The eigenblock and eigenmatrix decomposition of a matrix: its usefulness in Statistics – application to the likelihood ratio test for block-circularity

Carlos A. Coelho*

Departamento de Matemática and Centro de Matemática e Aplicações, Faculdade de Ciências e Tecnologia, Universidade Nova de Lisboa, Portugal

Abstract

We are all familiar with the spectral decomposition or eigenvalue-eigenvector decomposition of a matrix. This paper addresses cases where instead of using a single eigenvalue we may use a square non-diagonal matrix, which we will call eigenmatrix, which goes along with the concomitant set or matrix of block-eigenvectors in a number that has to match the dimension of the eigenmatrix. The usefulness of such construct or decomposition in statistical studies in multivariate analysis, more precisely, in the derivation of likelihood ratio statistics for tests of elaborate structures of covariance matrices, their moments and their distributions, is shown, and an application is made to the derivation and study of the likelihood ratio statistic to test block-circularity of covariance matrices. Through the use of such construct or decomposition, also near-exact distributions are easily obtained for this statistic and the relation of the test with other tests is easily derived.

Keywords: characteristic function, composition of hypotheses, decomposition of the null hypothesis, distribution of likelihood ratio statistics, near-exact distributions, product of independent Beta random variables, sum of independent Gamma random variables.

1. Introduction

We are all familiar with the eigenvalue-eigenvector or spectral decomposition of a square matrix, which commonly takes the form

$$\underset{(p^*\times p^*)}{A}\;\underline{u}_\alpha = \lambda_\alpha \underline{u}_\alpha\,, \qquad \alpha=1,\dots,p^*\,, \tag{1}$$

where A is a $p^*\times p^*$ square matrix, $\underline{u}_\alpha$ is the α-th eigenvector of A and λ_α its associated eigenvalue. The eigenvalues λ_α are all real in case A is symmetric and all positive in case A is positive-definite.

In this paper we will address the following questions: i) “are there cases where we may have a similar construct but with λ_α replaced by Λ_α, a $k_\alpha\times k_\alpha$ square matrix, and, concomitantly, with $\underline{u}_\alpha$ replaced by U_α, a $p^*\times k_\alpha$ matrix?”, and, in case the answer to this question is affirmative, ii) “are there applications for such construct?”, or rather, “are there ’things’ we can do with such construct that cannot be done without it, or which are rendered much easier when such construct is used?”.

*Corresponding author: Faculdade de Ciências e Tecnologia, Universidade Nova de Lisboa, Departamento de Matemática, Quinta da Torre, 2829-516 Caparica; phone: +351 21 294 8388, Fax: +351 21 294 8391.

Email address: `cmac@fct.unl.pt` (Carlos A. Coelho)

September 9, 2013

The answer to both questions above is affirmative. Namely, the answer to the second of the two questions above, is affirmative, in the sense that there are results in Statistics, more precisely, in Multivariate Analysis, which will be rendered much easier to obtain by using such construct, viz. the expressions for many l.r.t. (likelihood ratio test) statistics as well as the expressions for their moments and as such also the expressions for the c.f. (characteristic function) of its logarithm, besides the fact that then also the derivation of the l.r.t. statistics pertaining to bridging tests will become much easier to derive.

But then a third compound question that may arise is: "the results obtained by using such construct, cannot they also be obtained by using the common spectral decomposition?", and if not, "why not?". The answer to the first of these questions is indeed "No!" and the answer to the short last question will have to wait till we get a little further into the next section.

2. The eigenblock-eigenmatrix decomposition of a matrix

There are indeed matrices A, say of dimensions $p^*\times p^*$, for which we may write

$$\underset{(p^*\times p^*)}{A}\;\underset{(p^*\times k_\alpha)}{U_\alpha} = \underset{(p^*\times k_\alpha)}{U_\alpha}\;\underset{(k_\alpha\times k_\alpha)}{\Lambda_\alpha}\,, \qquad \alpha=1,\dots,p;\; p^*=\sum_{\alpha=1}^{p} k_\alpha\,, \tag{2}$$

or

$$\underset{(p^*\times p^*)}{A}\;\underset{(p^*\times p^*)}{U} = \underset{(p^*\times p^*)}{U}\;\underset{(p^*\times p^*)}{\Lambda} \tag{3}$$

where

$$U=\Big[\,U_1\,\Big|\dots\Big|\,U_\alpha\,\Big|\dots\Big|\,U_p\Big] \qquad \text{and} \qquad \Lambda=\textit{block-diagonal}\big(\Lambda_\alpha;\,\alpha=1,\dots,p\big)\,, \tag{4}$$

where Λ_α $(\alpha=1,\dots,p)$ are non-diagonal square matrices.

Hereinafter we will restrain ourselves to the case where A is a symmetric positive-definite matrix, in which case the matrices Λ_α $(\alpha=1,\dots,p)$ are also positive-definite matrices, since, ultimately, the set of eigenvalues of all the p matrices Λ_α will be the same as the set of eigenvalues of A.

