le comportement de la densité de probabilité associée à la fonction d'onde d'une particule dans un domaine mince dont les propriétés oscillent fortement. *Pour citer cet article : R. Ferreira, M.L. Mascarenhas, C. R. Acad. Sci. Paris, Ser. I ... (2007).*

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Notre but c'est l'étude du spectre d'un problème elliptique à coefficients périodiques dans un domaine de très petite épaisseur, en imposant des conditions de Dirichlet nulles au bord. Le domaine est de la forme $\Omega_\delta := \omega \times \delta I$ où ω est un domaine borné de $\mathbb{R}^2$, $I := (-1/2, 1/2)$ et δ est un petit paramètre

Email addresses: `ragf@ptmat.fc.ul.pt` (Rita Ferreira), `mlfm@fct.unl.pt` (M. Luísa Mascarenhas).

d'épaisseur. Les oscillations supposées indépendantes de la variable l'épaisseur sont décrites par un autre petit paramètre ε.

Une motivation physique est de comprendre le comportement de la densité de probabilité associée à la fonction d'onde d'une particule dans un domaine mince dont les propriétés oscillent fortement.

On choisit les caractères α et β pour représenter les indices $\{1,2\}$ et on utilise la convention de Einstein dans les indices répétés. On écrira $\bar{x}$ au lieu de (x_1, x_2). Soit $Y := (0,1)^2$ et considérons la matrice symétrique réelle, à coefficients Y-périodiques, $A = (a_{ij})_{1\leqslant i,j\leqslant 3} \in [L^\infty(\mathbb{R}^2)]^{3\times 3}$, telle que pour certains ζ et η dans $\mathbb{R}^+$, l'inégalité (1) est satisfaite pour tout $\xi \in \mathbb{R}^3$ et p.p. en $\bar{x} \in \mathbb{R}^2$. Pour des raisons techniques on simplifie le problème en supposant que $a_{\alpha 3} = 0$ p.p. en $\bar{x} \in \mathbb{R}^2$. Pour chaque $\varepsilon > 0$ on définit $a^\varepsilon_{ij}(\bar{x}) := a_{ij}\left(\frac{\bar{x}}{\varepsilon}\right)$ et $A_\varepsilon := (a^\varepsilon_{ij})_{1\leqslant i,j\leqslant 3}$. Notre problème est alors l'étude du comportement asymptotique de $\lambda^\delta_\varepsilon$ qui satisfait (2), quand $\varepsilon \to 0$ et $\delta \to 0$.

Une fois que Ω_δ est borné, le spectre $\sigma^\delta_\varepsilon$ du problème (2) se réduit à la suite de valeurs propres $0 < \lambda^{\delta,0}_\varepsilon \leqslant \lambda^{\delta,1}_\varepsilon \leqslant \cdots \leqslant \lambda^{\delta,i}_\varepsilon \leqslant \lambda^{\delta,i+1}_\varepsilon \leqslant \cdots$, $i \in \mathbb{N}_0$. Si on fait l'épaisseur δ tendre vers zéro il est clair que les valeurs propres tendent vers l'infini. On précise la caractérisation asymptotique du spectre dans Théorème 1.1 pour le cas $\varepsilon \approx \delta$ et dans Théorème 1.2 pour le cas $\varepsilon \ll \delta$.

La preuve des résultats repose sur la notion de Γ-convergence. On utilise un changement de variables classique pour réduire le problème à un domaine fixe. Après un changement d'échelles et de variable fonctionnelle (voir Proposition 2.2), on applique des résultats d'homogénéisation liés à la convergence spectrale (Lemme 2.1 et Proposition 2.3) pour obtenir la convergence désirés.

1. Introduction and Main results

Our aim is to study the asymptotic behavior of the spectrum of an elliptic problem with periodically oscillating coefficients in a thin domain, with zero Dirichlet boundary conditions.

We consider the domain $\Omega_\delta := \omega \times \delta I$, where ω is a bounded domain in $\mathbb{R}^2$, $I := (-1/2, 1/2)$ and δ is a small parameter of thickness. The oscillations, which we assume to be independent of the thickness variable, are described by another small parameter ε. We study the case when ε has the same order of δ and the case when ε goes to zero faster than δ.

A physical motivation can be to understand the behavior of the probability density associated to the wave function of a particle confined to a very thin domain, with periodically varying characteristics.

