Existence of Minimizers for Non-Level Convex Supremal Functionals

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Abstract

The paper is devoted to determine necessary and sufficient conditions for existence of solutions to the problem $\inf\left\{\operatorname*{ess\,sup}_{x\in\Omega} f(\nabla u(x)) : u \in u_0 + W_0^{1,\infty}(\Omega)\right\}$, when the supremand f is not necessarily level convex. These conditions are obtained through a comparison with the related level convex problem and are written in terms of a differential inclusion involving the boundary datum. Several conditions of convexity for the supremand f are also investigated.

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1 Introduction

The direct method of the calculus of variations requires some lower semicontinuity of the functional to minimize, which, in general, is related to some notion of convexity. In the lack of this *convexity*, the usual procedure is to consider the relaxed problem, related to the original one, obtained by *convexification* of the *non-convex* function. This leads in many problems to an understanding of the minimizing sequences and of the infimum to the original problem, but it doesn't ensure the problem has a solution.

In this paper we will investigate necessary and sufficient conditions for existence of solutions to

$$(P) \qquad \inf\left\{\operatorname*{ess\,sup}_{x\in\Omega} f\left(\nabla u\left(x\right)\right) : \ u \in u_0 + W_0^{1,\infty}(\Omega)\right\},$$

when f lacks of the appropriate convexity notion. We restrict our attention to the so-called scalar case, that is u is a scalar function, $u : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}$, $n \geq 1$, $u_0 \in W^{1,\infty}(\Omega)$. It is also possible to have u_0 Lipschitz only defined on $\partial\Omega$, $u_0 : \partial\Omega \longrightarrow \mathbb{R}$. In this case, u_0 shall be extended to Ω as a Lipschitz function and the study of problem (P) can be done according to the choice of the Lipschitz extension.

Functionals in the L^∞ form, as above, provide a realistic setting to many physical problems in a variety of contexts like nonlinear elasticity, chemotherapy or imaging. For a more detailed description see [16] due to Barron-Jensen-Wang or [18] due to Bocea-Nesi.

Minimizing the functional in problem (P) appears also as a generalization of the Lipschitz extension problem (this is the case when $f = |\cdot|$ and u_0 is a given Lipschitz function defined on $\partial\Omega$) and was intensively studied by Aronsson in the 1960's, cf. [4, 5, 6, 7], also developing a theory on absolute minimizers, see also the monograph of Aronsson-Crandall-Juutinen [8] and the references therein. The problem of existence and uniqueness of Lipschitz extension has been addressed by many authors with different tools, cf. for instance [33, 2, 3] among a wide literature. In recent years also other

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questions of the calculus of variations, like lower semicontinuity, relaxation, homogenization, L^p approximations, dimensional reduction, Γ-convergence and supremal representation, have been addressed for L^∞ functionals by several authors: Barron-Liu [17], Barron-Jensen-Wang [15], Acerbi-Buttazzo-Prinari [1], Briani-Garroni-Prinari [19], Bocea-Nesi [18], Prinari [39], [40], Cardialaguet-Prinari [21], Babadjian-Prinari-Zappale [9], Zappale [43].

The functional defined in problem (P) is known to be lower semicontinuous with respect to the weak* topology of $W^{1,\infty}$ (cf. [15] and [13], see also Theorems 3.1 and 3.2), if and only if f is a level convex function, that is, the level sets of f are convex (see Definition 2.1). This notion is usually known in areas like convex analysis, optimization, or economics as *quasiconvexity*. However, since *quasiconvexity* has a different meaning in the calculus of variations, we prefer to use the present terminology.

Our main interest is the case in which f is not necessarily a level convex function. This problem was addressed in the one dimensional case by Barron [12]. In the n-dimensional setting that we consider, the problem presents more involved features. We establish necessary and sufficient conditions for existence of solutions to (P). This is done through a differential inclusion which is obtained in turn relating the original problem (P) and the relaxed one

$$(P^{\mathrm{lc}}) \qquad \inf\left\{\operatorname*{ess\,sup}_{x\in\Omega} f^{\mathrm{lc}}\left(\nabla u\left(x\right)\right):\ u\in u_0+W_0^{1,\infty}(\Omega)\right\},$$

where f^{lc} denotes the level convex envelope of f (cf. Definition 2.1). This was the procedure applied to problems in the integral form

$$\inf\left\{\int_\Omega f\left(\nabla u\left(x\right)\right)dx:\ u\in u_0+W_0^{1,\infty}(\Omega)\right\},$$

and we refer to Cellina [22], [23] and Friesecke [32] in the scalar case and to Dacorogna-Marcellini [27] and Dacorogna-Pisante-Ribeiro [28] in the vectorial one. For further references, see also [25, Chapter 11].

In the present context of L^∞ functionals, to our knowledge very little is known, and although our problem is a scalar one, our approach is close to the one used in the vectorial case by Dacorogna-Marcellini [27] and Dacorogna-Pisante-Ribeiro [28] for integral functionals. Moreover, our results are sharp since differential inclusions in the scalar case are better understood than in the vectorial one. In particular, we characterize existence of solutions to problem (P) when the boundary datum is affine, say $u_0 := u_{\xi_0}$, with gradient $\xi_0 \in \mathbb{R}^n$, in terms of suitable *level convexity* properties of the relaxed density f^{lc} around ξ_0.

The paper is organized as follows. Section 2 is devoted to the study of the properties of a *level convex* function and the *level convex envelope* of a function. This includes supremal Jensen's inequality, Carathéodory type results and several notions of *strict level convexity*, which are explored in view of uniqueness results for the minimizing problems with affine boundary data. Moreover, this may also have an independent interest for optimization purposes. Some prerequisites concerning differential inclusions are also recalled in this section.

In Section 3 we state the relaxation result which will be one of the key results to achieve our necessary and sufficient condition for existence. In particular, under suitable hypothesis, we show in Corollary 3.6 that

$$\inf(P) = \inf(P^{\mathrm{lc}}).$$

Necessary and sufficient conditions for the existence of solutions to problem (P) are provided and discussed in Section 4. Our main general result is stated as Theorem 4.2 and it establishes that a necessary and sufficient condition for existence of solutions to (P) is

$$\exists\ u \in u_0 + W_0^{1,\infty}(\Omega):\ f(\nabla u(x)) \le \inf(P^{\mathrm{lc}}),\ a.e.\ x\in\Omega.$$

Moreover, making use of well known results on differential inclusions, a sufficient condition to this last one can be written as

$$\nabla u_0(x) \in L_{\inf(P^{\mathrm{lc}})}(f) \cup \operatorname{int} L_{\inf(P^{\mathrm{lc}})}(f^{\mathrm{lc}}),\ a.e.\ x\in\Omega,$$

where $L_c(g)$ denotes the set of level c of the function g, that is

$$L_c(g) := \{\xi \in \mathbb{R}^n : \ g(\xi) \leq c\}.$$

Then, with this characterization in mind, we explore both sufficient and necessary conditions. Regarding sufficient conditions we consider both the cases u_0 is an affine function or not. In particular, we can always get existence of solutions in dimension $n = 1$, cf. Corollary 4.4, and for arbitrary dimension n, and arbitrary data u_0, if we require some regularity on the solution of the relaxed problem (P^{lc}), together with some constant properties on f^{lc}, we can ensure existence of solutions to (P), see Theorem 4.5. The constant hypothesis on f^{lc} will be clarified later on Theorem 4.15, where necessary and sufficient conditions to have a solution to (P) with affine boundary datum $u_0(x) =< \xi_0, x > +c$, $\xi_0 \in \mathbb{R}^n$, will be explored in terms of the set $\{\xi \in \mathbb{R}^n : \ f^{\mathrm{lc}}(\xi) = f^{\mathrm{lc}}(\xi_0)\}$.

Concerning necessary conditions we follow the ideas of Marcellini [35], Dacorogna-Marcellini [27], and Dacorogna-Pisante-Ribeiro [28]. Our approach is done through uniqueness of solutions to a level convex problem of type (P). This can be achieved if the function f is strictly level convex (cf. Definition 2.13). However, as we observe, it is not reasonable to assume f^{lc} to satisfy such property and, mimicking Dacorogna-Marcellini [27], we introduce the notion of strict level convexity of a function f at a point $\xi_0 \in \mathbb{R}^n$ in at least one direction as: for some $\alpha \in \mathbb{R}^n \setminus \{0\}$

$$\left.\begin{array}{l} \xi_0 = t\gamma + (1-t)\eta, \ t \in (0,1) \\ f(\xi_0) = \max\{f(\gamma), f(\eta)\} \end{array}\right\} \Longrightarrow < \gamma - \eta, \alpha >= 0.$$

This condition turns out to be a sufficient one for uniqueness of solution to level convex problems with affine boundary datum, see Theorem 4.11. As a consequence, in Corollary 4.13, if $f^{\mathrm{lc}}(\xi_0) < f(\xi_0)$, we prove that f^{lc} satisfies the above condition if and only if the original problem (P) does not admit any solution.

The last part of Section 4 is devoted to discuss existence and uniqueness of absolute minimizers (cf. Definition 4.17) for level and non-level convex problems.

Finally, in the Appendix, we briefly address convexity notions in the supremal setting for the vectorial case. In [15], it was investigated the right notion to ensure lower semicontinuity of the supremal functionals in the vectorial case, together with supremal notions of polyconvexity and rank one convexity. Our goal here is to clarify the relations between these notions.

2 Level Convexity and Differential Inclusions

All through the paper we will use the following notation for affine functions. Given a vector $\xi_0 \in \mathbb{R}^n$, by $u_{\xi_0} : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ we denote a function such that $\nabla u_{\xi_0}(x) = \xi_0$, *a.e.* $x \in \Omega$, or equivalently $u_{\xi_0}(x) :=< \xi_0, x > +c$, for some $c \in \mathbb{R}$.

2.1 Properties of level convex functions and level convex envelopes

In this section we establish some results on level convex functions and level convex envelopes which are well known in the usual convex setting. The main reason to consider level convex functions here is that, as it was pointed out for the first time in [13], it is, together with the lower semicontinuity of the function, a sufficient and necessary condition to sequential weak* lower semicontinuity in $W^{1,\infty}(\Omega)$ for functionals in the supremal form (see Theorem 3.1 below due to Barron-Jensen-Wang [15] and Theorem 3.2).

We first recall definitions and properties on lower semicontinuity. We refer to [20, Chapter 1], [31, Chapter 3], [41, Section 7]. A function $f : \mathbb{R}^n \to [-\infty, +\infty]$ is said to be *lower semicontinuous* if the level sets $L_c(f) := \{\xi \in \mathbb{R}^n : \ f(\xi) \leq c\}$ are closed for every $c \in \mathbb{R}$. Equivalently, f is lower semicontinuous if it is sequentially lower semicontinuous, that is, if

$$f(\xi) \leq \liminf f(\xi_n), \ \text{for every} \ \ \xi_n \to \xi.$$

The lower semicontinuous envelope of a function $f : \mathbb{R}^n \to [-\infty, +\infty]$ is the function $\text{lsc}f : \mathbb{R}^n \to [-\infty, +\infty]$ defined by

$$\text{lsc}f(\xi) = \sup\{g(\xi) : \; g : \mathbb{R}^n \to [-\infty, +\infty], \; g \text{ lower semicontinuous}, \; g \leq f\}.$$

In particular, it results that $\text{lsc}f$ is a lower semicontinuous function and

$$\text{lsc}f(\xi) = \inf\{\liminf f(\xi_n) : \; \xi_n \to \xi\}, \; \forall\, \xi \in \mathbb{R}^n. \tag{2.1}$$

Moreover, for every $\xi \in \mathbb{R}^n$ there exists a sequence ξ_n converging to ξ, such that $\text{lsc}f(\xi) = \lim f(\xi_n)$.

Now we recall the notion of *level convexity* and the related envelope. We observe that, in convex analysis and operational research, level convexity is usually referred as *quasiconvexity.* We avoid here this designation because, in the calculus of variations, quasiconvexity is known as a different concept. For a reference in operational research, see [34].

Definition 2.1. *(i) A function $f : \mathbb{R}^n \to [-\infty, +\infty]$ is said to be* level convex *if the level sets of f, $L_c(f)$, are convex for each $c \in \mathbb{R}$. Equivalently a function $f : \mathbb{R}^n \to [-\infty, +\infty]$ is level convex if and only if for every $\xi, \eta \in \mathbb{R}^n$ and $t \in [0, 1]$*

$$f(t\xi + (1-t)\eta) \leq \max\{f(\xi), f(\eta)\}.$$

(ii) The level convex envelope of a function $f : \mathbb{R}^n \to [-\infty, +\infty]$ is the function $f^{\text{lc}} : \mathbb{R}^n \to [-\infty, +\infty]$ defined by

$$f^{\text{lc}}(\xi) = \sup\{g(\xi) : \; g : \mathbb{R}^n \to [-\infty, +\infty], \; g \textit{ level convex}, \; g \leq f\}.$$

(iii) The lower semicontinuous level convex envelope of a function $f : \mathbb{R}^n \to [-\infty, +\infty]$ is the function $f^{\text{lslc}} : \mathbb{R}^n \to [-\infty, +\infty]$ defined by

$$f^{\text{lslc}}(\xi) = \sup\{g(\xi) : \; g : \mathbb{R}^n \to [-\infty, +\infty], \; g \textit{ lower semicontinuous and level convex}, \; g \leq f\}.$$

Remark 2.2. *(i) It is easily seen that f^{lc} is a level convex function and that f^{lslc} is a lower semicontinuous and level convex function. Therefore, we can call these envelopes respectively the greatest level convex function below f and the greatest lower semicontinuous level convex function below f.*

(ii) It is easy to verify that $f^{\text{lslc}} \leq f^{\text{lc}} \leq f$ and $f^{\text{lslc}} \leq \text{lsc}f \leq f$.

(iii) The function $f(\xi) = -\xi^2$ defined in $\mathbb{R}$ provides an example of a function whose envelopes take the $-\infty$ value, indeed $f^{\text{lc}} \equiv f^{\text{lslc}} \equiv -\infty$.

(iv) In general, f^{lc} and f^{lslc} don't coincide. Indeed the characteristic function of $\mathbb{R} \setminus (0, 1)$ is a level convex function, but it is not lower semicontinuous.

(v) In general a level convex function defined in $\mathbb{R}^n$, with $n > 1$, may not be Borel measurable, in fact one may consider the characteristic function of the complement of a convex set which is not Borel measurable.

We establish some preliminary properties, whose proof can be deduced applying the above definitions.