It is true that ultimately, the set of eigenvalues of all the p matrices Λ_α will be the same as the set of eigenvalues of A, but the block construct in (2) or (3), when possible to be implemented in such a way that the matrices U_α are only functions of the dimension of A and not a function of the elements in A, will be extremely useful, whenever A has some elaborate block-structure, in: (i) the derivation of l.r.t.'s and corresponding statistics, (ii) the study of the distributions of such statistics, (iii) the design of bridging tests to other related block-structures, (iv) developing the implementation of the test for complex random variables, and (v) in avoiding the need for elaborate estimation procedures for patterned covariance matrices in order to be able to build the l.r.t's. Furthermore, since this approach will enable us to obtain, in most cases, the l.r.t. statistic as the product of l.r.t. statistics used to test the independence of groups of variables, equality of covariance matrices and sphericity, it will enable us to extend the results obtained to elliptically contoured distributions, based on the results in Chapters 8–10 in [1]. It happens that, in most cases when the construct in (2) or (3) is possible to be implemented, if we take the full common eigenvalue-eigenvector decomposition of the matrix A, then the eigenvectors of A will be functions of the elements in A, thus rendering this common eigenvalue-eigenvector decomposition useless for our purpose and this way bringing about the need for the construct or decomposition in (2) or

(3), rendering it extremely useful. These facts will be illustrated in the next section, with the application of this construct to the block-circularity test for a covariance matrix.

The construct in (2) or (3) is possible to be implemented when A has some elaborate block-structure, which will be made precise shortly.

In Multivariate Analysis, it is rather common to test hypotheses of the form

$$H_0 : \Sigma = \Sigma_0 \tag{5}$$

on the covariance matrix Σ of some random vector, where Σ_0 is specified through some structure or pattern [18]. Most often it is useful to be able to write the null hypothesis in (5) as

$$H_0 : \Sigma^* = \Sigma_0^* \tag{6}$$

where

$$\Sigma^* = \Gamma\Sigma\Gamma' \qquad \text{and} \qquad \Sigma_0^* = \Gamma\Sigma_0\Sigma'$$

for some orthogonal matrix Γ which is either completely specified 'a priori' or otherwise is only function of the dimensions of Σ or Σ_0, but not function of the elements of Σ_0.

The use of the construct in (2) in Statistics is indeed not completely new. Although it has never been explicitly written in the form in (2), equivalent constructs have been implicitly used by [16] and [19]. However, these authors never used such constructs in order to obtain either the expressions for the l.r.t. statistics or their moments. The authors in [7] implicitly use a construct of the type of the one in (2) to build the l.r.t. statistic for block compound symmetry and obtain its moments, but the construct was never stated in the form in (2) and a systematic approach to this construct was never undertaken.

We need now to systematize the situations in which we will be able to build for A a construct or decomposition as the one in (2) or (3).

There are many interesting structured $p{\times}p$ matrices A which may be written as

$$A = \sum_{\ell=1}^{s} a_\ell M_\ell \tag{7}$$

for some positive integer $s \leq p$, some reals a_ℓ and some $p{\times}p$ symmetric matrices M_ℓ ($\ell = 1, \ldots, s$), such that the s matrices M_ℓ have common eigenvectors $\underline{u}_\alpha$, which are only function of p and not of the elements in each matrix M_ℓ, as it is the case of, for example, the equivariance-equicorrelation and circular matrices [10, 16, 17]. We will call these matrices as scalar-structured matrices.

Let then $\lambda_{\alpha\ell}$ be the α-th eigenvalue of M_ℓ ($\alpha = 1, \ldots, p$), possibly function of the elements in M_ℓ. Then we may write

$$\begin{aligned} A\underline{u}_\alpha = \lambda_\alpha \underline{u}_\alpha \quad &\Longleftrightarrow \quad \left(\sum_{\ell=1}^{s} a_\ell M_\ell\right) \underline{u}_\alpha = \lambda_\alpha \underline{u}_\alpha \\ &\Longleftrightarrow \quad \sum_{\ell=1}^{s} a_\ell M_\ell \underline{u}_\alpha = \lambda_\alpha \underline{u}_\alpha \\ &\Longleftrightarrow \quad \sum_{\ell=1}^{s} a_\ell \lambda_{\alpha\ell}\, \underline{u}_\alpha = \lambda_\alpha \underline{u}_\alpha \end{aligned}$$

so that we have

$$\lambda_\alpha = \sum_{\ell=1}^{s} \lambda_{\alpha\ell}\, a_\ell\,.$$

But, since a_ℓ are scalars, we may write (7) as

$$A = \sum_{\ell=1}^{s} M_\ell \otimes a_\ell\,. \tag{8}$$

But then, if A has a structure such that instead of the scalars a_ℓ we may use $m{\times}m$ square symmetric matrices A_ℓ, keeping the same matrices M_ℓ in such a way that A remains positive-definite, with the matrix blocks appearing on the diagonal of A being positive-definite matrices. In this case A will have dimensions $mp{\times}mp$ and we will write

$$A = \sum_{\ell=1}^{s} M_\ell \otimes A_\ell\,, \tag{9}$$

having A what we call a block-structure.