In what follows the greek characters α and β take their values in the set $\{1,2\}$ and the Einstein convention of summation over repeated indices is adopted. We will write often $\bar{x}$ instead of (x_1, x_2). For $Y := (0,1)^2$ let $A = (a_{ij})_{1\leqslant i,j\leqslant 3} \in [L^\infty(\mathbb{R}^2)]^{3\times 3}$ be a real symmetric and Y-periodic matrix, satisfying, for ζ and η in $\mathbb{R}^+$,

$$\zeta\|\xi\|^2 \leqslant (A(\bar{x})\xi, \xi) \leqslant \eta\|\xi\|^2, \tag{1}$$

for all $\xi \in \mathbb{R}^3$ and for a.e. $\bar{x} \in \mathbb{R}^2$. For technical reasons we simplify the problem, assuming that $a_{\alpha 3} = 0$ a.e. in $\mathbb{R}^2$.

For each $\varepsilon > 0$, define $a^\varepsilon_{ij}(\bar{x}) := a_{ij}\left(\frac{\bar{x}}{\varepsilon}\right)$ and $A_\varepsilon := (a^\varepsilon_{ij})_{1\leqslant i,j\leqslant 3}$. The problem under study is then the asymptotic behavior of $\lambda^\delta_\varepsilon$, as $\varepsilon \to 0$ and $\delta \to 0$:

$$-\mathrm{div}(A_\varepsilon \nabla \tilde{v}^\delta_\varepsilon) = \lambda^\delta_\varepsilon\, \tilde{v}^\delta_\varepsilon, \text{ in } \Omega_\delta, \quad \tilde{v}^\delta_\varepsilon \in H^1_0(\Omega_\delta). \tag{2}$$

Since Ω_δ is bounded, the spectrum $\sigma^\delta_\varepsilon$ of problem (2) is discrete and can be written as $\sigma^\delta_\varepsilon := \{\lambda^{\delta,i}_\varepsilon \in \mathbb{R}^+ : i \in \mathbb{N}_0\}$, where $0 < \lambda^{\delta,0}_\varepsilon \leqslant \lambda^{\delta,1}_\varepsilon \leqslant \cdots \leqslant \lambda^{\delta,i}_\varepsilon \leqslant \lambda^{\delta,i+1}_\varepsilon \leqslant \cdots$. As the thickness of the domain goes to zero ($\delta \to 0$), it is clear that all the eigenvalues go to infinity. A precise characterization of the asymptotic behavior of $\sigma^\delta_\varepsilon$ is given in Theorem 1.1 for the case $\varepsilon \approx \delta$, and in Theorem 1.2 for the case $\varepsilon \ll \delta$.

We will use the Γ-convergence as main tool. Consider the quadratic energy $\widetilde{E}_\varepsilon^\delta : L^2(\omega\times\delta I) \to (-\infty, +\infty]$ associated with the self-adjoint operator $-\mathrm{div}(A_\varepsilon \nabla \cdot)$ from $L^2(\omega \times \delta I)$ into itself,

$$\widetilde{E}_\varepsilon^\delta(\tilde{v}) := \begin{cases} \int_{\omega\times\delta I} A_\varepsilon(\bar{x})\nabla\tilde{v}(\tilde{x})\nabla\tilde{v}(\tilde{x})\,\mathrm{d}\tilde{x}, & \text{if } \tilde{v} \in H_0^1(\omega \times \delta I), \\ +\infty, & \text{otherwise.} \end{cases} \tag{3}$$

As it is usual in the dimension reduction framework, the first step is to perform a rescaling and a change of variables in order to transform problem (2) into an equivalent one defined in the fixed domain $\omega \times I$. To each point $\tilde{x} = (\bar{x}, \tilde{x}_3) \in \omega \times \delta I$ we associate the point $x = (\bar{x}, x_3) = (\bar{x}, \delta^{-1}\tilde{x}_3) \in \omega \times I$ and we define $v \in H_0^1(\omega \times I)$ by $v(x) := \tilde{v}(\tilde{x})$, if $\tilde{v} \in H_0^1(\omega \times \delta I)$. Accordingly, we rescale the energy in (3) dividing it by δ so that the new energy becomes $E_\varepsilon^\delta : L^2(\omega \times I) \to (-\infty, +\infty]$,

$$E_\varepsilon^\delta(v) := \begin{cases} \int_{\omega\times I} a_{\alpha\beta}^\varepsilon(\bar{x})\partial_\alpha v(x)\partial_\beta v(x) + \frac{a_{33}^\varepsilon(\bar{x})}{\delta^2}|\partial_3 v(x)|^2\,\mathrm{d}x, & \text{if } v \in H_0^1(\omega \times I), \\ +\infty, & \text{otherwise,} \end{cases} \tag{4}$$

where ∂_i stands for $\partial/\partial x_i$, $i = 1, 2, 3$. The rescaled spectral problem is then

$$-\mathrm{div}(C_\varepsilon^\delta \nabla v_\varepsilon^\delta) = \lambda_\varepsilon^\delta v_\varepsilon^\delta, \text{ in } \omega \times I, \quad v_\varepsilon^\delta \in H_0^1(\omega \times I), \tag{5}$$

where $C_\varepsilon^\delta := A_\varepsilon I^\delta$, being I^δ the 3×3 diagonal matrix $\mathrm{diag}(1, 1, 1/\delta^2)$. We stress that problems (2) and (5) are equivalent.