Proposition 2.3. *Let $f : \mathbb{R}^n \to [-\infty, +\infty]$.*

(i) If f is a level convex function, then the lower semicontinuous envelope of f is still level convex, that is $\text{lsc}f$ is level convex.

(ii) The following identity holds: $f^{\text{lslc}} = \text{lsc}(f^{\text{lc}})$.

Next we relate the level convexity of a function with a generalization of Jensen's inequality for the supremal setting. The proof can be found in Barron [12, Theorem 30], (see [15, Theorem 1.2], where the theorem is stated under a lower semicontinuity hypothesis, and [14, Lemma 2.4] where one implication has been shown in order to provide a Hopf-Lax formula).

Theorem 2.4. *A Borel measurable function $f : \mathbb{R}^n \to \mathbb{R}$ is level convex if and only if it verifies the supremal Jensen's inequality:*

$$f\left(\int_\Omega \varphi \, d\mu\right) \leq \mu - \operatorname*{ess\,sup}_{x \in \Omega} f(\varphi(x))$$

for every probability measure μ on $\mathbb{R}^d$ supported on the open set $\Omega \subseteq \mathbb{R}^d$, and every $\varphi \in L^1_\mu(\Omega; \mathbb{R}^n)$.

In particular, considering the Lebesgue measure, if Ω is a set with finite Lebesgue measure,

$$f\left(\frac{1}{|\Omega|}\int_\Omega \varphi(x)\, dx\right) \leq \operatorname*{ess\,sup}_{x \in \Omega} f(\varphi(x)), \ \forall \ \varphi \in L^1(\Omega; \mathbb{R}^n).$$

From Carathéodory's theorem, it follows the next characterization of the level convex envelope of a function. For this characterization, under slightly different assumptions, we refer to [17, Theorem 5.5].

Theorem 2.5. *Let $f : \mathbb{R}^n \to (-\infty, +\infty]$ be a function such that $f^{\text{lc}} > -\infty$. Then*

$$f^{\text{lc}}(\xi) = \inf\left\{\max_{1 \leq i \leq n+1} f(\xi_i) : \ \xi = \sum_{i=1}^{n+1} \lambda_i \xi_i, \ \lambda_i \geq 0, \ \sum_{i=1}^{n+1} \lambda_i = 1\right\}, \ \forall \ \xi \in \mathbb{R}^n.$$

Moreover, if f is continuous and $\lim_{|\xi| \to +\infty} f(\xi) = +\infty$, then the infimum above is indeed a minimum and f^{lc} is a continuous function. In particular, in this case, $f^{\text{lslc}} = f^{\text{lc}}$.

Proof. Let

$$h(\xi) := \inf\left\{\max_{1 \leq i \leq I} f(\xi_i) : \ \xi = \sum_{i=1}^{I} \lambda_i \xi_i, \ \lambda_i \geq 0, \ \sum_{i=1}^{I} \lambda_i = 1, \ I \in \mathbb{N}\right\}, \ \forall \ \xi \in \mathbb{R}^n.$$

We start showing that $f^{\text{lc}} = h$.

Observe that, since f^{lc} is level convex and $f^{\text{lc}} \leq f$, if $\xi = \sum_{i=1}^{I} \lambda_i \xi_i$ for some $\lambda_i \geq 0$ such that $\sum_{i=1}^{I} \lambda_i = 1$, then $f^{\text{lc}}(\xi) \leq \max_{1 \leq i \leq I} f(\xi_i)$. From this we conclude that $-\infty < f^{\text{lc}} \leq h$.

Now we prove that h is level convex. Once this is proved, we achieve the identity $f^{\text{lc}} = h$ by definition of f^{lc} and because $h \leq f$. Let $c \in \mathbb{R}$, we need to show that $L_c(h)$ is convex. Let $\xi, \eta \in L_c(h)$ and $\lambda \in (0, 1)$, we have to show that $h(\lambda\xi + (1-\lambda)\eta) \leq c$. Since $h > -\infty$ we just need to show that, given $\varepsilon > 0$, we can find $I \in \mathbb{N}$, $\lambda_i \geq 0$, with $\sum_{i=1}^{I} \lambda_i = 1$ and $z_i \in \mathbb{R}^n$ such that $\lambda\xi + (1-\lambda)\eta = \sum_{i=1}^{I} \lambda_i z_i$ and $\max_{1 \leq i \leq I} f(z_i) \leq c + \varepsilon$. This follows easily from the fact that $\xi, \eta \in L_c(h)$ and thus we have that $f^{\text{lc}} = h$.

Next we show that I can be reduced to $n+1$ achieving the first assertion of the theorem. This follows from Carathéodory's theorem. Indeed, let $\xi \in \mathbb{R}^n$ and assume $\xi = \sum_{i=1}^{I} \lambda_i \xi_i$ for some $I > n+1$, $\xi_i \in \mathbb{R}^n$ and $\lambda_i \geq 0$, with $\sum_{i=1}^{I} \lambda_i = 1$. In particular, $\xi \in \text{co}\{\xi_1, \xi_2, ..., \xi_I\} \subset \mathbb{R}^n$ and, by Carathéodory's theorem, we can write $\xi = \sum_{j=1}^{n+1} \mu_j \xi_{\gamma(j)}$ for some $\mu_j \geq 0$, with $\sum_{i=1}^{n+1} \mu_j = 1$ and $\gamma : \{1, 2, ..., n+1\} \to \{1, 2, ..., I\}$ an into function. Defining $\eta_j = \xi_{\gamma(j)}$ we obviously have $\max_{1 \leq j \leq n+1} f(\eta_j) \leq \max_{1 \leq i \leq I} f(\xi_i)$ which shows our goal.

Now we show the assertion of the theorem saying that the infimum is attained as a minimum. Let $\xi \in \mathbb{R}^n$ and let $\lambda_i^\nu \geq 0$, with $\sum_{i=1}^{n+1} \lambda_i^\nu = 1$, and $\xi_i^\nu \in \mathbb{R}^n$ be such that $\xi = \sum_{i=1}^{n+1} \lambda_i^\nu \xi_i^\nu$, $f^{\text{lc}}(\xi) = \lim_{\nu \to \infty} \max_{1 \leq i \leq n+1} f(\xi_i^\nu)$.

Without loss of generality we can assume $\max_{1\leq i\leq n+1} f(\xi_i^\nu) = f(\xi_1^\nu)$. By the assumption on the limit of f at infinity we can reduce to the case where the sequences ξ_i^ν are bounded otherwise $f^{\mathrm{lc}}(\xi) = +\infty$, thus $f(\xi) = +\infty$ and the minimum is attained through the trivial convex combination of ξ: $\xi = 1 \cdot \xi$. In the case where the sequences ξ_i^ν are bounded, we have, up to a subsequence, $\lim_{\nu\to\infty} \xi_i^\nu = \xi_i$ and $\lim_{\nu\to\infty} \lambda_i^\nu = \lambda_i$, for every $i = 1, \ldots, n+1$. Clearly $\sum_{i=1}^{n+1} \lambda_i = 1$ and $\sum_{i=1}^{n+1} \lambda_i \xi_i = \xi$. Using the continuity hypothesis on f, we get $f(\xi_1) = \lim_{\nu\to\infty} f(\xi_1^\nu) \geq \lim_{\nu\to\infty} f(\xi_i^\nu) = f(\xi_i)$ for every $i = 1, ..., n+1$ and $f^{\mathrm{lc}}(\xi) = f(\xi_1)$.

Finally we show that f^{lc} is continuous, under the continuity assumption on f and its behavior at infinity. Let $\xi, \xi_\nu \in \mathbb{R}^n$ be such that $\lim_{\nu\to\infty} \xi_\nu = \xi$. First we show that $f^{\mathrm{lc}}(\xi) \leq \liminf f^{\mathrm{lc}}(\xi_\nu)$. Without loss of generality assume $\liminf f^{\mathrm{lc}}(\xi_\nu) = \lim f^{\mathrm{lc}}(\xi_\nu)$. From what was already proved, we can consider, for each ν, $\lambda_i^\nu \geq 0$, with $\sum_{i=1}^{n+1} \lambda_i^\nu = 1$, and $\xi_i^\nu \in \mathbb{R}^n$ such that $\xi_\nu = \sum_{i=1}^{n+1} \lambda_i^\nu \xi_i^\nu$ and $f^{\mathrm{lc}}(\xi_\nu) = \max_{1\leq i\leq n+1} f(\xi_i^\nu)$. Re-ordering if necessary the elements ξ_i^ν, we can assume $\max_{1\leq i\leq n+1} f(\xi_i^\nu) = f(\xi_1^\nu)$ and thus $f^{\mathrm{lc}}(\xi_\nu) = f(\xi_1^\nu)$. We consider two cases. If, up to a subsequence, for some i, $\lim_{\nu\to\infty} |\xi_i^\nu| = +\infty$ then the desired inequality follows from the assumption that $\lim_{|z|\to\infty} f(z) = +\infty$. Otherwise we can write, up to a subsequence, that $\lim_{\nu\to\infty} \xi_i^\nu = \xi_i$ and $\lim_{\nu\to\infty} \lambda_i^\nu = \lambda_i$. Observe that, the continuity of f implies that $\max_{1\leq i\leq n+1} f(\xi_i) = f(\xi_1)$ and thus

$$\lim f^{\mathrm{lc}}(\xi_\nu) = \lim f(\xi_1^\nu) = f(\xi_1) = \max_{1\leq i\leq n+1} f(\xi_i) \geq f^{\mathrm{lc}}(\xi)$$

this last inequality following from the fact that $\xi = \sum_{i=1}^{n+1} \lambda_i \xi_i$ and the first assertion of the present theorem. To establish the continuity of f^{lc} on ξ, it remains to show that $f^{\mathrm{lc}}(\xi) \geq \limsup f^{\mathrm{lc}}(\xi_\nu)$. Again, let's assume $\limsup f^{\mathrm{lc}}(\xi_\nu) = \lim f^{\mathrm{lc}}(\xi_\nu)$. Then, as before, we have $\xi = \sum_{i=1}^{n+1} \lambda_i \xi_i$ for some $\lambda_i \geq 0$, with $\sum_{i=1}^{n+1} \lambda_i = 1$, $\xi_i \in \mathbb{R}^n$ and $f^{\mathrm{lc}}(\xi) = \max_{1\leq i\leq n+1} f(\xi_i)$. Defining $\xi_i^\nu = \xi_i + \xi_\nu - \xi$, we have $\xi_\nu = \sum_{i=1}^{n+1} \lambda_i \xi_i^\nu$ and thus $f^{\mathrm{lc}}(\xi_\nu) \leq \max_{1\leq i\leq n+1} f(\xi_i^\nu)$. Since $\lim_{\nu\to\infty} \xi_i^\nu = \xi_i$, the continuity of f implies that $\lim_{\nu\to\infty} f(\xi_i^\nu) = f(\xi_i)$ and thus

$$\lim f^{\mathrm{lc}}(\xi_\nu) \leq \limsup \max_{1\leq i\leq n+1} f(\xi_i^\nu) = \max_{1\leq i\leq n+1} f(\xi_i) = f^{\mathrm{lc}}(\xi),$$

as desired. □

In particular, we can easily get the following characterization of the convex hulls of the level sets of a function.

Corollary 2.6. *Let $f : \mathbb{R}^n \to (-\infty, +\infty]$ be a continuous function such that $f^{\mathrm{lc}} > -\infty$ and $\lim_{|\xi|\to+\infty} f(\xi) = +\infty$, then*

$$\operatorname{co}\{\xi \in \mathbb{R}^n : \ f(\xi) \leq c\} = \left\{\xi \in \mathbb{R}^n : \ f^{\mathrm{lc}}(\xi) \leq c\right\}, \ \forall\, c \in \mathbb{R}.$$

Remark 2.7. *We can get the same assertion of the corollary if we assume f lower semicontinuous, bounded from below and such that $\lim_{|\xi|\to\infty} \frac{f(\xi)}{|\xi|} = +\infty$. This is achieved if we use Theorem 2.8 below, instead of Theorem 2.5.*

In the same spirit of [31, Theorem 4.98] for the convex setting, we can also get the following result which provides in particular a characterization of f^{lslc}. The proof is omitted.

Theorem 2.8. *Let $f:\mathbb{R}^n \to (-\infty,+\infty]$ be a function bounded from below such that* $\lim\limits_{|\xi|\to\infty} \dfrac{f(\xi)}{|\xi|} = +\infty$.

(i) If f is lower semicontinuous, then the level convex envelope of f is still lower semicontinuous, that is f^{lc} is lower semicontinuous.

(ii) The following identities hold:

$$f^{\mathrm{lslc}}(\xi) = \mathrm{lsc}(f^{\mathrm{lc}})(\xi) = (\mathrm{lsc}f)^{\mathrm{lc}}(\xi) =$$
$$= \min\left\{ \max_{1\le i\le n+1} \mathrm{lsc}f(\xi_i) :\ \xi = \sum_{i=1}^{n+1} \lambda_i\xi_i,\ \lambda_i \ge 0,\ \sum_{i=1}^{n+1}\lambda_i = 1 \right\},\ \forall\, \xi\in\mathbb{R}^n.$$

Remark 2.9. *We observe that, in comparison with Theorem 2.5, there is no continuity hypothesis in Theorem 2.8, but the growth hypothesis at infinity is stronger than the one considered in Theorem 2.5.*

In analogy with [23, page 346], we can obtain another representation for f^{lslc}. Indeed, the proof below makes use of the results contained in Subsection 2.3, which, in turn, rely on the pyramidal construction proposed by Cellina, and which provides, in particular, a representation for the convex envelope Cf of f, through the formula $Cf(\xi_0) = \inf\left\{\int_\Omega f(\xi_0 + \nabla\varphi)\,dx : \varphi\in W_0^{1,\infty}(\Omega)\right\}$, at least when $f:\mathbb{R}^n\to\mathbb{R}$ is continuous and satisfies suitable coerciveness assumptions.