In (8) and (9), each a_ℓ or each A_ℓ should appear in a single term, that is, in case the construction in (8) or (9) has more than one term where some a_ℓ or A_ℓ appears, then these terms should be combined into a single term by adding the corresponding matrices M_ℓ. We will call such a construction a 'legitimate' one, since only for this case it will be possible to develop the diagonalizations that follow. In the next section an example is given.

From (9), we may then write

$$\underset{(pm\times pm)}{A}\;\underset{(pm\times m)}{U_\alpha} = \underset{(pm\times m)}{U_\alpha}\;\underset{(m\times m)}{\Lambda_\alpha} \qquad (\alpha = 1,\ldots,p) \tag{10}$$

for

$$U_\alpha = \underline{u}_\alpha \otimes I_m \qquad \text{and} \qquad \Lambda_\alpha = \sum_{\ell=1}^{s} \lambda_{\alpha\ell}\, A_\ell\,, \tag{11}$$

since, if we take U_α defined as in (11), using the relation $(A\otimes B)(C\otimes D) = AC\otimes BD$, we have

$$\begin{aligned}(M_\ell\otimes A_\ell)\,U_\alpha &= (M_\ell\otimes A_\ell)\,(\underline{u}_\alpha\otimes I_m)\\ &= (M_\ell\,\underline{u}_\alpha)\otimes A_\ell\\ &= (\lambda_{\alpha\ell}\,\underline{u}_\alpha)\otimes A_\ell\\ &= \lambda_{\alpha\ell}\,(\underline{u}_\alpha\otimes I_m)\,A_\ell = \lambda_{\alpha\ell}\,U_\alpha\,A_\ell\end{aligned}$$

and as such

$$A\,U_\alpha = \left(\sum_{\ell=1}^{s} M_\ell\otimes A_\ell\right) U_\alpha = U_\alpha \left(\sum_{\ell=1}^{s} \lambda_{\alpha\ell} A_\ell\right) = U_\alpha \Lambda_\alpha\,, \qquad (\alpha=1,\ldots,p)$$

being thus the eigenmatrices Λ_α $(\alpha = 1,\ldots,p)$, 'similar' functions of the matrices A_ℓ, as the eigenvalues λ_α were of the scalars a_ℓ.

Then unless the matrices Λ_α have some scalar-structure, which allows for their eigenvectors to be functions of only their dimension m, their eigenvectors will be function of their elements, and as such, function of the elements in A.

3. An example of application of the eigenblock-eigenmatrix decomposition — the block-circular structure

Let us suppose that we have a random vector $\underline{X}$, of dimension $mp\times 1$, where there are n exact linear relations among its components. Let $\underline{X}$ be split into p subvectors $\underline{X}_k$ $(k=1,\ldots,p)$, each of dimension $m\times 1$, that is, let

$$\underline{X} = \left[\underline{X}_1,\ldots,\underline{X}_k,\ldots,\underline{X}_p\right]',$$

with

$$\Theta = Var(\underline{X}), \tag{12}$$

and

$$Cov(\underline{X}_k,\underline{X}_i) = R_{|k-i|}, \qquad (k,i=1,\ldots,p) \tag{13}$$

and

$$R_{p-j} = R_j, \qquad j=1,\ldots,p-1. \tag{14}$$

Then the matrix Θ in (12) will have what we call a block-circular structure. For example, for $p=6$ and $p=7$, we will have

$$\Theta = \begin{bmatrix} R_0 & R_1 & R_2 & R_3 & R_2 & R_1 \\ R_1 & R_0 & R_1 & R_2 & R_3 & R_2 \\ R_2 & R_1 & R_0 & R_1 & R_2 & R_3 \\ R_3 & R_2 & R_1 & R_0 & R_1 & R_2 \\ R_2 & R_3 & R_2 & R_1 & R_0 & R_1 \\ R_1 & R_2 & R_3 & R_2 & R_1 & R_0 \end{bmatrix}, \quad \text{and} \quad \Theta = \begin{bmatrix} R_0 & R_1 & R_2 & R_3 & R_3 & R_2 & R_1 \\ R_1 & R_0 & R_1 & R_2 & R_3 & R_3 & R_2 \\ R_2 & R_1 & R_0 & R_1 & R_2 & R_3 & R_3 \\ R_3 & R_2 & R_1 & R_0 & R_1 & R_2 & R_3 \\ R_3 & R_3 & R_2 & R_1 & R_0 & R_1 & R_2 \\ R_2 & R_3 & R_3 & R_2 & R_1 & R_0 & R_1 \\ R_1 & R_2 & R_3 & R_3 & R_2 & R_1 & R_0 \end{bmatrix}.$$

Let $I_{p,s}$ denote an identity matrix of order p, with the elements in each row shifted by s columns, or, more precisely, let it be a matrix of zeros, where in row i the element 1 appears in column

$$\text{mod}^*(i+s,p) = \begin{cases} \text{mod}(i+s,p) & \text{if mod}(i+s,p)\neq 0 \\ p & \text{if mod}(i+s,p)=0, \end{cases}$$

with $I_{p,0}=I_{p,p}=I_p$.