Let (μ_0, ψ_0) be the normalized first eigenpair for the following periodic spectral problem:

$$-\partial_{y_\alpha}(a_{\alpha\beta}(\bar{y})\partial_{y_\beta}\psi) + a_{33}(\bar{y})\pi^2\psi = \mu\psi, \text{ in } Y, \quad \psi \in H_\#^1(Y),$$

and consider the following bidimensional homogenized spectral problem:

$$-\partial_\alpha(\bar{a}_{\alpha\beta}^h\partial_\beta\varphi) = \nu\varphi, \text{ in } \omega, \ \varphi \in H_0^1(\omega), \tag{6}$$

where the constant matrix $(\bar{a}_{\alpha\beta}^h)$ is the homogenized matrix of the periodic sequence $\{(\bar{a}_{\alpha\beta}^\varepsilon)\}$, $\bar{a}_{\alpha\beta}^\varepsilon(\bar{x}) := \big((\psi_0^\varepsilon)^2 a_{\alpha\beta}^\varepsilon\big)(\bar{x})$, with $\psi_0^\varepsilon(\bar{x}) := \psi_0(\bar{x}/\varepsilon)$. We recall that the eigenvalue μ_0 is real, positive and simple and the associated eigenfunction ψ_0 is a strictly positive function which lies in $H_\#^1(Y) \cap C_\#^{0,s}(Y)$, for some $0 < s < 1$ (see [4]). The next theorem characterizes the behavior of the spectrum $\sigma_\varepsilon^\delta$ when $\delta \approx \varepsilon$.

Theorem 1.1 *Let $(\lambda_\varepsilon^k, v_\varepsilon^k)$ be a k^{th} eigenpair associated with problem (5) for $\delta = \varepsilon$ and let $\theta_0 \in H_0^1(I)$ be the normalized eigenfunction associated with the first eigenvalue, π^2, for the problem $-\theta'' = \lambda\theta$, in I, $\theta \in H_0^1(I)$, that is $\theta_0(x_3) := \sqrt{2}\cos(\pi x_3)$. Then,*

$$\lambda_\varepsilon^k = \frac{\mu_0}{\varepsilon^2} + \nu_\varepsilon^k, \qquad v_\varepsilon^k(\bar{x}, x_3) = \psi_0^\varepsilon(\bar{x})u_\varepsilon^k(\bar{x}, x_3), \ \text{for a.e. } (\bar{x}, x_3) \in \omega \times I, \tag{7}$$

where, as $\varepsilon \to 0$, $\nu_\varepsilon^k \to \nu^k$ and, up to a subsequence, $u_\varepsilon^k \rightharpoonup u^k$ weakly in $H_0^1(\omega \times I)$, with $u^k(\bar{x}, x_3) = \varphi^k(\bar{x})\theta_0(x_3)$, being (ν^k, φ^k) a k^{th} eigenpair associated with (6). Conversely, any eigenfunction $u^k = \varphi^k\theta_0$ is the weak limit of a particular sequence of eigenfunctions associated with ν_ε^k.

Before stating the theorem characterizing the behavior of the spectrum $\sigma_\varepsilon^\delta$ when $\varepsilon \ll \delta$, we introduce some notation. Let $\delta = \varepsilon^\tau$, $\tau \in (0, 1)$, and let $(\mu_{\varepsilon,0}, \psi_{\varepsilon,0})$ be the normalized first eigenpair for the problem

$$-\varepsilon^{2(\tau-1)}\partial_{y_\alpha}(a_{\alpha\beta}(\bar{y})\partial_{y_\beta}\psi_\varepsilon) + a_{33}(\bar{y})\pi^2\psi_\varepsilon = \mu_\varepsilon\psi_\varepsilon, \text{ in } Y, \quad \psi_\varepsilon \in H_\#^1(Y), \tag{8}$$

and, for $j \in \mathbb{N}_0$, define $\check{\mu}_j := \int_Y \pi^2 a_{33}(\bar{y})\check{\psi}_j(\bar{y})\,\mathrm{d}\bar{y}$, where $\check{\psi}_0 \equiv 1$ in Y and, for $j \geqslant 1$, $\check{\psi}_j$ are the solutions of the recurrence problems in $H_\#^1(Y)$

$$-\partial_{y_\alpha}(a_{\alpha\beta}(\bar{y})\partial_{y_\beta}\check{\psi}_j) = -a_{33}(\bar{y})\pi^2\check{\psi}_{j-1} + \sum_{k=0}^{j-1}\check{\mu}_k\check{\psi}_{j-1-k}, \quad \int_Y \check{\psi}_j(\bar{y})\,\mathrm{d}\bar{y} = 0. \tag{9}$$