Proposition 2.10. *Let $\Omega\subset\mathbb{R}^n$ be a bounded open set with Lipschitz boundary and let $f:\mathbb{R}^n\to\mathbb{R}$ be a continuous function such that $f^{\mathrm{lc}} > -\infty$ and $\lim\limits_{|\xi|\to+\infty} f(\xi) = +\infty$. For every $\xi_0\in\mathbb{R}^n$, it results that*

$$f^{\mathrm{lslc}}(\xi_0) = \inf\left\{ \operatorname*{ess\,sup}_{x\in\Omega} f\left(\nabla u\left(x\right)\right) :\ u\in u_{\xi_0} + W_0^{1,\infty}(\Omega)\right\}. \tag{LC}$$

Remark 2.11. *One says that a function $f:\mathbb{R}^n\to\mathbb{R}$ is weak Morrey quasiconvex (see Definition 5.1 (ii)) if it coincides with the right hand side of* (LC) *when Ω is the unit cube Q. In particular, the above result entails that, at least in the scalar case, in the notion of weak Morrey quasiconvexity the set Q can be replaced by any bounded open set with Lipschitz boundary. Moreover, we observe that we don't expect the identity* (LC) *to provide a characterization to f^{lslc} in the vectorial case. Indeed, the notions of level convexity and weak Morrey quasiconvexity, in general, don't coincide and we don't expect them to coincide even under coercivity assumptions. On the other hand, the proof of* (LC) *relies on scalar arguments.*

Proof. By the level convexity of f^{lslc} and by Theorem 2.4, the inequality

$$f^{\mathrm{lslc}}(\xi_0) \le \inf\left\{ \operatorname*{ess\,sup}_{x\in\Omega} f\left(\nabla u\left(x\right)\right) :\ u\in u_{\xi_0} + W_0^{1,\infty}(\Omega)\right\}$$

is trivial. It remains to prove the opposite one. To this end, it will be enough, for every $\varepsilon > 0$ to determine $u\in u_{\xi_0} + W_0^{1,\infty}(\Omega)$ such that

$$f^{\mathrm{lslc}}(\xi_0) + \varepsilon \ge \operatorname*{ess\,sup}_{x\in\Omega} f(\nabla u(x)).$$

Equivalently it will be sufficient to determine $u\in u_{\xi_0} + W_0^{1,\infty}(\Omega)$ such that $\nabla u(x)\in L_{f^{\mathrm{lslc}}(\xi_0)+\varepsilon}(f)$ for a.e. $x\in\Omega$. Thus we are led to solve a differential inclusion. By virtue of Theorems 2.29 and 2.30, the inclusion will admit a solution if and only if

$$\xi_0 \in L_{f^{\mathrm{lslc}}(\xi_0)+\varepsilon}(f) \cup \mathrm{int}L_{f^{\mathrm{lslc}}(\xi_0)+\varepsilon}(f^{\mathrm{lslc}}),$$

where we have exploited Corollary 2.6. Therefore, it remains to prove that, if $\xi_0\notin \mathrm{int}L_{f^{\mathrm{lslc}}(\xi_0)+\varepsilon}(f^{\mathrm{lslc}})$, then $\xi_0\in L_{f^{\mathrm{lslc}}(\xi_0)+\varepsilon}(f)$. Clearly

$$\xi_0 \in L_{f^{\mathrm{lslc}}(\xi_0)+\varepsilon}(f^{\mathrm{lslc}}) = \{\eta : f^{\mathrm{lslc}}(\eta) \le f^{\mathrm{lslc}}(\xi_0)+\varepsilon\}.$$

Thus, if $\xi_0 \notin \mathrm{int} L_{f^{\mathrm{lslc}}(\xi_0)+\varepsilon}(f^{\mathrm{lslc}})$, then

$$\xi_0 \in \partial L_{f^{\mathrm{lslc}}(\xi_0)+\varepsilon}(f^{\mathrm{lslc}}) = \partial(\{\eta : f^{\mathrm{lslc}}(\eta) \leq f^{\mathrm{lslc}}(\xi_0)+\varepsilon\}) \subseteq \{\eta : f^{\mathrm{lslc}}(\eta) = f^{\mathrm{lslc}}(\xi_0)+\varepsilon\},$$

where the last inclusion follows from the continuity of f^{lslc} ensured by Theorem 2.5. This concludes the proof. □

The end of this subsection is devoted to recall other notions of envelopes, which have found their application in economics, in order to compare with the present definitions. In particular, we refer to [36], where it was firstly introduced this type of notions, and to [38, 42] where these concepts have been further investigated. We will adopt the same notations as in [17] and not those in [36, 38], because of the applications to L^∞- variational minimum problems (cf. Section 3).

For every p and $\xi \in \mathbb{R}^n$, we denote by $< p, \xi >$ the standard scalar product in $\mathbb{R}^n$. A function of the form $\min\{< p, \xi >, \gamma\} - b$ for some vector $p \in \mathbb{R}^n$ and constants $\gamma, b \in \mathbb{R}$ is called *level-affine*. It is easily seen that level-affine functions are level convex and lower semicontinuous.

Let $f : \mathbb{R}^n \to [-\infty, +\infty]$. The first conjugate of f, denoted by $f^* : \mathbb{R}^n \times \mathbb{R} \to [-\infty, +\infty]$ is defined by

$$f^*(p, \gamma) = \sup\{\min\{< p, \xi >, \gamma\} - f(\xi) : \xi \in \mathbb{R}^n\}.$$

The second conjugate of f, $f^{**} : \mathbb{R}^n \to [-\infty; +\infty]$ is defined by

$$f^{**}(\xi) = \sup\{\min\{< p, \xi >, \gamma\} - f^*(p, \gamma) : p \in \mathbb{R}^n, \gamma \in \mathbb{R}\}. \tag{2.2}$$

It is easily verified that f^{**} is level convex, lower semicontinuous and $f^{**} \leq f$. Consequently

$$f^{**}(\xi) \leq f^{\mathrm{lslc}}(\xi), \text{ for every } \xi \in \mathbb{R}^n. \tag{2.3}$$

The following result, which has been proved in [36, Corollary 4.4], provides a sufficient condition to achieve equality in (2.3).

Proposition 2.12. *If $f : \mathbb{R}^n \to [-\infty, \infty]$ is bounded from below, then*

$$f^{**}(\xi) = f^{\mathrm{lslc}}(\xi), \text{ for every } \xi \in \mathbb{R}^n. \tag{2.4}$$

2.2 Strict level convexity

Next we introduce the notion of *strict level convexity* that we will relate later with uniqueness of solutions to minimum problems.

Definition 2.13. *A level convex function $f : \mathbb{R}^n \to [-\infty; +\infty]$ is said to be strictly level convex if*

$$f(t\xi + (1-t)\eta) < \max\{f(\xi), f(\eta)\},$$

for every $t \in (0, 1)$ and every $\xi, \eta \in \mathbb{R}^n$, $\xi \neq \eta$.

The above definition can be given also if f is defined on a convex subset of $\mathbb{R}^n$.

Clearly Definition 2.13 is more stringent than level convexity. In the remaining part of this subsection we give a characterization of strict level convexity that will be exploited in the sequel. We also introduce weaker conditions than strict level convexity.

We begin recalling some definitions and some results.

(i) A set $C \subset \mathbb{R}^n$ is said to be strictly convex if for every $x, y \in \partial C$ with $x \neq y$, for every $z := tx + (1-t)y$, $t \in (0, 1)$, $z \in \mathrm{int}(C)$.

(ii) A point x in a convex set $C \subset \mathbb{R}^n$ is an extreme point of C if and only if there exists no points $y, z \in C$, both distinct from x such that $x = (1-t)y + tz$ for some $t \in (0, 1)$. The set of extreme points of C is called the profile of C and is denoted by $\mathrm{Ext}(C)$.

(iii) A point x in a convex set $C \subset \mathbb{R}^n$ is an exposed point of C if and only if there exists a supporting hyperplane H, such that $H \cap C = \{x\}$. The set of exposed points of a convex set C is denoted by $\mathrm{Exp}(C)$.

Remark 2.14. *(i) Of course* $\mathrm{Exp}(C) \subset \mathrm{Ext}(C)$ *for any convex set* $C \subset \mathbb{R}^n$*. We observe however that there may exist extreme points which are not exposed, even if, cf. [41, Theorem 18.6], the set* $\mathrm{Exp}(C)$ *is dense in* $\mathrm{Ext}(C)$*. Consider, for example, the set*

$$\left\{(x,y) \in \mathbb{R}^2 : \ x^2 + y^2 \leq 1\right\} \cup ([0,1] \times [-1,1]) .$$

(ii) When a convex set C is closed, strict convexity of C is equivalent to the condition that every boundary point of C is an extreme point of C.

Proposition 2.15. *Let* $C \subset \mathbb{R}^n$ *be a strictly convex set, then* $\mathrm{Ext}(C) = \mathrm{Exp}(C)$.

Remark 2.16. *The converse of Proposition 2.15 is false as one can easily see considering the set* $Q = [0,1]^2$ *in* $\mathbb{R}^2$*, where* $\mathrm{Ext}(Q) = \{(0,0),(0,1),(1,0),(1,1)\} = \mathrm{Exp}(Q)$ *and Q is not strictly convex.*

Proof. Let $x \in \mathrm{Ext}(C)$, and assume by contradiction that x is not exposed. Since x is not exposed, for every supporting hyperplane H at x, it results that there exists $y \neq x$, $y \in H \cap C$. Clearly $y \in \partial C$, and $ty + (1-t)x \in H \cap C \subset \partial C$, $t \in (0,1)$ (see for instance [41, Corollary 18.1.3]) and this contradicts the strict convexity of C. □

Let $f : \mathbb{R}^n \to [-\infty, +\infty]$ and $c \in [-\infty, +\infty]$. In the sequel we will make use of the following set

$$R_c(f) := \{x \in \mathbb{R}^n : f(x) = c\}. \tag{2.5}$$

The proof of the following result can be found in [30, Theorem 4.3].

Theorem 2.17. *Let f be a real valued function defined on a convex set C in* $\mathbb{R}^n$*. The function f is strictly level convex if and only if for every c in the range of f the following conditions are verified:*

(i) $L_c(f)$ *is convex,*

(ii) $R_c(f) \subseteq \mathrm{Ext}(L_c(f))$.

The previous characterization doesn't ensure the strict convexity of $L_c(f)$. To get this property we need to assume the continuity of f.

Proposition 2.18. *Let f be a real valued function defined on a convex set C in* $\mathbb{R}^n$*. If f is strictly level convex and continuous then for every c in the range of f,* $L_c(f)$ *is closed and strictly convex for every* $c \in \mathbb{R}$.

Proof. The closedness of $L_c(f)$ is a consequence of the lower semicontinuity of f. By the level convexity of f follows that $L_c(f)$ is convex. It remains to prove its strict convexity.

To this end, by Remark 2.14 (ii), it will be enough to show that $\partial L_c(f) \subseteq \mathrm{Ext}(L_c(f))$. By Theorem 2.17 it is known that $R_c(f) \subseteq \mathrm{Ext}(L_c(f))$. Thus it will suffice to prove that $\partial L_c(f) \subseteq R_c(f)$.

Let $y \in \partial L_c(f)$. Then, by the continuity of f, $f(y) \leq c$. Assume $y \notin R_c(f)$, that is $f(y) < c$. Again, the continuity of f would imply that $y \in \mathrm{int}(L_c(f))$ which is clearly a contradiction. That concludes the proof. □

Next we give several different characterizations of strict level convexity.

Proposition 2.19. *Let* $f : \mathbb{R}^n \to (-\infty, +\infty]$ *be a level convex function. Then f is strictly level convex if and only if one of the following conditions is satisfied.*

(i) $f(t\xi + (1-t)\eta) = \max\{f(\xi), f(\eta)\}$ *for some* $t \in (0,1)$ *implies* $\xi = \eta$.

(ii) $f\left(\xi + \frac{1}{2}\eta\right) = \max\left\{f(\xi), f(\xi + \eta)\right\}$ *implies* $\eta = 0$.

(iii) In the case f is a Borel measurable and finite function,

$$f\left(\int_\Omega \varphi\, d\mu\right) < \mu - \operatorname*{ess\,sup}_{x\in\Omega} f(\varphi(x)),$$

for every probability measure μ on $\mathbb{R}^n$ supported in the open set $\Omega \subset \mathbb{R}^d$ and every nonconstant $\varphi \in L^1_\mu(\Omega;\mathbb{R}^n)$.

Remark 2.20. *(i) In condition (ii), the value $1/2$ can be replaced by any $t \in (0,1)$.*

(ii) By Proposition 2.19 (iii), we observe that if a Borel measurable function f is strictly level convex then whenever

$$\operatorname*{ess\,sup}_{x\in\Omega} f(\xi_0 + \nabla\varphi(x)) = f(\xi_0), \tag{2.6}$$

for some $\varphi \in W^{1,\infty}_0(\Omega;\mathbb{R}^m)$, φ is necessarily 0. Notice that we include both the scalar and the vectorial case in this assertion.

Proof. The equivalence between strict level convexity, (i), and (ii) follows by standard arguments.

Concerning condition (iii), first we observe that if f satisfies (iii), then for every $t \in (0,1)$ we can take a function φ with value ξ on a set of μ-measure t and η ($\eta \neq \xi$) on a set of μ-measure $(1-t)$ to get $f(t\xi + (1-t)\eta) < \max\{f(\xi), f(\eta)\}$, and showing the strict level convexity of f.

To prove the converse, we argue by contradiction. Assume f is strictly level convex and there exists a nonconstant function $\varphi \in L^1_\mu(\Omega;\mathbb{R}^n)$ such that

$$f\left(\int_\Omega \varphi\, d\mu\right) = \mu - \operatorname*{ess\,sup}_{x\in\Omega} f(\varphi(x)) = c.$$

If $K = \{\zeta : f(\zeta) \leq c\}$, i.e. $K = L_c(f)$, the level convexity of f guarantees that K is convex and in particular $f(\varphi(x)) \leq c$ for μ-a.e. $x \in \Omega$, i.e. $\varphi(x) \in K$ for μ-a.e. $x \in \Omega$. Observe that by Theorem 2.17, since $\int_\Omega \varphi\, d\mu \in R_c(f)$, $\int_\Omega \varphi\, d\mu \in \mathrm{Ext} L_c(f)$.

Now if $E = \{x : \varphi(x) \neq \int_\Omega \varphi\, d\mu\}$, since φ is assumed nonconstant, $\mu(E) > 0$. Moreover, observe that $\frac{1}{\mu(E)}\int_E \varphi\, d\mu = \int_\Omega \varphi\, d\mu$. Clearly for a.e. $x \in E : \varphi(x) \in K' := L_c(f) \setminus \{\int_\Omega \varphi\, d\mu\}$, which is still a convex set (because it is a convex without an extreme point of it, cf. Theorem 2.17). Since K' is convex and $\varphi(x) \in K'$ for a.e. $x \in E$, it results that $\frac{1}{\mu(E)}\int_E \varphi\, d\mu \in K'$, and this is obviously a contradiction. □

In the remainder of this section we will investigate weaker conditions than strict level convexity.

Definition 2.21. *A level convex function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be strictly level convex at $\xi_0 \in \mathbb{R}^n$ if for every $t \in (0,1)$ and for every $\xi \neq \eta$: $\xi_0 = t\xi + (1-t)\eta \Longrightarrow f(\xi_0) < \max\{f(\xi), f(\eta)\}$.*

With the following result, we introduce a stronger notion than the one in Definition 2.21.