Then, we may write

$$\Theta = \sum_{\ell=0}^{p-1} I_{p,\ell}\otimes R_\ell,$$

or

$$\Theta = \sum_{\ell=0}^{\lfloor \frac{p}{2}\rfloor} M_\ell\otimes R_\ell, \tag{15}$$

where

$$M_\ell = \begin{cases} I_{p,0}=I_p & \ell=0 \\ I_{p,\ell}+I_{p,p-\ell} & \ell=1,\ldots,\left\lfloor \frac{p-1}{2}\right\rfloor \\ I_{p,\ell} & \ell=\frac{p}{2},\ \text{only for even } p, \end{cases} \tag{16}$$

given that from (13) and (14) it is clear that there will be only $1 + \lfloor \frac{p}{2} \rfloor$ different R_ℓ matrices, which are $R_0, \ldots, R_{\lfloor \frac{p}{2} \rfloor}$.

Moreover, given the comment right after expression (9), in the previous section, it follows that the construction of Θ in (15) is indeed the 'legitimate' one, since it is the one that uses only once each matrix R_ℓ and the one for which the diagonalizations that follow are valid.

Then, if we take

$$\underline{u}_\alpha = [u_{\alpha j},\, j = 1, \ldots, p]\,, \tag{17}$$

with (see [17])

$$u_{\alpha j} = \frac{1}{\sqrt{p}} \left\{ \cos\Big(2\pi(\alpha-1)(j-1)/p\Big) + \sin\Big(2\pi(\alpha-1)(j-1)/p\Big) \right\}, \quad j = 1, \ldots, p;\ \alpha = 1, \ldots, p\,, \tag{18}$$

we will have

$$M_\ell\, \underline{u}_\alpha = \lambda_{\alpha\ell}\, \underline{u}_\alpha\,, \qquad \alpha = 1, \ldots, p; \quad \ell = 0, \ldots, \left\lfloor \frac{p}{2} \right\rfloor$$

or, if we take the matrix

$$\underset{(p\times p)}{\Gamma} \;=\; \big[\, \underline{u}_1 \,|\, \ldots \,|\, \underline{u}_\alpha \,|\, \ldots \,|\, \underline{u}_p \,\big]\,, \tag{19}$$

it will diagonalize each matrix M_ℓ, with

$$M_\ell \Gamma = \Gamma \Lambda_\ell^*\,, \qquad \ell = 0, \ldots, \left\lfloor \frac{p}{2} \right\rfloor$$

with

$$\Lambda_\ell^* = diag(\lambda_{\alpha\ell},\, \alpha = 1, \ldots, p)\,, \qquad \ell = 0, \ldots, \left\lfloor \frac{p}{2} \right\rfloor$$

where, for $\alpha = 1, \ldots, p$,

$$\lambda_{\alpha\ell} = \begin{cases} 1 & \ell = 0 \\ 2\,\cos\big(2\pi(\alpha-1)\ell/p\big) & \ell = 1, \ldots \left\lfloor \frac{p-1}{2} \right\rfloor \\ \cos\big(2\pi(\alpha-1)\ell/p\big) & \ell = \frac{p}{2} \text{ only for even } p \end{cases} \tag{20}$$

and thus with

$$\begin{aligned} &\lambda_{\alpha\ell} = \lambda_{p-\alpha+2,\ell}\,, && \text{for } \alpha = 2, \ldots, p \\ &\lambda_{1\ell} = 2\,, && \ell = 1, \ldots, \left\lfloor \tfrac{p-1}{2} \right\rfloor \\ &\lambda_{1,\frac{p}{2}} = 1\,, && \ell = \tfrac{p}{2} \text{ for even } p\,. \end{aligned} \tag{21}$$

Note that $\underline{u}_\alpha$ are the eigenvectors, and $\lambda_\alpha = \sum_{\ell=0}^{\lfloor p/2 \rfloor} \lambda_{\alpha\ell} r_\ell$ the corresponding eigenvalues of the scalar-circular $p{\times}p$ matrix

$$A^* = \sum_{\ell=0}^{\lfloor \frac{p}{2} \rfloor} M_\ell \otimes r_\ell$$

for some scalars r_ℓ.

Now let us take

$$\underset{(pm\times pm)}{\Gamma^*} \;=\; \Gamma \otimes I_m \;=\; \Big[\, U_1 \,|\, \ldots \,|\, U_\alpha \,|\, \ldots \,|\, U_p \,\Big]$$

where, for $\alpha = 1, \ldots, p$, each eigenblock U_α is defined as

$$\underset{(pm\times m)}{U_\alpha} = \underset{(p\times 1)}{\underline{u}_\alpha} \otimes \underset{(m\times m)}{I_m}\,, \tag{22}$$

with $\underline{u}_\alpha$ given by (17)-(18).