Theorem 1.2 *Suppose that $a_{\alpha\beta}$ are uniformly Lipschitz continuous in Y. Let $(\check{\lambda}_\varepsilon^k, \check{v}_\varepsilon^k)$ be a k^{th} eigenpair associated with problem (5) for $\delta = \varepsilon^\tau$. Let $i \in \mathbb{N}$ be such that $\frac{i-1}{i} < \tau \leqslant \frac{i}{i+1}$ and let θ_0 be as in Thm 1.1. Then, as $\varepsilon \to 0$, $\mu_{\varepsilon,0} \to \pi^2 \int_Y a_{33}(\bar{y})\,\mathrm{d}\bar{y} = \check{\mu}_0$, $\psi_{\varepsilon,0}(\bar{x}/\varepsilon) \to \check{\psi}_0$ uniformly in ω and*

$$\check{\lambda}_\varepsilon^k = \sum_{j=0}^{i} \frac{\check{\mu}_j}{\varepsilon^{2[\tau(j+1)-j]}} + \check{\nu}_\varepsilon^k + \rho_\varepsilon^\tau, \quad \check{v}_\varepsilon^k(\bar{x}, x_3) = \psi_{\varepsilon,0}\Big(\frac{\bar{x}}{\varepsilon}\Big)\check{u}_\varepsilon^k(\bar{x}, x_3), \text{ for a.e. } (\bar{x}, x_3) \in \omega \times I,$$

where, as $\varepsilon \to 0$, $\check{\nu}_\varepsilon^k \to \check{\nu}^k$, $\rho_\varepsilon^\tau \to 0$ and, up to a subsequence, $\check{u}_\varepsilon^k \rightharpoonup \check{u}^k$ weakly in $H_0^1(\omega \times I)$, with $\check{u}^k(\bar{x}, x_3) = \check{\varphi}^k(\bar{x})\theta_0(x_3)$, being $(\check{\nu}^k, \check{\varphi}^k)$ a k^{th} eigenpair associated with the following bidimensional homogenized spectral problem: $-\partial_\alpha(a_{\alpha\beta}^h \partial_\beta \varphi) = \check{\nu}\varphi$, in ω, $\varphi \in H_0^1(\omega)$, where the constant matrix $(a_{\alpha\beta}^h)$ is the homogenized matrix of the sequence $\{(a_{\alpha\beta}^\varepsilon)\}$. Conversely, any eigenfunction $\check{u}^k = \check{\varphi}^k\theta_0$ is the weak limit of a particular sequence of eigenfunctions associated with $\check{\nu}_\varepsilon^k$.

2. Sketch of the proofs

The proof of Theorems 1.1 and 1.2 relies on a Γ-convergence argument applied to a sequence of quadratic functionals related to the energy associated with problem (5) for $\delta = \varepsilon$ and $\delta = \varepsilon^\tau$, respectively, by means of the following result, presented in [2].

Lemma 2.1 *Let $\mathcal{A}_\varepsilon : H_\varepsilon \to H_\varepsilon$ be a sequence of densely defined self-adjoint operators where H_ε coincides algebraically with a fixed Hilbert space H endowed with a scalar product $(\cdot,\cdot)_\varepsilon$ such that $c_1\|u\|^2 \leqslant (u,u)_\varepsilon \leqslant c_2\|u\|^2$, for suitable positive constants c_1, c_2, and $\lim_\varepsilon (u_\varepsilon, v_\varepsilon)_\varepsilon = (u, v)$ whenever $u_\varepsilon \to u$ and $v_\varepsilon \to v$ in H. Let $G_\varepsilon : H \to (-\infty, +\infty]$ be defined by $G_\varepsilon(u) := (\mathcal{A}_\varepsilon u, u)_\varepsilon$, if $u \in D(\mathcal{A}_\varepsilon)$, and $G_\varepsilon(u) := +\infty$, otherwise. Assume further that the three following conditions hold: (i) $G_\varepsilon(u) \geqslant c_0\|u\|^2$, for a suitable constant $c_0 > 0$; (ii) If $\sup_\varepsilon G_\varepsilon(u_\varepsilon) < +\infty$, then the sequence $\{u_\varepsilon\}$ is strongly relatively compact in H; (iii) G_ε Γ-converges to a certain functional G. Then, the limit functional G determines a unique closed linear operator $\mathcal{A}_0 : H \to H$ with compact resolvent such that $G(u) = (\mathcal{A}_0 u, u)$, for all $u \in D(\mathcal{A}_0)$. Furthermore, the spectral problems associated with $\mathcal{A}_\varepsilon$ converge in the following sense: let $(\nu_\varepsilon^k, u_\varepsilon^k)$ and (ν^k, u^k) be k^{th} eigenpairs associated with the operators $\mathcal{A}_\varepsilon$ and $\mathcal{A}$, respectively. Then, as $\varepsilon \to 0$, $\nu_\varepsilon^k \to \nu^k$, for every $k \in \mathbb{N}_0$. Moreover, up to a subsequence, $\{u_\varepsilon^k\}$ converges strongly to eigenfunctions associated to ν^k. Conversely, any eigenfunction u^k is the strong limit of a particular sequence of eigenfunctions of $\mathcal{A}_\varepsilon$ associated with ν_ε^k.*