Proposition 2.22. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a level convex function and let $\xi_0 \in \mathbb{R}^n$. Assume that*

$$\text{for every } t \in (0,1) \text{ and for every } \xi \neq \xi_0,\; f(t\xi_0 + (1-t)\xi) < \max\{f(\xi_0), f(\xi)\}. \tag{2.7}$$

Then f is strictly level convex at ξ_0.

Remark 2.23. *The reverse implication of Proposition 2.22 is not true, to this end it is enough to consider the function*

$$\xi \in \mathbb{R} \to f(\xi) := \begin{cases} -\xi & \text{if } \xi \leq 0, \\ 0 & \text{if } \xi > 0, \end{cases}$$

which is strictly level convex at 0, but it doesn't satisfy condition (2.7) with $\xi_0 = 0$.

Proof. Let $\xi_0 = t\xi + (1-t)\eta$ for $t \in (0,1)$ and $\xi, \eta \in \mathbb{R}^n$. Let θ and ζ be in the segments (ξ, ξ_0) and (ξ_0, η) respectively. By (2.7) $f(\theta) < \max\{f(\xi_0), f(\xi)\}$ and $f(\zeta) < \max\{f(\xi_0), f(\eta)\}$. Observe that $\xi_0 \in (\theta, \zeta)$. From the level convexity of f and the previous inequalities it results

$$f(\xi_0) \leq \max\{f(\theta), f(\zeta)\} < \max\{f(\xi), f(\eta), f(\xi_0)\}.$$

Therefore $\max\{f(\xi), f(\eta), f(\xi_0)\} = \max\{f(\xi), f(\eta)\}$ and thus

$$f(\xi_0) < \max\{f(\eta), f(\xi)\}$$

which concludes the proof. □

Proposition 2.24. *A level convex function $f : \mathbb{R}^n \to \mathbb{R}$ is strictly level convex at ξ_0 if and only if $\xi_0 \in \mathrm{Ext}L_{f(\xi_0)}(f)$.*

Proof. Arguing by contradiction, assume that f is strictly level convex at ξ_0 but $\xi_0 \notin \mathrm{Ext}L_{f(\xi_0)}(f)$, namely there exist ξ and $\eta \in L_{f(\xi_0)}(f)$ such that $\xi_0 = t\xi + (1-t)\eta$, $t \in (0,1)$ and $\xi \neq \eta$, then $\max\{f(\xi), f(\eta)\} > f(\xi_0)$, and this contradicts the fact that $\xi, \eta \in L_{f(\xi_0)}(f)$.

Now we want to prove that if $\xi_0 \in \mathrm{Ext}L_{f(\xi_0)}(f)$, then f is strict level convex at ξ_0. If this was not the case, there would exist ξ, $\eta \in \mathbb{R}^n$, with $\xi, \eta \neq \xi_0$ and $t \in (0,1)$ such that $\xi_0 = t\xi + (1-t)\eta$ and $f(\xi_0) = \max\{f(\xi), f(\eta)\}$ and so $\xi, \eta \in L_{f(\xi_0)}(f)$, which is a contradiction. □

We finish this section with a notion, weaker than the strict level convexity at a point either in the sense of Definition 2.21 or in the sense of (2.7). This will be useful to deal with the minimizing problems in Section 4.

Definition 2.25. *A level convex function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be strictly level convex at $\xi_0 \in \mathbb{R}^n$ in at least one direction if there exists $\alpha \in \mathbb{R}^n \setminus \{0\}$ such that: if for some γ and $\eta \in \mathbb{R}^n$*

$$\begin{cases} \xi_0 = t\gamma + (1-t)\eta, \ t \in (0,1) \\ f(\xi_0) = \max\{f(\gamma), f(\eta)\} \end{cases}$$

then

$$< \gamma - \eta, \alpha >= 0.$$

Remark 2.26. *One could also give a definition of strict level convexity in at least one direction in the spirit of (2.7). Precisely given $\xi_0 \in \mathbb{R}^n$ there exists $\alpha \in \mathbb{R}^n \setminus \{0\}$ such that: if for some $\eta \in \mathbb{R}^n$*

$$\begin{cases} \xi = t\xi_0 + (1-t)\eta, \ t \in (0,1) \\ f(\xi) = \max\{f(\xi_0), f(\eta)\} \end{cases} \tag{2.8}$$

then

$$< \xi_0 - \eta, \alpha >= 0.$$

An argument very similar to that employed to prove Proposition 2.22 and the fact that ξ_0, γ and η of Definition 2.25 are in the same line, guarantee that if f satisfies (2.8) at ξ_0 then it is also strictly level convex at ξ_0 in at least one direction.

On the other hand, the opposite implication is false, that is, there are strictly level convex functions at ξ_0 in at least one direction but not in the sense of (2.8). To this end consider the function $f(\xi) = \mathrm{dist}(\xi, \mathbb{R}^+ \times \mathbb{R})$ defined for $\xi \in \mathbb{R}^2$ and take $\xi_0 = (0,0)$. Note also that, this function, although being strictly level convex at ξ_0 in at least one direction, it is not strictly level convex at ξ_0.

We have the following characterization of strict level convexity at a point in at least one direction.

Proposition 2.27. *A lower semicontinuous and level convex function $f : \mathbb{R}^n \to \mathbb{R}$ is strictly level convex at ξ_0 in at least one direction if and only if $\xi_0 \in \partial L_{f(\xi_0)}(f)$.*

Remark 2.28. *Actually, the lower semicontinuity hypothesis is only required to show that $\xi_0 \in \partial L_{f(\xi_0)}(f)$ is a sufficient condition for strict level convexity of f at ξ_0 in at least one direction.*

Proof. Assume f is strictly level convex at ξ_0 in at least one direction. We will show that $\xi_0 \in \partial L_{f(\xi_0)}(f)$. Of course $\xi_0 \in L_{f(\xi_0)}(f)$. Assume, by contradiction, that $\xi_0 \in \mathrm{int}L_{f(\xi_0)}(f)$. Then $f(\xi) \leq f(\xi_0)$ in a neighborhood of ξ_0. Let α be the direction given by Definition 2.25 and let η_1, η_2 be of the form $\eta_i = \xi_0 + \varepsilon_i \alpha$ for some $\varepsilon_i \in \mathbb{R}$ such that η_i belong to the neighborhood referred above. Then, by the level convexity of f, $f(\xi_0) = \max\{f(\eta_1), f(\eta_2)\}$. But $< \eta_1 - \eta_2, \alpha >\neq 0$ which contradicts the hypothesis.

Next we show the opposite implication. Assume that $\xi_0 \in \partial L_{f(\xi_0)}(f)$. The lower semicontinuity of f ensures that $L_{f(\xi_0)}(f)$ is closed. Notice that, since $\xi_0 \in \partial L_{f(\xi_0)}(f)$, $L_{f(\xi_0)}(f) \neq \mathbb{R}^n$. Moreover, since $L_{f(\xi_0)}(f)$ is also convex, there is $\alpha \in \mathbb{R}^n \setminus \{0\}$, such that $< \alpha, \xi_0 > \geq < \alpha, \xi >$ for all $\xi \in L_{f(\xi_0)}(f)$. We will show that f is strictly level convex at ξ_0 in the direction α. Assume $\xi_0 = \lambda\xi + (1-\lambda)\eta$ for some $\lambda \in (0,1)$ and $\eta \neq \xi$ such that $\eta - \xi$ is collinear with α. Then at least one of ξ and η is not in the set $L_{f(\xi_0)}(f)$, let's say it is η. Then $f(\eta) > f(\xi_0)$ and thus $f(\xi_0) < \max\{f(\eta), f(\xi)\}$, showing that f is strictly level convex at ξ_0 in at least the direction α. □

2.3 Differential inclusions

We recall that $W_0^{1,\infty}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{1,1}(\Omega)$ intersected with $W^{1,\infty}(\Omega)$.

In the sequel we recall two classical results stating necessary and sufficient conditions for existence of solutions to differential inclusions for scalar valued functions. The results are due to Cellina [22], [23], Friesecke [32]. See also Bandyopadhyay-Barroso-Dacorogna-Matias [11]. We observe that $\operatorname{int}\operatorname{co} E$ stands for the interior of the convex hull of the set E and we refer respectively to [25, Theorem 10.24] and [26, Theorem 2.10] for the proofs.

Theorem 2.29. *Let $\Omega \subset R^n$ be a bounded open set, $E \subset \mathbb{R}^n$, $\xi_0 \in \mathbb{R}^n$ and denote by u_{ξ_0} an affine function such that $\nabla u_{\xi_0} = \xi_0$. If $u \in u_{\xi_0} + W_0^{1,\infty}(\Omega)$ is such that*

$$\nabla u(x) \in E, \text{ a.e. } x \in \Omega,$$

then

$$\xi_0 \in E \cup \operatorname{int}\operatorname{co} E.$$

Theorem 2.30. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and $E \subset \mathbb{R}^n$. Let $\varphi \in W^{1,\infty}(\Omega)$ satisfying*

$$\nabla\varphi(x) \in E \cup \operatorname{int}\operatorname{co} E, \text{ a.e. } x \in \Omega.$$

Then there exists $u \in \varphi + W_0^{1,\infty}(\Omega)$ such that

$$\nabla u(x) \in E, \text{ a.e. } x \in \Omega.$$

Moreover, given $\varepsilon > 0$, u can be chosen such that $||u - \varphi||_{L^\infty(\Omega)} \leq \varepsilon$.

Remark 2.31. *The last assertion of the previous theorem follows from a more careful pyramidal construction than the one present in [26, Theorem 2.10]. Namely, in the proof of Lemma 2.11 in [26], the L^∞-norm of the pyramids $v_{x_0}^r$ shall be controlled. This can be done considering in Step 4 of [26, Lemma 2.11] the family*

$$\mathcal{G} := \{G(x_0, r) : \ x_0 \in \Omega, \ 0 < r < \varepsilon\}.$$

Moreover, in the proof of [26, Theorem 2.10], one shall also invoke the L^∞ estimate of the norm shown in [26, Corollary 10.11].

3 Relaxation Theorem

Consider the following two minimum problems

$$(P) \qquad \inf\left\{\operatorname*{ess\,sup}_{x\in\Omega} f\left(\nabla u\left(x\right)\right) : \ u \in u_0 + W_0^{1,\infty}(\Omega)\right\} \tag{3.1}$$

and

$$(P^{\mathrm{lc}}) \qquad \inf\left\{\operatorname*{ess\,sup}_{x\in\Omega} f^{\mathrm{lslc}}\left(\nabla u\left(x\right)\right) : \ u \in u_0 + W_0^{1,\infty}(\Omega)\right\}, \tag{3.2}$$

where f is given, f^{lslc} is the lower semicontinuous and level convex envelope of f, introduced in (iii) of Definition 2.1, and $u_0 \in W^{1,\infty}(\Omega)$ is the boundary data. Notice that, if u_0 is only defined on the boundary of Ω, $u_0 : \partial\Omega \longrightarrow \mathbb{R}$, and it is a Lipschitz function, we can extend it to all Ω and get a $W^{1,\infty}$ function. In this case, and in view of Theorem 4.2 below, the existence of solutions to problem (P) can be ensured depending on the choice of the extension.

The goal of this section is to show that

$$\inf(P) = \inf(P^{\mathrm{lc}}).$$

This will result as a consequence of the relaxation Theorem 3.3 below.

Before that we recall that level convexity, together with lower semicontinuity is a necessary and sufficient condition for sequential weak* lower semicontinuity in $W^{1,\infty}(\Omega)$ for functionals in the supremal form. The sufficient part is due to Barron-Jensen-Wang cf. [15, Theorem 3.3].

Theorem 3.1. *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a level convex and lower semicontinuous function and let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Then the functional $F(u) = \operatorname*{ess\,sup}_{x\in\Omega} f(\nabla u(x))$ defined in $W^{1,\infty}(\Omega)$ is sequential weak* lower semicontinuous.*

The necessary condition follows from [13, Theorem 3.5] (see also [1, Theorem 4.1]).

Theorem 3.2. *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a Borel function and let $\Omega \subset \mathbb{R}^n$ be a bounded open set. If the functional $F(u) = \operatorname*{ess\,sup}_{x\in\Omega} f(\nabla u(x))$ defined in $W^{1,\infty}(\Omega)$ is sequentially weak* lower semicontinuous, then f is lower semicontinuous and level convex.*

Now we establish the relaxation theorem.

Theorem 3.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary and let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function satisfying*

$$f(\xi) \geq \gamma(|\xi|),\ \forall\ \xi \in \mathbb{R}^n, \tag{3.3}$$

with $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ a continuous and increasing function such that $\lim_{t\to+\infty} \gamma(t) = +\infty$. Let $u_0 \in W^{1,\infty}(\Omega)$. Define the functional $F : u \in u_0 + W_0^{1,\infty}(\Omega) \to \operatorname*{ess\,sup}_{x\in\Omega} f(\nabla u(x))$, and let $\overline{F}$ be the relaxed functional of F with respect to the weak convergence in $W^{1,\infty}(\Omega)$, namely for every $u \in u_0 + W_0^{1,\infty}(\Omega)$,*

$$\overline{F}(u) = \inf\left\{\liminf_{h\to+\infty} F(u_h) : u_h \in u_0 + W_0^{1,\infty}(\Omega), u_h \rightharpoonup u \text{ weakly* in } W^{1,\infty}(\Omega)\right\}.$$

Then

$$\overline{F}(u) = \operatorname*{ess\,sup}_{x\in\Omega} f^{\mathrm{lslc}}(\nabla u(x)),\ \forall\ u \in u_0 + W_0^{1,\infty}(\Omega).$$

Remark 3.4. *A result similar to Theorem 3.3 was proved by Prinari [40, Theorem 2.6], with no boundary condition. Our arguments are very similar so we omit them.*

In the integral context, this type of results can be proved directly thanks to piecewise affine approximation arguments. In the supremal setting, sets of arbitrarily small measure are important to the value of the functional and this kind of argument is not well suited. To show the relaxation Theorem 3.3 we need to pass through the sequential weak lower semicontinuous envelope of the functional involved in problem (P), and we extended first this functional to $C(\overline{\Omega})$, as $+\infty$ in the complement of $u_0 + W_0^{1,\infty}(\Omega)$. For the reader's convenience we recall that a result devoted to this extension can be found in [39, Proposition 3.1], see also [29, Theorem 8.10 and Corollary 8.12]. We also observe that our setting entails that the sequentially weak* lower semicontinuous envelope of F coincides with the lower semicontinuous envelope of the extended functional in $C(\overline{\Omega})$, with respect to the uniform tolopology.*

Moreover we emphasize that, by virtue of (2.4), Theorem 3.3 can be rephrased saying that

$$\overline{F}(u) = \operatorname*{ess\,sup}_{x\in\Omega} f^{**}(\nabla u(x)),\ \forall\ u \in u_0 + W_0^{1,\infty}(\Omega), \tag{3.4}$$

where f^{**} is the function introduced in (2.2).