Then, we may write

$$\underset{(pm\times pm)}{\Theta}\ \underset{(pm\times m)}{U_\alpha} = \underset{(pm\times m)}{U_\alpha}\ \underset{(m\times m)}{\Lambda_\alpha} \qquad (\alpha = 1, \ldots, p)\,, \tag{23}$$

or, since Γ in (19) is an orthonormal matrix, and so is Γ^*,

$$\Theta\Gamma^* = \Gamma^*\Lambda \tag{24}$$

or yet

$$\begin{aligned} \Gamma^{*\prime}\Theta\Gamma^* &= \Lambda \\ &= \textit{block-diagonal}\big(\Lambda_1, \ldots, \Lambda_\alpha, \ldots, \Lambda_p\big) \end{aligned} \tag{25}$$

where

$$\Lambda_\alpha = \sum_{\ell=0}^{\lfloor \frac{p}{2} \rfloor} \lambda_{\alpha\ell}\, R_\ell\,, \qquad \alpha = 1, \ldots, p \tag{26}$$

so that, since all all matrices R_ℓ are symmetric, also all matrices Λ_α are symmetric, and unless all the matrices R_ℓ have some particular scalar structure, the matrices Λ_α will be unstructured, and the eigenvectors of Θ will be functions of the elements in Θ, rendering a full common spectral decomposition of Θ useless for our goal, while the construct or decomposition in (23) or (24)–(25) is much useful. This is the situation we address in the next section.

Given the structure of the $\lambda_{\alpha\ell}$ in (20) and (21), the matrices Λ_α in (23)–(26) also verify the relations

$$\Lambda_1 = \begin{cases} R_0 + \sum\limits_{\ell=1}^{(p-1)/2} 2\,R_\ell & \text{odd } p \\ R_0 + R_{\frac{p}{2}} + \sum\limits_{\ell=1}^{p/2-1} 2\,R_\ell & \text{even } p \end{cases} \qquad \text{and} \qquad \Lambda_\alpha = \Lambda_{p-\alpha+2}\,, \qquad \alpha = 2, \ldots, p\,. \tag{27}$$

4. Testing for block-circularity of the covariance matrix

We are interested in the following hypothesis: once assumed

$$\underline{X} = \big[\underline{X}_1, \ldots, \underline{X}_k, \ldots, \underline{X}_p\big]' \sim N_{mp}(\mu, \Sigma)\,,$$

we want to test

$$H_0 : \Sigma = \Theta\,, \tag{28}$$

for Θ bearing the block-circular structure described in the previous section and given by (15)-(16).

But then, by (23)-(25), to test H_0 in (28) is the same as to test

$$H_0 : \Sigma^*\, U_\alpha = U_\alpha\, \Lambda_\alpha\,, \qquad \alpha = 1, \ldots, p \tag{29}$$

for U_α given by (22), or, equivalently,

$$H_0 : \Sigma^* = \textit{block-diagonal}\big(\Lambda_1, \dots, \Lambda_\alpha, \dots, \Lambda_p\big) \tag{30}$$

where

$$\Sigma^* = \Gamma^{*\prime}\Sigma\Gamma^*\,,$$

and where the matrices Λ_α ($\alpha = 1, \dots, p$) verify the relations in (27).

For $m = 1$ this test reduces to the circularity test for the covariance matrix in sec. 3.3 of [17], while for general m it may be seen as a generalization of the usual test of equality of p variance-covariance matrices based on independent samples, since we are indeed testing the equality of the p diagonal blocks of dimensions $m{\times}m$ of Σ, whose m.l.e.'s (maximum likelihood estimators) are now not independent.

4.1. Deriving the l.r.t. statistic and its moments

The form of the null hypothesis in (29) or (30), together with the property in (27), exhibited by the matrices Λ_α, allow for a much convenient split of this null hypothesis as

$$H_0 \equiv H_{0b|a} \circ H_{0a}$$

where '$\circ$' is to be read as 'after', and where

$$H_{0a}: \quad \Sigma^* = \text{block-diagonal}\,(\Sigma^*_\alpha,\ \alpha = 1, \dots, p) \quad \Big(\text{where } \Sigma^*_\alpha\ (\alpha = 1, \dots, p) \text{ are unspecified } m{\times}m \text{ matrices}\Big)$$

and

$$H_{0b|a}: \quad \bigwedge_{k=2}^{\lfloor \frac{p+1}{2} \rfloor} \Sigma^*_k = \Sigma^*_{p-k+2}\,, \text{ assuming } H_{0a}\,. \tag{31}$$

The l.r.t. statistic to test H_{0a}, based on a sample of size n, is

$$\boldsymbol{\Lambda}_a = \left(\frac{|A|}{\prod_{j=1}^p |A_j|} \right)^{n/2},$$

where A is the unstructured or common m.l.e. of Σ^* and A_j is its j-th $m{\times}m$ diagonal block.