The idea of the proof of Theorem 1.1 is the following: since we are expecting the asymptotic behavior mentioned in (7) for the shifted spectrum $\sigma_\varepsilon^\varepsilon - \frac{\mu_0}{\varepsilon^2}$, instead of the energy defined in (4) for $\delta = \varepsilon$, we consider the functional $E_\varepsilon : L^2(\omega \times I) \to (-\infty, +\infty]$, defined by

$$E_\varepsilon(v) := \begin{cases} \int_{\omega\times I} a_{\alpha\beta}^\varepsilon(\bar{x})\partial_\alpha v(x)\partial_\beta v(x) + \frac{a_{33}^\varepsilon(\bar{x})}{\varepsilon^2}|\partial_3 v(x)|^2 - \frac{\mu_0}{\varepsilon^2}|v(x)|^2\,\mathrm{d}x, & \text{if } v \in H_0^1(\omega \times I), \\ +\infty, & \text{otherwise.} \end{cases} \tag{10}$$

The following result allows us, using a classical change of unknowns introduced in [6] (see also [1]), to transform functional (10) in order to apply Lemma 2.1, obtaining the desired convergences.

Proposition 2.2 *For fixed $\varepsilon > 0$, consider the functions u and v related by $v(x) = \psi_0^\varepsilon(\bar{x})u(x)$, for a.e. $x = (\bar{x}, x_3) \in \omega \times I$. Then $v \in H_0^1(\omega \times I)$ if and only if $u \in H_0^1(\omega \times I)$. Moreover, if $v \in H_0^1(\omega \times I)$,*

$$\int_{\omega\times I} a_{\alpha\beta}^\varepsilon(\bar{x})\partial_\alpha v(x)\partial_\beta v(x) + \frac{a_{33}^\varepsilon(\bar{x})}{\varepsilon^2}\pi^2 v^2(x) - \frac{\mu_0}{\varepsilon^2}v^2(x)\,\mathrm{d}x = \int_{\omega\times I} \bar{a}_{\alpha\beta}^\varepsilon(\bar{x})\partial_\alpha u(x)\partial_\beta u(x)\,\mathrm{d}x.$$

Using Proposition 2.2, we get $E_\varepsilon(v) = G_\varepsilon(u)$, $G_\varepsilon(u) : L^2(\omega \times I) \to (-\infty, +\infty]$ given by

$$G_\varepsilon(u) := \begin{cases} \int_{\omega\times I} \bar a^\varepsilon_{\alpha\beta}(\bar x)\partial_\alpha u(x)\partial_\beta u(x) + \frac{\left((\psi_0^\varepsilon)^2 a^\varepsilon_{33}\right)(\bar x)}{\varepsilon^2}\Big(|\partial_3 u(x)|^2 - \pi^2|u(x)|^2\Big)\,\mathrm{d}x, & \text{if } u \in H^1_0(\omega\times I),\\ +\infty, & \text{otherwise.}\end{cases}$$

Proposition 2.3 *The sequence of functionals $\{G_\varepsilon\}$ Γ-converges, with respect to the strong topology of $L^2(\omega \times I)$, to the functional $G : L^2(\omega\times I) \to (-\infty, +\infty]$ defined by $G(u) := \int_\omega \bar a^h_{\alpha\beta}\partial_\alpha\varphi(\bar x)\partial_\beta\varphi(\bar x)\,\mathrm{d}\bar x$ if $u(\bar x, x_3) = \varphi(\bar x)\,\theta_0(x_3)$, $\varphi \in H^1_0(\omega)$, and $G(u) := +\infty$, otherwise.*