In the sequel we proceed in analogy with the definitions introduced at the end of Subsection 2.1, namely, given $u_0 \in W^{1,\infty}(\Omega)$, we will consider, for every $p \in (L^1(\Omega))^{n+1}$, $\gamma, b \in \mathbb{R}$, the *level-affine* mappings:

$$u \in u_0 + W_0^{1,\infty}(\Omega) \to \min\{< p, u >, \gamma\} - b,$$

where $< \cdot, \cdot >$ denotes the standard duality product between $W^{1,\infty}(\Omega)$ and $(L^1(\Omega))^{n+1}$, where $u \in W^{1,\infty}(\Omega)$ is identified with $(u, \nabla u) \in (L^\infty(\Omega))^{n+1}$. It is easily seen that level-affine functionals are level convex and weak* lower semicontinuous.

Next, for $F : u \in u_0 + W_0^{1,\infty}(\Omega) \to \operatorname*{ess\,sup}_{x\in\Omega} f(\nabla u(x))$, we consider, the functionals (cf. [17]) $F^* : (L^1(\Omega))^{n+1} \times \mathbb{R} \to [-\infty, +\infty]$, defined as

$$F^*(p, \gamma) = \sup\{\min\{< p, u >, \gamma\} - F(u) : u \in u_0 + W_0^{1,\infty}(\Omega)\}.$$

and $F^{**} : u \in u_0 + W_0^{1,\infty}(\Omega) \to [-\infty, +\infty]$, given by

$$F^{**}(u) = \sup\{\min\{< p, u >, \gamma\} - F^*(p, \gamma) : p \in (L^1(\Omega))^{n+1}, \gamma \in \mathbb{R}\}. \tag{3.5}$$

Moreover, in the same spirit of Definition 2.1 (iii), replacing the natural topology in $\mathbb{R}^n$ with the weak* one in $W^{1,\infty}(\Omega)$, we define

$$F^{\rm lslc}(u) = \sup\Big\{G(u) : G : u_0 + W_0^{1,\infty}(\Omega) \to [-\infty, +\infty],\ G \text{ weakly* lower semicontinuous}, \\ G \text{ level convex},\ G \le F\Big\}. \tag{3.6}$$

It has to be pointed out that F^{**} is level convex, weak* lower semicontinuous, $F^{**} \le F$, and it coincides with the so-called Γ-regularization of F, as proved in [17, Theorem 3.6], namely

$$F^{**}(u) = \sup\{G(u) : G \text{ is level-affine and } G(u) \le F(u)\},$$

thus, in particular, it results that $F^{**} \le F^{\rm lslc}$.

The following result provides a sufficient condition in order to have $F^{**} = \overline{F}$, in particular, it guarantees a supremal representation for F^{**}.

Proposition 3.5. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary and let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function satisfying* (3.3)*. Let $u_0 \in W^{1,\infty}(\Omega)$, let $F : u \in u_0 + W_0^{1,\infty}(\Omega) \to \operatorname*{ess\,sup}_{x\in\Omega} f(\nabla u(x))$, and let $\overline{F}$ be its relaxed functional with respect to the weak* convergence in $W^{1,\infty}(\Omega)$. Let F^{**} be the functional defined in* (3.5)*, then*

$$F^{**}(u) = \overline{F}(u), \forall\ u \in u_0 + W_0^{1,\infty}(\Omega), \tag{3.7}$$

and

$$F^{**}(u) = \operatorname*{ess\,sup}_{x\in\Omega} f^{**}(\nabla u(x)),\ \forall\ u \in u_0 + W_0^{1,\infty}(\Omega), \tag{3.8}$$

where f^{**} is the function defined in (2.2)*.*

Proof. The proof of (3.7) is obtained by double inequality. First we observe that it has been shown in [40] that $\overline{F}$ is level convex, more precisely Theorem 2.7 (pages 50-51) therein states that $\overline{F} = F^{\rm lslc}$, where the latter is the functional introduced in (3.6). Therefore, since, as observed above, $F^{**} \le F^{\rm lslc}$, one has

$$F^{**}(u) \le \overline{F}(u),\ \forall\ u \in u_0 + W_0^{1,\infty}(\Omega).$$

The opposite inequality follows from [17, Corollary 3.5 and Theorem 3.6], which, in turn, provide $F^{\rm lslc} \le F^{**}$, thus we obtain the identity $F^{**} = F^{\rm lslc} = \overline{F}$.

Finally (3.8) follows from (3.7) and (3.4). □

Now we achieve the desired condition

$$\inf(P) = \inf(P^{\rm lc}).$$

Corollary 3.6. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary and let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function satisfying condition* (3.3)*. Let $u_0 \in W^{1,\infty}(\Omega)$ and let (P) and $(P^{\rm lc})$ be the problems (3.1) and (3.2), respectively. Then*

$$\inf(P) = \inf(P^{\rm lc}).$$

Moreover, if the boundary condition is affine, say $u_0(x) = u_{\xi_0}(x)$ for some $\xi_0 \in \mathbb{R}^n$, then

$$\inf(P) = f^{\rm lslc}(\xi_0).$$

Remark 3.7. *From this result we recover the characterization* (LC)*, stated in Proposition 2.10, for the lower semicontinuous and level convex envelope of f, namely that, for every $\xi_0 \in \mathbb{R}^n$,*

$$f^{\rm lslc}(\xi_0) = \inf \left\{ \operatorname*{ess\,sup}_{x\in\Omega} f\left(\nabla u\left(x\right)\right) : \ u \in u_{\xi_0} + W_0^{1,\infty}(\Omega) \right\}.$$

Proof. We start proving the first equality. Clearly $\inf(P) \geq \inf(P^{\rm lc})$. To prove the converse inequality, let $\overline{F}$ be the functional introduced in Theorem 3.3 and observe that the same theorem entails that $\overline{F}(u) = \operatorname*{ess\,sup}_{\Omega} f^{\rm lslc}(\nabla u)$. Therefore, by the direct method of the calculus of variations, the infimum appearing in $(P^{\rm lc})$ is indeed a minimum, since (3.3) entails the required coercivity condition and since $\overline{F}$ is sequential weak* lower semicontinuous. Thus we can write

$$\min(P^{\rm lc}) = \overline{F}(\overline{u}),$$

for some $\overline{u} \in u_0 + W_0^{1,\infty}(\Omega)$.

By definition of $\overline{F}$, we also have that there exists a sequence $\{u_n\} \subset u_0 + W_0^{1,\infty}(\Omega)$, such that

$$\overline{F}(\overline{u}) = \liminf_{n\to+\infty} F(u_n) \geq \inf(P),$$

and that proves the first equality in the claim.

To prove the last assertion of the corollary we only need to observe that, if the boundary condition is affine then the supremal Jensen's inequality in Theorem 2.4 guarantees that u_{ξ_0} is a solution to problem $(P^{\rm lc})$. □

4 Existence of Minimizers

This section is devoted first to investigate necessary and sufficient conditions for existence of solutions to the non-level convex problem (P) introduced in Section 3, see equation (3.1). Then, in Subsection 4.2, we start exploring conditions to ensure existence and uniqueness of absolute minimizers for level convex and non-level convex problems.

4.1 Necessary and Sufficient Conditions

In order to analyze sufficient and necessary conditions for minimizers for supremal functionals with non level convex supremand, we start with some considerations on level convex problems.

Observe that, if f is a level convex function, then the solutions u to the related problem (P) are completely characterized by the following condition

$$\begin{cases} u \in u_0 + W_0^{1,\infty}(\Omega) \\ f(\nabla u(x)) \leq \inf(P), \ \text{a.e. } x \in \Omega. \end{cases} \tag{4.1}$$

This characterization shall be compared with Theorem 1 in [22]. In the present supremal context and for affine boundary condition $u_0 = u_{\xi_0}$, the analogous result we should obtain is that for any solution u to problem (P), one has $\nabla u(x) \in F$, *a.e.* $x \in \Omega$ if F is a face of the convex set $L_{f(\xi_0)}(f)$ containing ξ_0 in its relative interior. Since the relative interiors of the faces are disjoint, cf. [41, Theorem 18.2], Proposition 4.1 below, shows that $L_{f(\xi_0)}(f)$ is the only possible face containing ξ_0 in its relative interior.

Therefore the analogous result to [22, Theorem 1] doesn't give more information than what was stated in condition (4.1).

The next Proposition 4.1, concerns uniqueness of solution to a level convex problem.

Proposition 4.1. *Let* $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ *be a Borel measurable level convex function and let* (P) *be the related problem defined in (3.1) with an affine boundary condition* $u_0 = u_{\xi_0}$, $\xi_0 \in \mathbb{R}^n$.

If (P) *admits a solution* $u \neq u_{\xi_0}$, *then* $\xi_0 \in \operatorname{int} L_{f(\xi_0)}(f)$.

Proof. Since f is a level convex function, by the supremal Jensen's inequality $\inf(P) = f(\xi_0)$. Therefore, if u is a solution to (P), $u \neq u_{\xi_0}$, then, by (4.1), $f(\nabla u(x)) \leq f(\xi_0)$, $\forall\, x \in \Omega \setminus A$, where A is a null measure subset of Ω. Thus

$$S := \{\nabla u(x) :\ x \in \Omega \setminus A\} \subset \{\xi :\ f(\xi) \leq f(\xi_0)\},$$

this last set being convex. On the other hand, since

$$\xi_0 = \frac{1}{|\Omega|} \int_\Omega \nabla u(x)\, dx,$$

we get $\xi_0 \in \overline{\mathrm{co}S}$. Repeating the argument of Theorem 10.24 in [25], we get $\xi_0 \in \operatorname{int} \overline{\mathrm{co}S} = \operatorname{int} \mathrm{co}S \subset \operatorname{int}\{\xi :\ f(\xi) \leq f(\xi_0)\}$. □

Now, we pass to the case of non-level convex problems. In the same spirit as before, we have the following result. We observe that, in this case, we need to assume the continuity of the function in order to apply the relaxation theorem of the previous section.

Theorem 4.2. *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded open set with Lipschitz boundary and let* $f : \mathbb{R}^n \to \mathbb{R}$ *be a continuous function satisfying (3.3). Let* $u_0 \in W^{1,\infty}(\Omega)$ *and let* (P) *and* (P^{lc}) *be the problems (3.1) and (3.2), respectively.*

Then problem (P) *has a solution if and only if there exists* $u \in u_0 + W_0^{1,\infty}(\Omega)$ *such that*

$$f(\nabla u\,(x)) \leq \inf(P^{\mathrm{lc}}),\ \ a.e.\ x \in \Omega. \tag{4.2}$$

Moreover, if

$$\nabla u_0(x) \in L_{\inf(P^{\mathrm{lc}})}(f) \cup \operatorname{int} L_{\inf(P^{\mathrm{lc}})}(f^{\mathrm{lslc}}),\ \ a.e.\ x \in \Omega \tag{4.3}$$

then (P) *has a solution.*

In particular, if u_0 *is affine, say* $u_0 = u_{\xi_0}$ *with* $\nabla u_{\xi_0}(x) = \xi_0 \in \mathbb{R}^n$, *then condition* (4.2) *reads*

$$f(\nabla u\,(x)) \leq f^{\mathrm{lslc}}(\xi_0),\ \ a.e.\ x \in \Omega. \tag{4.4}$$

Moreover, still under the assumption that $u_0 = u_{\xi_0}$ *is affine, problem (P) admits a solution if and only if*

$$\xi_0 \in L_{f^{\mathrm{lslc}}(\xi_0)}(f) \cup \operatorname{int} L_{f^{\mathrm{lslc}}(\xi_0)}(f^{\mathrm{lslc}}). \tag{4.5}$$

Remark 4.3. *(i) By the supremal Jensen's inequality, the existence of* $u \in u_0 + W_0^{1,\infty}(\Omega)$ *satisfying* (4.4) *is equivalent to the existence of* $u \in u_0 + W_0^{1,\infty}(\Omega)$ *such that*

$$\operatorname*{ess\,sup}_{x\in\Omega} f\,(\nabla u\,(x)) = f^{\mathrm{lslc}}(\xi_0).$$

(ii) Observe that, for general u_0, (4.3) *is only a sufficient condition, while, for affine functions* u_0 *it is necessary and sufficient (cf.* (4.5)*).*

(iii) If u_0 *is a Lipschitz function only defined on the boundary of* Ω, $u_0 : \partial\Omega \longrightarrow \mathbb{R}$, *then condition* (4.3) *can be replaced by condition (2.62) in [26, Theorem 2.17] to get a sufficient condition. Notice that, from* (4.2), *we need to solve the differential inclusion* $\nabla u \in E$ *where* $E = \{\xi \in \mathbb{R}^n :\ f(\xi) \leq \inf(P^{\mathrm{lc}})\}$.

Proof. We start proving the first equivalence stated in the theorem. Let $u \in u_0 + W_0^{1,\infty}(\Omega)$ be a solution to problem (P). Then, by Corollary 3.6, it results that

$$\operatorname*{ess\,sup}_{x\in\Omega} f\left(\nabla u\left(x\right)\right) = \inf(P^{\mathrm{lc}}),$$

hence $f(\nabla u(x)) \leq \inf(P^{\mathrm{lc}})$ for a.e. $x \in \Omega$.

To prove the reverse implication, it is enough to observe that

$$\inf(P^{\mathrm{lc}}) \leq \inf(P).$$

Then, that condition (4.3) is sufficient for existence of solutions to (P), follows from (4.2), Theorem 2.30, and from Corollary 2.6, where the set E is given by $L_{\inf(P^{\mathrm{lc}})}(f)$.

Regarding condition (4.4), it suffices to observe that, by the supremal Jensen's inequality, cf. Theorem 2.4, $f^{\mathrm{lslc}}(\xi_0) = \inf(P^{\mathrm{lc}})$.

For what concerns the last statement, condition (4.5), we observe that it follows from Theorems 2.29 and 2.30 and from Corollary 2.6, where the set E is given by $L_{f^{\mathrm{lslc}}(\xi_0)}(f)$. □

As we will see in the next result, in dimension $n = 1$, for sufficiently regular f, problem (P) with an affine boundary condition, always admits a solution. This is not true for $n > 1$ as Example 4.7 shows. The following result shall be compared with [12, Theorem 31].