The l.r.t. statistic to test $H_{0b|a}$ is then

$$\boldsymbol{\Lambda}_{b|a} = \prod_{k=2}^{\lfloor \frac{p+1}{2} \rfloor} \boldsymbol{\Lambda}_{bk|a}$$

where, since the m.l.e.'s of Σ^*_k and Σ^*_{p-k+2} are independent under H_{0a}, and, under $H_{0b|a}$, both are Wishart $W_m(n-1, \Lambda_k)$ matrices,

$$\boldsymbol{\Lambda}_{bk|a} = \left(2^{2m}\, \frac{|A_k|\,|A_{p-k+2}|}{|A_k + A_{p-k+2}|^2} \right)^{n/2}.$$

The l.r.t. statistic to test H_0 in (3) is thus

$$
\begin{aligned}
\mathbf{\Lambda} &= \mathbf{\Lambda}_a \, \mathbf{\Lambda}_{b|a} = \mathbf{\Lambda}_a \prod_{k=2}^{\lfloor (p+1)/2 \rfloor} \mathbf{\Lambda}_{bk|a} \\
&= \left(2^{2m\lfloor (p-1)/2 \rfloor} \frac{|A|}{\left(|A_{\lfloor p/2 \rfloor +1}|\right)^{1-\mathrm{mod}(p,2)} \prod_{k=2}^{\lfloor (p+1)/2 \rfloor} |A_k + A_{p-k+2}|^2} \right)^{n/2},
\end{aligned}
$$

which, for $m=1$ yields exactly the l.r.t. statistic in (3.5) of section 3.3 of [17].

The h-th moment of $\mathbf{\Lambda}_a$ may be written as [1, Chap. 9][3, 15]

$$
\begin{aligned}
E\left(\mathbf{\Lambda}_a^h\right) &= \prod_{k=1}^{p-1} \prod_{j=1}^{m} \frac{\Gamma\left(\frac{n-j}{2}\right) \Gamma\left(\frac{n-j-m(p-k)}{2} + \frac{n}{2}h\right)}{\Gamma\left(\frac{n-j}{2} + \frac{n}{2}h\right) \Gamma\left(\frac{n-j-m(p-k)}{2}\right)} \\
&= \left\{ \prod_{j=3}^{mp} \left(\frac{n-j}{n}\right)^{r_j} \left(\frac{n-j}{n} + h\right)^{-r_j} \right\} \left(\frac{\Gamma\left(\frac{n-1}{2}\right) \Gamma\left(\frac{n-2}{2} + \frac{n}{2}h\right)}{\Gamma\left(\frac{n-1}{2} + \frac{n}{2}h\right) \Gamma\left(\frac{n-2}{2}\right)} \right)^{k^*}
\end{aligned}
$$

where

$$
k^* = \begin{cases} 0 & m \text{ even} \\ \lfloor p/2 \rfloor & m \text{ odd}, \end{cases}
$$

and

$$
r_j = \begin{cases} h_{j-2} + (-1)^j k^* & j = 3, 4 \\ r_{j-2} + h_{j-2} & j = 5, \dots, mp \end{cases} \qquad \text{with} \qquad h_j = \begin{cases} p-1, & j = 1, \dots, m \\ -1 & j = m+1, \dots, mp-2 \,. \end{cases} \tag{32}
$$

Then, given that $\Lambda_{bk|a}$ is the l.r.t. statistic to test the equality of two covariance matrices, whose m.l.e.'s, under $H_{0b|a}$ in (31), are both Wishart $W_m(n-1, \Lambda_k)$ matrices, from [1, Chap. 10] and [5, 15] the h-th moment of $\mathbf{\Lambda}_{bk|a}$ is

$$
\begin{aligned}
E\left(\mathbf{\Lambda}_{bk|a}^h\right) &= \prod_{j=1}^{m} \prod_{\ell=1}^{2} \frac{\Gamma\left(\frac{n-1}{2} - \frac{j-1}{4} + \frac{\ell-1}{2}\right) \Gamma\left(\frac{n-j}{2} + \frac{n}{2}h\right)}{\Gamma\left(\frac{n-1}{2} - \frac{j-1}{4} + \frac{\ell-1}{2} + \frac{n}{2}h\right) \Gamma\left(\frac{n-j}{2}\right)} \\
&= \left\{ \prod_{j=2}^{m} \left(\frac{n-j}{n}\right)^{s_j} \left(\frac{n-j}{n} + h\right)^{-s_j} \right\} \\
&\quad \times \left\{ \prod_{j=1}^{\lfloor m/2 \rfloor} \frac{\Gamma\left(n-j-\frac{1}{2}\right) \Gamma(n-j-1+nh)}{\Gamma\left(n-j-\frac{1}{2}+nh\right) \Gamma(n-j-1)} \right\} \\
&\qquad \times \left(\frac{\Gamma\left(\frac{n-m}{2} + \left\lfloor \frac{m-1}{4} \right\rfloor + \frac{1}{2}\right) \Gamma\left(\frac{n-m}{2} + \left\lfloor \frac{m-1}{4} \right\rfloor + \frac{n}{2}h\right)}{\Gamma\left(\frac{n-m}{2} + \left\lfloor \frac{m-1}{4} \right\rfloor + \frac{1}{2} + \frac{n}{2}h\right) \Gamma\left(\frac{n-m}{2} + \left\lfloor \frac{m-1}{4} \right\rfloor\right)} \right)^{m \perp\!\!\!\perp 2}
\end{aligned}
\tag{33}
$$