Idea of the proof. Using the periodic homogenization result, $(\bar a^\varepsilon_{\alpha\beta})$ G-converges to $(\bar a^h_{\alpha\beta})$. Consequently (see [3, Thm 13.12]) the associated quadratic functional $\overline{G}_\varepsilon(\varphi) := \int_\omega \bar a^\varepsilon_{\alpha\beta}\partial_\alpha\varphi\partial_\beta\varphi\,\mathrm{d}\bar x$, $\varphi \in H^1_0(\omega)$, Γ-converges, with respect to the strong topology of $L^2(\omega)$, to the functional $\overline{G}(\varphi) := \int_\omega \bar a^h_{\alpha\beta}\partial_\alpha\varphi\partial_\beta\varphi\,\mathrm{d}\bar x$. We will now proceed in two steps.

Step 1. We prove that for any $\{u_\varepsilon\}$ and u in $L^2(\omega\times I)$ such that $u_\varepsilon \to u$ in $L^2(\omega \times I)$, one has $G(u) \leqslant \liminf\limits_{\varepsilon\to 0} G_\varepsilon(u_\varepsilon)$.

Up to a subsequence (not relabeled), we may assume without loss of generality that $\liminf_\varepsilon G_\varepsilon(u_\varepsilon) = \lim_\varepsilon G_\varepsilon(u_\varepsilon) < +\infty$. Then $\{u_\varepsilon\} \subset H^1_0(\omega\times I)$ and $\sup_\varepsilon G_\varepsilon(u_\varepsilon) < +\infty$. Since $u_\varepsilon(\bar x, \cdot) \in H^1_0(I)$, for a.e. $\bar x \in \omega$, and π^2 is the first eigenvalue associated with the problem $-\theta'' = \lambda\theta$, in I, $\theta \in H^1_0(I)$, we have $\int_I (|\partial_3 u_\varepsilon|^2 - \pi^2|u_\varepsilon|^2)\,\mathrm{d}x_3 \geqslant 0$, for a.e. $\bar x \in \omega$. Using the uniform ellipticity of $\left((\psi_0^\varepsilon)^2 a^\varepsilon_{ij}\right)_{1\le i,j\le 3}$ and the strong convergence in $L^2(\omega\times I)$ of $\{u_\varepsilon\}$, we get

$$\int\limits_{\omega\times I} |\partial_\alpha u_\varepsilon|^2\,\mathrm{d}x \leqslant C, \qquad \int\limits_{\omega\times I} |\partial_3 u_\varepsilon|^2\,\mathrm{d}x \leqslant C\,\varepsilon^2 + \pi^2 \int\limits_{\omega\times I} |u_\varepsilon|^2\,\mathrm{d}x \leqslant \overline{C}, \tag{11}$$

where C and $\overline{C}$ are constants independent of ε. Consequently, $\sup_\varepsilon \|u_\varepsilon\|_{H^1_0(\omega\times I)} < +\infty$ and $u_\varepsilon \rightharpoonup u$ weakly in $H^1_0(\omega\times I)$. The l.s.c. of the L^2-norm with respect to the weak topology, together with (11), implies that $\int_I \left(|\partial_3 u|^2 - \pi^2|u|^2\right)\mathrm{d}x_3 \leqslant 0$ for a.e. $\bar x \in \omega$. Since the opposite inequality has to be satisfied as well, we conclude the existence of a function $\varphi \in H^1_0(\omega)$ such that $u(\bar x, x_3) = \varphi(\bar x)\,\theta_0(x_3)$, for a.e. $(\bar x, x_3) \in \omega\times I$. Finally, using Fatou's Lemma and the Γ-convergence result mentioned in the beginning of this proof, we obtain $\liminf_{\varepsilon\to0} G_\varepsilon(u_\varepsilon) \geqslant \liminf_{\varepsilon\to0}\int_{\omega\times I}\bar a^\varepsilon_{\alpha\beta}\partial_\alpha u_\varepsilon\partial_\beta u_\varepsilon\,\mathrm{d}x \geqslant G(u)$.

Step 2. We prove that for any $u \in L^2(\omega\times I)$, there exists a sequence $\{u_\varepsilon\} \subset L^2(\omega\times I)$ satisfying $u_\varepsilon \to u$ in $L^2(\omega\times I)$ and $G(u) = \lim_{\varepsilon\to0} G_\varepsilon(u_\varepsilon)$.

Given $u \in L^2(\omega\times I)$, the only nontrivial case is when $u(\bar x, x_3) = \varphi(\bar x)\,\theta_0(x_3)$, with $\varphi \in H^1_0(\omega)$, otherwise, considering Step 1, it's enough to take $u_\varepsilon \equiv u$.