Corollary 4.4. *Let $\Omega \subset \mathbb{R}$ be a bounded open set and let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying* (3.3)*. Then problem (P) introduced in* (3.1) *admits a solution for every $u_0 : \partial\Omega \longrightarrow \mathbb{R}$.*

Proof. Let $u_0 : \partial\Omega \longrightarrow \mathbb{R}$. Since $\Omega \subset \mathbb{R}$, we can write $\Omega = \cup_{i\in\mathbb{N}}\Omega_i$ with Ω_i disjoint open intervals. In each of these intervals consider the affine functions u_{ξ_i}, for some $\xi_i \in \mathbb{R}$, such that $u_{\xi_i} = u_0$ on $\partial\Omega_i$. Once proved the existence of solution to each problem

$$(P_i) \quad \inf\left\{\operatorname*{ess\,sup}_{x\in\Omega_i} f\left(u'(x)\right) : \ u \in u_{\xi_i} + W_0^{1,\infty}(\Omega_i)\right\},$$

say $u_i \in u_{\xi_i} + W_0^{1,\infty}(\Omega_i)$, one gets the existence of solution to (P) patching together the functions u_i in each interval Ω_i.

It remains to prove the existence of solution to each problem (P_i). To achieve this it will be enough to show that every $\xi_i \in \mathbb{R}$ verifies (4.5) with ξ_0 replaced by ξ_i. Of course, $\xi_i \in L_{f^{\mathrm{lslc}}(\xi_i)}(f^{\mathrm{lslc}}) = \operatorname{co} L_{f^{\mathrm{lslc}}(\xi_i)}(f)$, by Corollary 2.6. If $\xi_i \in L_{f^{\mathrm{lslc}}(\xi_i)}(f)$, the existence is proved. So, without loss of generality, we may assume that $\xi_i \in \operatorname{co} L_{f^{\mathrm{lslc}}(\xi_i)}(f) \setminus L_{f^{\mathrm{lslc}}(\xi_i)}(f)$. Since we are working on the real line, we immediately conclude that $\xi_i \in \operatorname{int} L_{f^{\mathrm{lslc}}(\xi_i)}(f^{\mathrm{lslc}}) = \operatorname{int}\operatorname{co} L_{f^{\mathrm{lslc}}(\xi_i)}(f)$: in $\mathbb{R}$, the elements of the boundary of a convex hull belong either to the original set or to the complement of the convex hull. Thus we proved (4.5) and the proof is finished. □

The following result provides a sufficient condition for existence of solutions to a non-level convex problem with more general boundary data.

Theorem 4.5. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary and let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function satisfying (3.3). Let $u_0 \in W^{1,\infty}(\Omega)$ and let (P) and (P^{lc}) be the problems (3.1) and (3.2), respectively. Assume that problem (P^{lc}) admits a solution $\overline{u} \in C^1_{piec}(\Omega)$.*

Then, if f^{lslc} is constant in each connected component of the set where $f^{\mathrm{lslc}} < f$, problem (P) has a solution.

Remark 4.6. *The same assertion can be proved under the weaker assumption that f^{lslc} is constant in the connected components of $\{\xi : \ f^{\mathrm{lslc}}(\xi) < f(\xi)\}$ whose intersection with $\{\nabla\overline{u}(x) : \ x \in \Omega'\}$ is nonempty for some $\Omega' \subset \Omega$ with positive measure.*

Proof. By the continuity hypothesis on f and the coercivity condition (3.3), Theorem 2.5 implies that f^{lslc} is also continuous and thus the set

$$A := \left\{\xi \in \mathbb{R}^n : \ f^{\text{lslc}}(\xi) < f(\xi)\right\}$$

is open. Therefore we can split A in a disjoint countable union of open sets A_i, $i \in \mathbb{N}$. By hypothesis, in each of these sets A_i, the function f^{lslc} is constant. Since the function $\xi \mapsto \gamma(|\xi|)$ is level convex and continuous, $f^{\text{lslc}} \geq \gamma(|\cdot|)$. Thus $\lim\limits_{|\xi|\to+\infty} f^{\text{lslc}}(\xi) = +\infty$ and this, together with the fact that f^{lslc} is constant in A_i implies that the sets A_i are bounded.

Now, we split the set Ω in several parts:

$$\Omega_0 := \{x \in \Omega : \ \nabla\overline{u}(x) \notin \cup_{i\in\mathbb{N}} A_i\}$$
$$\Omega_i := \{x \in \Omega : \ \nabla\overline{u}(x) \in A_i\}, \ i \in \mathbb{N}$$

and we construct the solution in the following way. For each $i \in \mathbb{N}$ consider $v_i \in \overline{u} + W_0^{1,\infty}(\Omega_i)$ such that $\nabla v_i \in \partial A_i$, this exists by Theorem 2.30 and since A_i being open and bounded, is contained in $\operatorname{int}\operatorname{co}\partial A_i$.

Define

$$u(x) := \begin{cases} \overline{u}(x), \text{ if } x \in \Omega_0, \\ v_i(x), \text{ if } x \in \Omega_i, \ i \in \mathbb{N}. \end{cases}$$

One has $\overline{u} \in u_0 + W_0^{1,\infty}(\Omega)$. Moreover, since f^{lslc} is constant on each A_i, by the continuity of f^{lslc}, it is constant on $\overline{A_i}$. On the other hand, since A_i are open, on their boundary, f^{lslc} coincides with f. Hence, since $\nabla v_i \in \partial A_i$,

$$\operatorname*{ess\,sup}_{x\in\Omega_i} f\left(\nabla v_i\left(x\right)\right) = \operatorname*{ess\,sup}_{x\in\Omega_i} f^{\text{lslc}}\left(\nabla v_i\left(x\right)\right) = \operatorname*{ess\,sup}_{x\in\Omega_i} f^{\text{lslc}}\left(\nabla \overline{u}\left(x\right)\right)$$

where we have used the fact that $\nabla\overline{u} \in A_i$ in Ω_i.

Consequently

$$\begin{aligned} \operatorname*{ess\,sup}_{x\in\Omega} f\left(\nabla u\left(x\right)\right) &= \max\left\{\operatorname*{ess\,sup}_{x\in\Omega_0} f\left(\nabla\overline{u}\left(x\right)\right), \ \operatorname*{ess\,sup}_{x\in\Omega_i} f\left(\nabla v_i\left(x\right)\right), \ i \in \mathbb{N}\right\} \\ &= \operatorname*{ess\,sup}_{x\in\Omega} f^{\text{lslc}}\left(\nabla\overline{u}\left(x\right)\right) = \inf(P^{\text{lc}}) \end{aligned}$$

and the existence of solution to problem (P) follows from Theorem 4.2. ☐

The following example shows many of the features of the results stated in this section.

Example 4.7. *Let $f : (\xi_1, \xi_2) \in \mathbb{R}^2 \to (\xi_1^2 - 1)^2 + \xi_2^2 \in \mathbb{R}$. Clearly $f \geq 0$, and $f(\xi_1, \xi_2) = 0$ if and only if $(\xi_1, \xi_2) = \pm(1, 0)$. Since $f \geq 0$ then $f^{\text{lslc}} \geq 0$, and thus, using Corollary 2.6,*

$$\left\{f^{\text{lslc}} = 0\right\} = L_0(f^{\text{lslc}}) = \operatorname{co} L_0(f) = [-1, 1] \times \{0\},$$

which has empty interior.

This example also shows that in dimension $n > 1$ there are cases where the gradient of the boundary datum doesn't belong either to $L_{f^{\text{lslc}}(\xi_0)}(f)$ or to $\operatorname{int} L_{f^{\text{lslc}}(\xi_0)}(f^{\text{lslc}})$ and thus the inclusion (4.5) doesn't admit any solution. By Theorem 4.2, if $u_0 = u_{\xi_0}$ is such that $\nabla u_0(x) \equiv \xi_0 \in (-1, 1) \times \{0\}$, considering the minimizing problem (P) related to the function f with the boundary data $u_0 = u_{\xi_0}$, we can ensure that (P) doesn't have a solution.

We also observe that, in this example, problem (P^{lc}) with affine boundary condition $u_0 = u_{\xi_0}$, such that $\nabla u_0(x) \equiv \xi_0 \in (-1, 1) \times \{0\}$ has exactly one solution. Indeed, if $u \in u_0 + W_0^{1,\infty}(\Omega)$ is a solution to (P^{lc}), then $f^{\text{lslc}}(\nabla u(x)) = 0$, a.e. $x \in \Omega$ and thus $\nabla u(x) \in [-1, 1] \times \{0\}$, a.e. $x \in \Omega$. In particular, $\nabla u(x) - \xi_0$ is orthogonal to the vector $(0, 1)$, a.e. $x \in \Omega$. It then follows by [25, Lemma 11.17], that $u \equiv u_{\xi_0}$, showing that u_{ξ_0} is the only solution to (P^{lc}).

As we will see in Proposition 4.8, the fact that (P^{lc}) admits a unique solution, and the boundary condition is such that $\xi_0 \in \{f^{\text{lslc}} < f\}$ implies that (P) has no solution.

We will now turn our attention to necessary conditions for existence of solutions to a non-level convex problem of the form (P) with affine boundary data. Our strategy will be based on uniqueness of solutions to the relaxed problem (P^{lc}), as considered for problems in the integral form by Marcellini [35], Dacorogna-Marcellini [27], and Dacorogna-Pisante-Ribeiro [28]. The basis to our research is the following result.

Proposition 4.8. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary and let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function satisfying (3.3). Let (P) and (P^{lc}) be the problems (3.1) and (3.2), respectively, with the affine boundary condition $u_{\xi_0}(x) = \xi_0 \cdot x + c$. Assume that $f^{\mathrm{lslc}}(\xi_0) < f(\xi_0)$ and assume f^{lslc} satisfies the condition*

$$\left.\begin{array}{c} \operatorname*{ess\,sup}_{x\in\Omega} f^{\mathrm{lslc}}\left(\nabla u\left(x\right)\right) = f^{\mathrm{lslc}}(\xi_0) \\ u \in u_{\xi_0} + W_0^{1,\infty}(\Omega) \end{array}\right\} \Rightarrow \;\; u = u_{\xi_0},$$

which means that problem (P^{lc}) has a unique solution.

Then problem (P) has no solution.

Proof. Since $f^{\mathrm{lslc}} \leq f$, if u is a solution to (P) then, by Corollary 3.6, it is also a solution to (P^{lc}). Therefore, by the uniqueness of solutions to (P^{lc}) stated in the hypothesis, $u = u_{\xi_0}$. This contradicts the fact that $f^{\mathrm{lslc}}(\xi_0) < f(\xi_0)$ and we conclude that a solution to (P) cannot exist. □

In view of the result just stated, we want to find conditions ensuring uniqueness of solution to level convex problems of type (P), defined in (3.1). We start by observing that the strict level convexity of the function related to the minimizing problem (P) provides that uniqueness, if the boundary condition is affine. Indeed this follows from Remark 2.20 and applies to both the scalar and the vectorial cases. It is interesting to observe that condition (2.6), involved in Proposition 4.8 is the counterpart in the supremal setting of the notion of strict quasiconvexity at ξ_0 which guarantees uniqueness of solutions in the integral framework (see Definition 11.9 and Theorem 11.11 in [25].) Therefore, if f^{lslc} was strictly level convex then, for affine boundary conditions, problem (P^{lc}), defined in (3.1), would have a unique solution and problem (P), defined in (3.2), would have no solution. However, we show in the next proposition that we can't expect $f^{\mathrm{lslc}} \neq f$ to be strictly level convex and thus we will work with a weaker notion.

Proposition 4.9. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function such that $f^{\mathrm{lc}} > -\infty$ and $\lim_{|\xi|\to+\infty} f(\xi) = +\infty$ and let $\xi_0 \in \mathbb{R}^n$ be such that $f^{\mathrm{lslc}}(\xi_0) < f(\xi_0)$. Then f^{lslc} is constant in a segment line containing ξ_0 (possibly ξ_0 is an extremity of the segment line).*

Remark 4.10. *The same assertion can be proved if we assume f to be lower semicontinuous, bounded from below, and such that $\lim_{|\xi|\to+\infty} \dfrac{f(\xi)}{|\xi|} = +\infty$. The proof in this case is the same that we present below, but one shall invoke Theorem 2.8 (ii), instead of Theorem 2.5.*

Proof. By Theorem 2.5, $f^{\mathrm{lslc}}(\xi_0) = \max\{f(\xi_1), \ldots, f(\xi_{n+1})\}$ for some $\xi_i \in \mathbb{R}^n,\ i = 1, \ldots, n+1$, such that $\xi_0 = \lambda_1\xi_1 + \cdots + \lambda_{n+1}\xi_{n+1}$ with $\lambda_i \geq 0,\ i = 1, \ldots, n+1$ and $\sum_{i=1}^{n+1} \lambda_i = 1$. Moreover, we can assume $\lambda_i > 0$ for every $i = 1, \ldots n+1$ (notice that some ξ_i can be equal) and since $f^{\mathrm{lslc}}(\xi_0) < f(\xi_0)$, we conclude that ξ_0 belongs to the relative interior of the convex hull $\mathrm{co}\{\xi_1, \ldots, \xi_{n+1}\}$. Therefore we can consider a segment line $[\eta, \zeta]$ contained in this relative interior such that $\xi_0 \in (\eta, \zeta)$. By the level convexity of f^{lslc},

$$f^{\mathrm{lslc}}(\xi) \leq \max\{f^{\mathrm{lslc}}(\xi_1), \ldots, f^{\mathrm{lslc}}(\xi_{n+1})\} \leq \max\{f(\xi_1), \ldots, f(\xi_{n+1})\} = f^{\mathrm{lslc}}(\xi_0),\ \forall\ \xi \in [\eta, \zeta].$$

Finally, again by the level convexity of f^{lslc}, one has $f^{\mathrm{lslc}} \equiv f^{\mathrm{lslc}}(\xi_0)$ either in $[\eta, \xi_0]$ or in $[\xi_0, \zeta]$, as desired. □

In the spirit of [27], we will deal with a weaker notion of strict level convexity, the strict level convexity in at least one direction, which was introduced in Section 2.2. We have the following result.