where

$$s_j = \begin{cases} \left.\begin{array}{ll} s^*_{j-1} & j = 2, \ldots, m \end{array}\right\} \text{if } \bmod(m,4) \neq 3 \\ \left.\begin{array}{ll} s^*_{j-1} & j = 2, \ldots, m \\ & \text{except } j = (m+3)/2 \\ s^*_{j-1}+1 & j = (m+3)/2 \end{array}\right\} \text{if } \bmod(m,4) = 3 \end{cases} \tag{34}$$

with

$$s^*_j = \begin{cases} j-1+\bmod(j,2) & j = 1, \ldots, \alpha \\ 2\alpha_1 + \bmod(m+1,2) & j = \alpha+1 \\ 2\left(\left\lfloor \frac{m}{2} \right\rfloor - \left\lfloor \frac{j}{2} \right\rfloor\right) & j = \alpha+2, \ldots, \min(m-2\alpha_1, m-1) \\ & j = 2+m-2\alpha_1, \ldots, 2\left\lfloor \frac{m}{2} \right\rfloor - 1, \ \textit{step } 2 \\ 2\left(\left\lfloor \frac{m+1}{2} \right\rfloor - \left\lfloor \frac{j}{2} \right\rfloor\right) & j = 1+m-2\alpha_1, \ldots, m-1, \ \textit{step } 2 \end{cases} \tag{35}$$

for

$$\alpha = \left\lfloor \frac{m-1}{2} \right\rfloor \qquad \text{and} \qquad \alpha_1 = \left\lfloor \frac{m-1}{4} \right\rfloor. \tag{36}$$

4.2. The characteristic function of $W = -\log \Lambda$

Given that under H_{0a} the $\left\lfloor \frac{p}{2} \right\rfloor - 1$ statistics $\mathbf{\Lambda}_{bk|a}$ are independent and given that $\mathbf{\Lambda}_b = \prod_{k=2}^{\lfloor p/2 \rfloor} \mathbf{\Lambda}_{bk|a}$ is only function of $A_2, \ldots, A_p$ and in [6] it is shown that $\mathbf{\Lambda}_a$ is independence of $A_1, \ldots, A_p$, $\mathbf{\Lambda}_a$ and $\mathbf{\Lambda}_b$ are independent. As such, we have

$$E\left(\mathbf{\Lambda}^h\right) = E\left(\mathbf{\Lambda}_a^h\right) E\left(\mathbf{\Lambda}_b^h\right) = E\left(\mathbf{\Lambda}_a^h\right) \prod_{k=2}^{\lfloor (p+1)/2 \rfloor} E\left(\mathbf{\Lambda}_{bk|a}^h\right),$$

so that, for $W = -\log \mathbf{\Lambda}$ we may write

$$\begin{aligned} \Phi_W(t) &= E\left(e^{\mathrm{i}tW}\right) = E\left(e^{-\mathrm{i}t \log \mathbf{\Lambda}}\right) = E\left(\mathbf{\Lambda}^{-\mathrm{i}t}\right) \\ &= \underbrace{\left\{ \prod_{j=2}^{mp} \left(\frac{n-j}{n}\right)^{z_j} \left(\frac{n-j}{n} - \mathrm{i}t\right)^{-z_j} \right\}}_{\Phi_{1,W}(t)} \\ &\quad \underbrace{\begin{aligned} &\times \left(\frac{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{n-2}{2} - \frac{n}{2}\mathrm{i}t\right)}{\Gamma\left(\frac{n-1}{2} - \frac{n}{2}\mathrm{i}t\right)\Gamma\left(\frac{n-2}{2}\right)} \right)^{k^*} \left(\prod_{j=1}^{\lfloor m/2 \rfloor} \frac{\Gamma\left(n-j-\frac{1}{2}\right)\Gamma(n-j-1-n\mathrm{i}t)}{\Gamma\left(n-j-\frac{1}{2}-n\mathrm{i}t\right)\Gamma(n-j-1)} \right)^{\lfloor (p-1)/2 \rfloor} \\ &\times \left(\frac{\Gamma\left(\frac{n-m}{2} + \left\lfloor \frac{m-1}{4} \right\rfloor + \frac{1}{2}\right)\Gamma\left(\frac{n-m}{2} + \left\lfloor \frac{m-1}{4} \right\rfloor - \frac{n}{2}\mathrm{i}t\right)}{\Gamma\left(\frac{n-m}{2} + \left\lfloor \frac{m-1}{4} \right\rfloor + \frac{1}{2} - \frac{n}{2}\mathrm{i}t\right)\Gamma\left(\frac{n-m}{2} + \left\lfloor \frac{m-1}{4} \right\rfloor\right)} \right)^{(m \perp\!\!\perp 2)\left\lfloor \frac{p-1}{2} \right\rfloor} \end{aligned}}_{\Phi_{2,W}(t)}, \end{aligned} \tag{37}$$

where, since in (33) we have always $s_1 = 1$,

$$z_j = \begin{cases} \lfloor (p-1)/2 \rfloor & j = 2 \\ r_j + \lfloor (p-1)/2 \rfloor s_j & j = 3, \dots, m \\ r_j & j = m+1, \dots, mp \end{cases}$$

with r_j given by (32) and s_j by (34)–(36).