By the Γ-convergence result mentioned before, we know that there exists a sequence $\{\varphi_\varepsilon\} \subset H^1_0(\omega)$ converging in $L^2(\omega)$ to φ and such that $\lim_{\varepsilon\to0}\int_\omega \bar a^\varepsilon_{\alpha\beta}\partial_\alpha\varphi_\varepsilon\partial_\beta\varphi_\varepsilon\,\mathrm{d}\bar x = \int_\omega \bar a^h_{\alpha\beta}\partial_\alpha\varphi\partial_\beta\varphi\,\mathrm{d}\bar x$. In order to obtain the intended equality, it's enough to define, for each $\varepsilon > 0$, $u_\varepsilon(\bar x, x_3) := \varphi_\varepsilon(\bar x)\,\theta_0(x_3)$. □

On the proof of Theorem 1.1. We use Lemma 2.1 with $H = L^2(\omega\times I)$ and $(u, v)_\varepsilon := \int_{\omega\times I}(\psi_0^\varepsilon)^2 uv\,\mathrm{d}x$. For each $\varepsilon > 0$, G_ε is a nonnegative l.s.c. quadratic form in $L^2(\omega\times I)$. Consequently, the associated operator, $\mathcal{A}_\varepsilon$, is a self-adjoint operator in $L^2(\omega\times I)$ endowed with the scalar product $(\,\cdot\,,\cdot\,)_\varepsilon$ (see [3, Thm 12.13]). On the other hand, using Proposition 2.3, it's easy to see that G_ε satisfies the hypotheses of Lemma 2.1. Furthermore, representing by ν^k_ε the k^{th} eigenvalue associated with $\mathcal{A}_\varepsilon$, we have $\nu^k_\varepsilon = \lambda^k_\varepsilon - \frac{\mu_0}{\varepsilon^2}$. Having in mind the change of unknowns introduced in Proposition 2.2, Theorem 1.1 follows. □

On the proof of Theorem 1.2. Replacing ψ_0 by $\psi_{\varepsilon,0}$ and μ_0 by $\mu_{\varepsilon,0}$, the steps of the proof of Theorem 1.2 are similar to those of Theorem 1.1, provided we establish the convergence $\psi_{\varepsilon,0}(\bar x/\varepsilon) \to 1$ uniformly in ω and the asymptotic expansion $\mu_{\varepsilon,0} = \breve\mu_0 + \varepsilon^{2(1-\tau)}\breve\mu_1 + \cdots + \varepsilon^{2i(1-\tau)}\breve\mu_i + o\big(\varepsilon^{2i(1-\tau)}\big)$.

We begin by proving that $\mu_{\varepsilon,0} \to \check{\mu}_0 > 0$ and that all the others eigenvalues of problem (8) tend to $+\infty$. By the Rayleigh's formula, $\mu_{\varepsilon,0} := \inf \left\{ \int_Y \frac{a_{\alpha\beta}}{\varepsilon^{2(1-\tau)}} \partial_\alpha \psi \partial_\beta \psi + a_{33}\pi^2\psi^2 \, \mathrm{d}\bar{y} : \psi \in H^1_\#(Y), \|\psi\|_{L^2(Y)} = 1 \right\}$. Using (1) and $\psi = 1$ as a test function, we conclude that $\zeta\pi^2 \leqslant \mu_{\varepsilon,0} \leqslant \check{\mu}_0$. Since $\psi_{\varepsilon,0}$ is a minimizer for $\mu_{\varepsilon,0}$, one gets $\|\nabla\psi_{\varepsilon,0}\|_{L^2(Y)} \to 0$. Consequently, $\psi_{\varepsilon,0} \to 1$ in $H^1_\#(Y)$. Moreover, $\limsup_{\varepsilon\to 0} \mu_{\varepsilon,0} \leqslant \check{\mu}_0$. On the other hand, $\liminf_{\varepsilon\to 0} \mu_{\varepsilon,0} \geqslant \liminf_{\varepsilon\to 0} \int_Y a_{33}\pi^2\psi^2_{\varepsilon,0} \, \mathrm{d}\bar{y} = \check{\mu}_0$. Therefore, $\mu_{\varepsilon,0} \to \check{\mu}_0$. Similarly, using the Rayleigh's formula for $\mu_{\varepsilon,1}$ and admitting that the latter is bounded, we are led to a contradiction, since we would conclude that any minimizing sequence of eigenfunctions convergence, on the one hand to the constant function $\check{\psi}_0 \equiv 1$, on the other hand to a function having zero mean. So, except the first, all the eigenvalues of problem (8) tend to $+\infty$.