Theorem 4.11. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, let $\xi_0 \in \mathbb{R}^n$, and let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a Borel measurable and level convex function which is strictly level convex at ξ_0 in at least one direction. Then, u_{ξ_0} is the only solution to the problem*

$$\inf \left\{ \operatorname*{ess\,sup}_{x\in\Omega} f\left(\nabla u\left(x\right)\right) : \; u \in u_{\xi_0} + W_0^{1,\infty}(\Omega) \right\}. \tag{4.6}$$

Proof. Of course, by the supremal Jensen's inequality, u_{ξ_0} is a solution of the minimizing problem and thus another solution u satisfies $\operatorname*{ess\,sup}_{x\in\Omega} f\left(\nabla u\left(x\right)\right) = f(\xi_0)$. Let us fix a representative of u still denoted by u.

One has, in particular, $f(\nabla u(x)) \leq f(\xi_0)$, $\forall\, x \in \Omega \setminus A$, where A is a null measure subset of Ω. Thus

$$S := \{\nabla u(x) : \; x \in \Omega \setminus A\} \subset \{\xi : \; f(\xi) \leq f(\xi_0)\}.$$

On the other hand, as in the proof of Proposition 4.1 we get $\xi_0 \in \operatorname{int} \overline{\operatorname{co} S} = \operatorname{int} \operatorname{co} S$.

Now let $x \in \Omega \setminus A$. Then either $\nabla u(x) = \xi_0$ or $\nabla u(x) \neq \xi_0$. If x is in this last case we do the following. Since $\xi_0 \in \operatorname{int} \operatorname{co} S$, we can write

$$\xi_0 = t\, \nabla u(x) + (1-t)\eta$$

for some $\eta \in \operatorname{co} S$ with $\eta \neq \xi_0$ and $t \in (0,1)$. By the level convexity of f

$$f(\xi_0) \leq \max\{f(\nabla u(x)), f(\eta)\} \leq f(\xi_0).$$

Thus, $\max\{f(\nabla u(x)), f(\eta)\} = f(\xi_0)$ and by the strict level convexity of f at ξ_0 in at least one direction one gets $< \nabla u(x) - \eta, \gamma >= 0$ for some $\gamma \in \mathbb{R}^n \setminus \{0\}$. Since $\nabla u(x) - \eta$ and $\nabla u(x) - \xi_0$ are colinear, then $< \nabla u(x) - \xi_0, \gamma >= 0$.

So we have obtained $< \nabla u(x) - \xi_0, \gamma >= 0$, $\forall\, x \in \Omega \setminus A$ and repeating the argument in [27, Theorem 5.1] one gets $u = u_{\xi_0}$ which proves the desired uniqueness of solution. □

Remark 4.12. *Another proof for Theorem 4.11 can be obtained via geometric arguments on the level sets. Namely Proposition 2.27 ensures that $\xi_0 \in \partial L_{f(\xi_0)}(f)$, and thus by Proposition 4.1,* (4.6) *admits just the affine solution.*

It is worth to observe also that Theorem 4.11 provides a result analogous to [25, Proposition 11.14] (where the notion of strict quasiconvexity has been introduced and compared with strict convexity in at least one direction in order to guarantee uniqueness of solutions to integral vectorial minimum problems). In fact one can read (2.6) *as a strict weak Morrey quasiconvexity, and deduce that this condition is weaker than strict convexity at a point in at least one direction.*

Going back to the non-level convex problems we can state the following result which shows that, the strict level convexity at ξ_0 in at least one direction for f^{lslc}, with $f^{\mathrm{lslc}}(\xi_0) < f(\xi_0)$, is the characterizing feature for non existence of solutions to problem (P).

Corollary 4.13. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary, let $\xi_0 \in \mathbb{R}^n$, and let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a continuous function satisfying* (3.3). *Assume that $f^{\mathrm{lslc}}(\xi_0) < f(\xi_0)$. Consider problem (P) with $u_0 = u_{\xi_0}$. Then problem (P) admits a solution if and only if f^{lslc} is not strictly level convex at ξ_0 in any direction.*

Remark 4.14. *Under the assumptions of the corollary, we can state more precisely that, if f^{lslc} is strictly level convex at ξ_0 in at least one direction, then problem (P^{lc}) has exactly one solution and problem (P) has no solution. This follows from Theorem 4.11 and Proposition 4.8, as mentioned in the proof below. If f is also lower semicontinuous, Propositions 4.1 and 2.27 ensure that strict level convexity at ξ_0 in at least one direction is equivalent to strict weak Morrey quasiconvexity at ξ_0.*

Proof. The fact that if (P) has a solution then f^{lslc} is not strictly level convex at ξ_0 in any direction is an immediate consequence of Theorem 4.11 and Proposition 4.8.

Now we prove the reverse implication. Assume f^{lslc} is not strictly level convex at ξ_0 in any direction. Then, by Propostion 2.27, $\xi_0 \notin \partial L_{f^{\mathrm{lslc}}(\xi_0)}(f^{\mathrm{lslc}})$, being an interior point of this set. Theorem 4.2, (4.5) ensures then that (P) has a solution. □

Our previous results lead to the following theorem, which intends to characterize the set $R_{f^{\rm lslc}(\xi_0)}(f^{\rm lslc})$ (cf. (2.5)) near the point ξ_0, for non level convex problems admitting a minimizer and with affine boundary data $u_0 = u_{\xi_0}$. The result is the analogous version to [25, Theorem 11.26] for the supremal setting.

Theorem 4.15. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary, let $\xi_0 \in \mathbb{R}^n$, and let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a continuous function satisfying* (3.3)*. Assume that $f^{\rm lslc}(\xi_0) < f(\xi_0)$. Let $K := \{\xi \in \mathbb{R}^n : f^{\rm lslc}(\xi) < f(\xi)\}$ and assume that K is connected, otherwise we replace K by its connected component containing ξ_0. Consider problem (P) with $u_0 = u_{\xi_0}$.*

(i) [Necessary condition.] If (P) has a minimizer, then there exists $\nu \in \mathbb{R}^n \setminus \{0\}$ and $\varepsilon > 0$ such that $f^{\rm lslc}$ is constant in the set $\{\xi \in B_\varepsilon(\xi_0) : < \xi - \xi_0, \nu > \geq 0\} \subset R_{f^{\rm lslc}(\xi_0)}(f^{\rm lslc})$.

(ii) [Sufficient condition.] If there exists $E \subset \partial K$ such that $\xi_0 \in {\rm int\,co}(E)$ and $f^{\rm lslc}\big|_{\{\xi_0\}\cup E}$ is constant then (P) has a solution.

Proof. To prove the necessary part we start observing that, by Corollary 4.13, if (P) admits a minimizer, then $f^{\rm lslc}$ is not strictly level convex at ξ_0 in any direction. Then, by Proposition 2.27, $\xi_0 \in {\rm int} L_{f^{\rm lslc}(\xi_0)}(f^{\rm lslc})$. Let $\varepsilon > 0$ be such that $B_\varepsilon(\xi_0) \subset L_{f^{\rm lslc}(\xi_0)}(f^{\rm lslc})$ and consider, for each $n \in \mathbb{N}$, the convex sets

$$C_n := \left\{ \xi \in B_\varepsilon(\xi_0) : \ f^{\rm lslc}(\xi) \leq f^{\rm lslc}(\xi_0) - \frac{1}{n} \right\}.$$

Let

$$C := \bigcup_{n\in\mathbb{N}} C_n = \left\{ \xi \in B_\varepsilon(\xi_0) : \ f^{\rm lslc}(\xi) < f^{\rm lslc}(\xi_0) \right\}.$$

Observe that C, being an increasing sequence of convex sets, it is also a convex set. Moreover, by Theorem 2.5, $f^{\rm lslc}$ is continuous and thus C is open. If C is empty, it means that $f^{\rm lslc}$ is constant in $B_\varepsilon(\xi_0)$. Otherwise, applying a separation result for the convex open set C, one gets the existence of $\nu \in \mathbb{R}^n \setminus \{0\}$ such that $< \xi_0 - \xi, \nu >< 0$ for all $\xi \in C$. Therefore, for all $\xi \in B_\varepsilon(\xi_0)$ such that $< \xi_0 - \xi, \nu > \geq 0$ one has $f^{\rm lslc}(\xi) = f^{\rm lslc}(\xi_0)$, as wished.

The sufficient part is proved observing that, by Theorem 2.30, there exists $\overline{u} \in u_{\xi_0} + W_0^{1,\infty}(\Omega)$ such that $\nabla\overline{u} \in E \subset \partial K$ for a.e. $x \in \Omega$. Since $f^{\rm lslc} = f$ on ∂K, we have $f(\nabla\overline{u}(x)) = f^{\rm lslc}(\nabla\overline{u}(x))$ for a.e. $x \in \Omega$ and since $f^{\rm lslc}$ is constant in $\{\xi_0\} \cup E$, we have

$$\operatorname*{ess\,sup}_{x\in\Omega} f(\nabla\overline{u}(x)) = \operatorname*{ess\,sup}_{x\in\Omega} f^{\rm lslc}(\nabla\overline{u}(x)) = f^{\rm lslc}(\xi_0),$$

which, by Theorem 4.2, ensures that (P) has a solution. □

Remark 4.16. *We observe that (as very well emphasized by Crandall [24] and Aronsson-Crandall-Juutinen in [8]) that given $u_0 \in W^{1,\infty}(\Omega)$, and $f : \mathbb{R}^n \to \mathbb{R}$ continuous and strict level convex, then the functional*

$$F : u \in C(\overline{\Omega}) \to \begin{cases} \operatorname*{ess\,sup}_{x\in\Omega} f(\nabla u(x)) & \text{if } u \in u_0 + W_0^{1,\infty}(\Omega), \\ \\ +\infty & \text{otherwise,} \end{cases}$$

is in general not strictly level convex as a functional, i.e. it does not satisfy $F(u) < \max\{F(u_1), F(u_2)\}$, for every $u = \lambda u_1 + (1-\lambda)u_2$, $u_1, u_2 \in u_0 + W_0^{1,\infty}(\Omega)$, $\lambda \in (0,1)$, even when $n = 1$. This is in fact the case of the minimum problem arising when looking for the minimal Lipschitz extension, where in fact the density f defined as $f(\cdot) := |\cdot|$ is strictly level convex, but the minimizer is not unique. On the other hand, our previous results (cf. in particular Remark 2.20, Proposition 4.8 and Theorem 4.11) show (also in the vectorial case) that if u_0 is affine, namely $u_0 := u_{\xi_0}$ then we have a unique solution to the problem $\inf\left\{ \operatorname*{ess\,sup}_{x\in\Omega} f(\nabla u(x)) : u \in u_{\xi_0} + W_0^{1,\infty}(\Omega) \right\}$.

We also observe that the continuity and strict level convexity of a function $f : \mathbb{R}^n \to \mathbb{R}$, satisfying (3.3)*, is sufficient to ensure the uniqueness of solution to the minimization problem*

$$\inf\left\{ \operatorname*{ess\,sup}_{x\in\Omega} f(u(x)) : \ u \in L^\infty(\Omega; \mathbb{R}^n) \right\}.$$

Moreover, this unique solution is a constant function. Indeed, the coercivity condition and the continuity of f implies the existence of a global minimum to f and the strict level convexity ensures that this minimum is attained in only one point, say $\xi_0 \in \mathbb{R}^n$. Therefore $u(x) \equiv \xi_0$ is a solution to the minimization problem and it is obviously the only solution, since any other solution v satisfies $\operatorname{ess\,sup}_{x\in\Omega} f(v(x)) = f(\xi_0)$ *and this implies that $v(x) = \xi_0$ for a.e. $x \in \Omega$.*

4.2 Some remarks about absolute minimizers

Absolute minimizers are of great importance due to their relationships with Aronsson-Euler equation, cf. [6, 33, 15, 16]. The aim of this subsection is to analyze the immediate consequences of our results for the existence of absolute minimizers for level convex and non level convex problems. Regarding this, several questions are raised that require a deeper treatment and that we leave for future research.

We start by recalling the following definition. (As in [16] we require $u \in W^{1,\infty}(\Omega)$, but in general other definitions are possibile, i.e. $u \in C(\Omega)$ is what Crandall assumes for absolute minimizers in [24, Proposition 6.1]. Also observe that if Ω has regular boundary $W^{1,\infty}(\Omega) \subset C(\overline{\Omega})$.)

Definition 4.17. *A function $u^* \in W^{1,\infty}(\Omega)$ is an absolute minimizer of $F(u,\Omega) := \operatorname{ess\,sup}_{x\in\Omega} f(\nabla u(x))$, if, for any open set $\Omega' \subset \overline{\Omega'} \subset \Omega$, u^* is a solution to*

$$\inf \left\{ \operatorname*{ess\,sup}_{x\in\Omega'} f\left(\nabla u(x)\right) : \ u \in u^* + W_0^{1,\infty}(\Omega') \right\}.$$

It is worth to observe that such a property is automatically inherited by minimizers in the integral setting.

4.2.1 Minimizers and absolute minimizers for level convex and strictly level convex problems.

It is natural to look for conditions on the supremand f which ensure existence and/or uniqueness of absolute minimizers for functionals $F(u,\Omega) := \operatorname{ess\,sup}_{x\in\Omega} f(\nabla u(x))$.

For what concerns the existence of absolute minimizers to level convex problems, we start by observing that in Definition 4.17 no boundary condition on $\partial\Omega$ is imposed. Therefore, it easily follows from Theorem 2.4 that, if f is level convex then, for every $\xi_0 \in \mathbb{R}^n$, u_{ξ_0} is an absolute minimizer. If one considers in problem (P) an affine boundary condition on $\partial\Omega$, combining our previous results, one still gets the following.

(i) If f is level convex and (P) has u_{ξ_0} as boundary condition, then u_{ξ_0} is a solution to (P) which is an absolute minimizer.

(ii) If f is strictly level convex at ξ_0 in at least one direction and (P) has u_{ξ_0} as boundary condition, then u_{ξ_0} is an absolute minimizer and u_{ξ_0} is the only solution to (P). The uniqueness of the solution to (P) is a consequence of Theorem 4.11. On the other hand, there is no uniqueness of absolute minimizers since, as it has been observed above, all the other affine functions are absolute minimizers.

It is also interesting to understand if strict level convexity of f guarantees that minimizers to (P) are also absolute minimizers. In general this is not true, as the following example shows.

Example 4.18. *(Lipschitz extension problem.) Let Ω be a convex set with regular boundary, let $u_0 \in C(\partial\Omega)$ be a Lipschitz function, and consider the problem of finding a minimizer in $C(\overline{\Omega})$ for the functional*

$$\mathcal{F}_\infty(u,\Omega) := \operatorname*{ess\,sup}_{x\in\Omega} |\nabla u(x)|$$

among the functions $u \in C(\overline{\Omega})$ whose boundary datum coincides with u_0.