Since the coefficients $a_{\alpha\beta}$ are uniformly Lipschitz continuous in Y, $\{\psi_{\varepsilon,0}\}$ is uniformly bounded in $H^2(Y)$ (see, [4, Thm 8.8]). Due to the compact injection of $H^2(Y)$ in $C^0(\overline{Y})$, we conclude that $\psi_{\varepsilon,0}(\bar{y}) \to 1$ uniformly in $\overline{Y}$. From the Y-periodicity of $\psi_{\varepsilon,0}$, we derive $\psi_{\varepsilon,0}(\bar{x}/\varepsilon) \to 1$ uniformly in ω.

The rest of the proof is based on the following result (see [5]): Let $L : H \to H$ be a linear compact self-adjoint operator in a Hilbert space H, and suppose that there exists a real $\lambda > 0$ and a vector $f \in H$, $f \neq 0$, such that $\|Lf - \lambda f\|_H \leqslant \gamma$, for a constat $\gamma > 0$. Then there exists an eigenvalue $\bar{\lambda}$ of L such that $|\bar{\lambda} - \lambda| \leqslant \gamma\|f\|^{-1}_H$.

For the sake of simplicity we will stretch the proof only for $i = 1$, the argument being easily generalized for $i > 1$. Considering $H = L^2_\#(Y)$ and setting $\bar{\varepsilon} := \varepsilon^{2(1-\tau)}$, $b(\bar{y}) := \pi^2 a_{33}(\bar{y})$ and $A_{\bar{\varepsilon}}\varphi := -\frac{1}{\bar{\varepsilon}}\partial_\alpha(a_{\alpha\beta}(\bar{y})\partial_\beta\varphi) + b(\bar{y})$, $\varphi \in H^1_\#(Y)$, we apply the above result to $L_{\bar{\varepsilon}} : L^2_\#(Y) \to L^2_\#(Y)$ such that $L_{\bar{\varepsilon}} g = \varphi$, solution of $A_{\bar{\varepsilon}}\varphi = g$, to $f_{\bar{\varepsilon}} := A_{\bar{\varepsilon}}\psi_{\bar{\varepsilon}}$, $\psi_{\bar{\varepsilon}} := \check{\psi}_0 + \bar{\varepsilon}\check{\psi}_1 + \bar{\varepsilon}^2\check{\psi}_2$, and to $\lambda_{\bar{\varepsilon}} = (\check{\mu}_0 + \bar{\varepsilon}\check{\mu}_1)^{-1}$. Since $L_{\bar{\varepsilon}} f_{\bar{\varepsilon}} - \lambda_{\bar{\varepsilon}} f_{\bar{\varepsilon}} = \psi_{\bar{\varepsilon}} - \lambda_{\bar{\varepsilon}} A_{\bar{\varepsilon}}\psi_{\bar{\varepsilon}} =: w_{\bar{\varepsilon}}$, using (9) we easily obtain that $w_{\bar{\varepsilon}} = \left(\bar{\varepsilon}^2[(b - \check{\mu}_0)\check{\psi}_2 - \check{\mu}_1\check{\psi}_1] - \bar{\varepsilon}^3\check{\mu}_1\check{\psi}_2\right)(\check{\mu}_0 + \bar{\varepsilon}\check{\mu}_1)^{-1}$ and since $\check{\mu}_0 > 0$ we get, for a constant c independent of $\bar{\varepsilon}$, $\|w_{\bar{\varepsilon}}\|_H \leqslant \bar{\varepsilon}^2 c$ and, consequently, the existence of an eigenvalue $\bar{\lambda}_{\bar{\varepsilon}}$ satisfying, for another constant c independent of $\bar{\varepsilon}$, $|\bar{\lambda}_{\bar{\varepsilon}} - (\check{\mu}_0 + \bar{\varepsilon}\check{\mu}_1)^{-1}| \leqslant \bar{\varepsilon}^2 c$. We used the fact that $\|A_{\bar{\varepsilon}}\psi_{\bar{\varepsilon}}\| \to \check{\mu}_0 > 0$. Since all the eigenvalues of $L_{\bar{\varepsilon}}$ tend to zero, except the first, which converges to $\check{\mu}_0^{-1} > 0$, we conclude that, for $\bar{\varepsilon}$ small enough, $\bar{\lambda}_{\bar{\varepsilon}} = \mu_{\bar{\varepsilon},0}{}^{-1}$ and then $|\mu_{\bar{\varepsilon},0} - (\check{\mu}_0 + \bar{\varepsilon}\check{\mu}_1)| \leqslant \bar{\varepsilon}^2 C$, for C still independent of $\bar{\varepsilon}$. This concludes the proof for $i = 1$. □

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