Observe that the density of $\mathcal{F}_\infty$ is the Euclidean norm $|\cdot|$, which is strictly level convex. Moreover, the convexity of Ω guarantees that the problem of finding a minimum to $\mathcal{F}_\infty$ is equivalent to find a Lipschitz extension of u_0 to all Ω with the least Lipschitz constant

$$\mathrm{Lip}(u,\Omega) := \inf\{L \in \mathbb{R} : |u(x) - u(y)| \leq L|x-y|, \forall\, x, y \in \Omega\}.$$

Therefore, for our purposes, it is enough to observe that u_0 admits a Lipschitz extension with the least Lipschitz constant which is not an absolute minimizer to $\mathcal{F}_\infty$.

In general (see [24, Example 2.1]), given a Lipschitz function $u_0 \in C(\partial\Omega)$, there may exist infinitely many solutions to

$$\begin{cases} u \in C(\overline{\Omega}),\ u = u_0 \text{ on } \partial\Omega, \\ \mathrm{Lip}(u,\overline{\Omega}) = \min\{\mathrm{Lip}(v,\overline{\Omega}) : v \in C(\overline{\Omega}), v = u_0 \text{ on } \partial\Omega\}, \end{cases}$$

that in addition satisfy $\mathrm{Lip}(u,\overline{\Omega}) = \mathrm{Lip}(u_0,\partial\Omega)$. So, actually, the Lipschitz constant of u_0 is preserved.

Finally, we recall that for this problem, as proved by Jensen [33, Corollary 3.14], there is uniqueness of absolute minimizers satisfying the boundary condition, concluding that a minimizer is not necessarily an absolute minimizer.

Now we address the converse question, that is whether an absolute minimizer is also a minimizer. Since the definition of absolute minimizer does not prescribe the boundary value on $\partial\Omega$, from what observed at the beginning of this subsection, i.e in (i) and (ii), there might be a unique minimizer but more than an absolute one, even if the density is $|\cdot|$, in particular even if f is strictly level convex, one can have absolute minimizers which are not minimizers.

On the other hand if one requires the absolute minimizers to satisfy also an affine boundary condition, i.e. $u = u_{\xi_0}$ on $\partial\Omega$, then if f is strictly level convex at ξ_0 in at least one direction, there is a unique affine absolute minimizer, wich is of course u_{ξ_0}. If f is strictly level convex at ξ_0 in at least one direction and $u = u_{\xi_0}$ on $\partial\Omega$ is not affine, then it is not a minimizer, but we do not know if there might exist non-affine absolute minimizers, which verify the boundary condition.

4.2.2 Absolute minimizers for non level convex problems

Finally we focus on the existence of absolute minimizers to non-level convex problems, i.e. to

$$F(u,\Omega) := \operatorname*{ess\,sup}_{x\in\Omega} f(\nabla u(x)),$$

when f is not level convex.

By Theorem 4.2 we know that if $u_0 = u_{\xi_0}$ and $f^{\mathrm{lslc}}(\xi_0) = f(\xi_0)$, then u_{ξ_0} is a solution to (P) , when the affine boundary datum u_{ξ_0} has been imposed on $\partial\Omega$, and it is also an absolute minimizer, arguing as in the beginning of Subsection 4.2.1. Clearly we cannot ensure that this is either the only minimizer or the only absolute minimizer to the above problem, even when a boundary condition is imposed.

Furthermore we also stress that, since Definition 4.17 does not prescribe a boundary condition on $\partial\Omega$, the requirement $f(\xi_0) = f^{\mathrm{lslc}}(\xi_0)$ is enough to ensure that, for any choice of the set Ω, u_{ξ_0} is an absolute minimizer to $F(u,\Omega) := \operatorname{ess\,sup}_{x\in\Omega} f(\nabla u(x))$.

Clearly, (4.5) may ensure that problem (P) admits a solution even if $f(\xi_0) \neq f^{\mathrm{lslc}}(\xi_0)$, indeed it is enough to require that $\xi_0 \in \mathrm{int}(L_{f^{\mathrm{lslc}}(\xi_0)}(f^{\mathrm{lslc}}))$. However, in general one cannot deduce that the minimizer u, given by Theorem 4.2 is an absolute minimizer. Observe that, if u is a minimizer to (P) (either with an affine or non-affine boundary condition), then, by Theorem 4.2, for every open set $\Omega' \subset \overline{\Omega'} \subset \Omega$, if $(P_{\Omega'})$ is the problem

$$(P_{\Omega'}) \qquad \inf\left\{\operatorname*{ess\,sup}_{x\in\Omega'} f(\nabla v(x)) : v = u \text{ on } \partial\Omega'\right\},$$

then u solves $(P_{\Omega'})$ if and only if $f(\nabla u(x)) \leq \inf(P^{\mathrm{lc}}_{\Omega'})$, where $(P^{\mathrm{lc}}_{\Omega'})$ stands for the relaxed problem of $(P_{\Omega'})$.

Next we give an example in which one can construct minimizers to some problem (P) which are also absolute minimizers. This example includes the case where an affine boundary condition u_{ξ_0} is considered in (P) with $f^{\mathrm{lslc}}(\xi_0) < f(\xi_0)$, but it is not restricted to Dirichlet problems with affine boundary conditions. The main feature of this example is that it is possible to construct functions whose gradients take values in the set of minimum points of the supremand f and it provides a new paradigm for further research.

Example 4.19. *Let $f : \xi \in \mathbb{R}^2 \to (|\xi|^2 - 1)^2 \in \mathbb{R}$. It is easy to see that*

$$f^{\mathrm{lslc}}(\xi) = \begin{cases} 0, & \text{if } |\xi| \leq 1, \\ (|\xi|^2 - 1)^2, & \text{if } |\xi| > 1. \end{cases}$$

If we consider problem (P) with a boundary datum u_0 satisfying $|\nabla u_0(x)| < 1$ a.e. $x \in \Omega$ (in particular we can take $u_0 = u_{\xi_0}$ with $\xi_0 = 0$ that we observe satisfies $f^{\mathrm{lslc}}(\xi_0) < f(\xi_0)$) then, by Theorem 2.30, we can construct (infinitely many) solutions to (P), by constructing functions $u \in u_0 + W_0^{1,\infty}(\Omega)$ such that $|\nabla u(x)| = 1$ a.e. $x \in \Omega$. It is also clear that, according to Definition 4.17, all these solutions are absolute minimizers.

5 Appendix

In this section we make some considerations concerning the vectorial case, that is when $f : \mathbb{R}^{m\times n} \to \mathbb{R}$. We also correct some statements by Barron-Jensen-Wang in [15]. In the vectorial case the necessary and sufficient condition for sequential weak* lower semicontinuity of the supremal functional is the so called (strong) Morrey quasiconvexity as proved by Barron-Jensen-Wang [15, Theorems 2.6 and 2.7]. We start recalling this notion together with other notions also introduced in [15, Definitions 2.1, 2.2 and 3.7]. We denote by Q the unitary cube of $\mathbb{R}^n$.

Definition 5.1. *(i) A Borel measurable function $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ is said to be strong Morrey quasiconvex if for any $\varepsilon > 0$, for any $\xi \in \mathbb{R}^{m\times n}$, and any $K > 0$, there exists a $\delta = \delta(\varepsilon, K, \xi) > 0$ such that if $\varphi \in W^{1,\infty}(Q;\mathbb{R}^m)$ satisfies*

$$\|\nabla\varphi\|_{L^\infty(Q)} \leq K, \qquad \max_{x\in\partial Q} |\varphi(x)| \leq \delta,$$

then,

$$f(\xi) \leq \operatorname*{ess\,sup}_{x\in Q} f(\xi + \nabla\varphi(x)) + \varepsilon.$$

(ii) A function $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ is said to be weak Morrey quasiconvex if

$$f(\xi) \leq \operatorname*{ess\,sup}_{x\in Q} f(\xi + \nabla\varphi(x)), \tag{5.1}$$

for every $\xi \in \mathbb{R}^{m\times n}$ and every $\varphi \in W_0^{1,\infty}(Q;\mathbb{R}^m)$.

(iii) A function $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ is level convex if $f(t\xi + (1-t)\eta) \leq \max\{f(\xi), f(\eta)\}$, for every $t \in [0,1]$ and for every $\xi, \eta \in \mathbb{R}^{m\times n}$.

(iv) A function $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ is rank one quasiconvex (rank one level convex) if for any $\xi, \eta \in \mathbb{R}^{m\times n}$ with $\mathrm{rank}(\xi - \eta) \leq 1$, $f(t\xi + (1-t)\eta) \leq \max\{f(\xi), f(\eta)\}$, for every $t \in [0,1]$.

Remark 5.2. *Clearly, as observed in [15], strong Morrey quasiconvexity implies weak Morrey quasiconvexity. However, it is not true that weak Morrey quasiconvexity implies rank one quasiconvexity, as it was wrongly stated in [15, Proposition 3.8 and Corollary 3.9]. See Example 5.3. We will show in Theorem 5.5 below, that this statement is true if we assume the function to be upper semicontinuous.*

Example 5.3. *Let $m \geq 1$ and $n > 1$. Let $S := \{\xi, \eta\} \subset \mathbb{R}^{m\times n}$ such that $\mathrm{rank}(\xi - \eta) = 1$ and let $f := 1 - \chi_S$, where χ_S is the characteristic function of S. Of course f is not rank one quasiconvex. Let's see that f is weak Morrey quasiconvex:*

$$f(\zeta) \leq \operatorname*{ess\,sup}_{x\in Q} f(\zeta + \nabla\varphi(x)), \ \forall\, \zeta \in \mathbb{R}^{m\times n}, \ \varphi \in W_0^{1,\infty}(Q;\mathbb{R}^m).$$

To this end it is enough to consider the case where $\zeta \notin S$. Then, the inequality follows from the fact that, there is no $\varphi \in W_0^{1,\infty}(Q;\mathbb{R}^m)$ such that $\nabla\varphi(x) \in \{\xi - \zeta, \eta - \zeta\}$ a.e. in Q. Actually, if $m = 1$, this is a consequence of Theorem 2.29. In the vectorial case $m > 1$, this follows from [10, Propositions 1 and 2].

We also observe that f is lower semicontinuous. So lower semicontinuity and weak Morrey quasiconvexity is not enough to ensure rank one quasiconvexity.

Next we show that if a function is upper semicontinuous and weak Morrey quasiconvex then it is rank one quasiconvex. We start recalling a lemma due to Müller-Šverák [37, Lemma 2.1], which is a generalization of a classical one and which will be useful for our proof.

Lemma 5.4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Let $t \in [0,1]$ and $\xi, \eta \in \mathbb{R}^{m\times n}$ with $\operatorname{rank}(\xi - \eta) = 1$. Let φ be an affine map such that*

$$\nabla\varphi(x) = t\xi + (1-t)\eta,\ x \in \overline{\Omega}.$$

Then, for every $\varepsilon > 0$, there exists $u \in Aff_{piec}(\overline{\Omega};\mathbb{R}^m)$ such that

$$\begin{cases} \operatorname{dist}(\nabla u(x), \{\xi,\eta\}) \leq \varepsilon,\ \text{a.e. } x \in \Omega, \\ \sup\limits_{x\in\Omega} |u(x) - \varphi(x)| \leq \varepsilon, \\ u(x) = \varphi(x),\ x \in \partial\Omega. \end{cases}$$

Theorem 5.5. *(i) Let $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ be an upper semicontinuous and weak Morrey quasiconvex function, then f is rank one quasiconvex. In particular, for $m = 1$, if f is upper semicontinuous and weak Morrey quasiconvex, then f is level convex.*

(ii) For $m = 1$, and f continuous, it results that f is weak Morrey quasiconvex if and only if f is level convex.

(iii) Let $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ be a Borel measurable function. If $n = 1$ and f is weak Morrey quasiconvex then f is level convex.

Remark 5.6. *Clearly, if a function $f : \mathbb{R}^{m\times n} \longrightarrow \mathbb{R}$ satisfies the supremal Jensen's inequality then it is weak Morrey quasiconvex. The converse being true in the scalar case $n = 1$, that is when Ω is an interval. This follows from the present theorem combined with Theorem 2.4.*

Proof. Once the first assertion of the theorem is proved, the remainder of condition (i) and condition (ii) follow immediately since evidently rank one quasiconvexity reduces to level convexity if $m = 1$ and since, level convex functions satisfy the supremal Jensen's inequality (cf. Theorem 2.4).

We prove that weak Morrey quasiconvexity implies Morrey rank one quasiconvexity for upper semicontinuous functions. Let $\xi, \eta \in \mathbb{R}^{m\times n}$ be such that $\operatorname{rank}(\xi - \eta) = 1$ and let $t \in (0,1)$. We want to show that $f(t\xi + (1-t)\eta) \leq \max\{f(\xi), f(\eta)\}$. Fix $\delta > 0$. By the upper semicontinuity of f, there exists $\varepsilon > 0$ such that

$$|X - \xi| \leq \varepsilon,\ |Y - \eta| \leq \varepsilon\ \Rightarrow\ f(X) \leq \delta + f(\xi),\ f(Y) \leq \delta + f(\eta).$$

Applying Lemma 5.4, we get $\psi \in W_0^{1,\infty}(Q,\mathbb{R}^m)$ such that

$$\operatorname{dist}(\nabla\psi(x), \{(1-t)(\xi-\eta), -t(\xi-\eta)\}) \leq \varepsilon,\ \text{a.e. } x \in Q.$$

Using the weak Morrey quasiconvexity and the upper semicontinuity of f, we get

$$f(t\xi + (1-t)\eta) \leq \operatorname*{ess\,sup}_{x\in Q} f(t\xi + (1-t)\eta + \nabla\psi(x)) \leq \delta + \max\{f(\xi), f(\eta)\}.$$

The result is now achieved letting δ go to zero.

It remains to prove condition (iii). Let $f : \mathbb{R}^{m\times 1} \to \mathbb{R}$ be a Borel measurable and weak Morrey quasiconvex function, let $\xi, \eta \in \mathbb{R}^{m\times 1}$, and let $t \in [0,1]$ be arbitrary. Define

$$\varphi(x) = \begin{cases} (1-t)(\xi-\eta)x, & \text{if } 0 \leq x \leq t, \\ t(\xi-\eta)(1-x), & \text{if } t \leq x \leq 1. \end{cases}$$

Clearly $\varphi \in W_0^{1,\infty}((0,1);\mathbb{R}^m)$ and applying (5.1) one gets $f(t\xi + (1-t)\eta) \leq \max\{f(\xi), f(\eta)\}$, which gives the level convexity of f. □

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