In (37) $\Phi_{1,W}(t)$ is the c.f. of a GIG (Generalized Integer Gamma) distribution of depth $pm-1$, with rate parameters $\frac{n-j}{n}$ and integer shape parameters z_j $(j = 2, \dots, pm)$, that is the c.f. of the sum of $pm-1$ independent Gamma r.v.'s with those shape and rate parameters, while $\Phi_{2,W}(t)$ is the c.f. of a sum of $k^* + \lfloor (p-1)/2 \rfloor (\lfloor m/2 \rfloor + \text{mod}(m, 2))$ independent Logbeta r.v.'s.

4.3. Near-exact distributions for the l.r.t. statistic

Since the exact distributions of both $\mathbf{\Lambda}$ and W are not manageable due to the part that in the distribution of $W = -\log \mathbf{\Lambda}$ is represented by $\Phi_{2,W}(t)$, we will approximate the distribution of W by leaving $\Phi_{1,W}(t)$ unchanged and approximating $\Phi_{2,W}(t)$ by the c.f. of a finite mixture of Gamma distributions, all with the same rate parameter.

This approximation is based on the results in Section 5 of [20] which state that we may asymptotically approximate any Logbeta(a, b) distribution by an infinite mixture of $\Gamma(b+k, a)$ distributions $(k = 0, 1, \dots)$. Given that the first parameters in the Logbeta distributions in $\Phi_{2,W}(t)$ are indeed not all that different, mainly for large n, we will use a common rate parameter λ for the Gamma distributions, which will be the parameter λ in

$$\Phi^*(t) = \theta \lambda^{r_1} (\lambda - \mathrm{i}t)^{-r_1} + (1-\theta) \lambda^{r_2} (\lambda - \mathrm{i}t)^{-r_2}$$

which is the c.f. of a mixture of two Gamma distributions with a common rate parameter. The parameter λ, together with the parameters r_1, r_2 and θ, will be determined in such a way that, for $h = 1, \dots, 4$,

$$\left. \frac{d^h}{dt^h} \Phi^*(t) \right|_{t=0} = \left. \frac{d^h}{dt^h} \Phi_{2,W}(t) \right|_{t=0} .$$

Then, we will approximate $\Phi_{2,W}(t)$ with

$$\Phi_2^*(t) = \sum_{k=0}^{m^*} p_k \lambda^{r+k} (\lambda - \mathrm{i}t)^{-(r+k)} , \tag{38}$$

which is the c.f. of a finite mixture of Gamma distributions, all with the same rate parameter λ, and shape parameters $r + k$, with

$$r = 1/2 \left(\left\lfloor \frac{p}{2} \right\rfloor \text{mod}(m, 2) + \left\lfloor \frac{p-1}{2} \right\rfloor \left\lfloor \frac{m+1}{2} \right\rfloor \right),$$

which is the sum of all the second parameters of the Logbeta distributions in $\Phi_{2,W}(t)$. The weights p_k in (38) will be determined in such a way that

$$\left. \frac{d^h}{dt^h} \Phi_2^*(t) \right|_{t=0} = \left. \frac{d^h}{dt^h} \Phi_{2,W}(t) \right|_{t=0} ,$$

for $h=1,\dots,m^*$, and with $p_{m^*}=1-\sum_{k=0}^{m^*-1}p_k$.

This approach yields for W a near-exact c.f. of the form

$$\Phi^*_W(t)=\sum_{k=0}^{m^*}p_k\left\{\lambda^{r+k}(\lambda-\mathrm{i}t)^{-(r+k)}\prod_{j=2}^{mp}\left(\frac{n-j}{n}\right)^{z_j}\left(\frac{n-j}{n}-\mathrm{i}t\right)^{-z_j}\right\}$$

which is the c.f. of a mixture of m^*+1 Generalized Near-Integer Gamma (GNIG) distributions (see [4, 5, App. 1] for the GNIG distribution) of depth mp, with with integer shape parameters z_j $(j=2,\dots,pm)$ and non-integer shape parameter r, and corresponding rate parameters $\frac{n-j}{n}$ $(j=2,\dots,pm)$ and λ. Using the notation in Appendix 1 of [5] for the p.d.f. and the c.d.f. of the GNIG distribution, which formulation is also shown in that reference, or in [4], these distributions have p.d.f.'s of the form

$$f^*_W(w)=\sum_{k=0}^{m^*}p_k\,f^{GNIG}\left(w\,|\,\{z_2,\dots,z_{pm}\},r+k;\left\{\frac{n-2}{n},\dots,\frac{n-pm}{n}\right\},\lambda;pm\right),\qquad w>0$$

and c.d.f.'s of the form

$$F^*_W(w)=\sum_{k=0}^{m^*}p_k\,F^{GNIG}\left(w\,|\,\{z_2,\dots,z_{pm}\},r+k;\left\{\frac{n-2}{n},\dots,\frac{n-pm}{n}\right\},\lambda;pm\right),\qquad w>0\,,$$

which for Λ yield near-exact distributions with p.d.f.'s

$$f^*_\Lambda(\ell)=\sum_{k=0}^{m^*}p_k\,f^{GNIG}\left(-\log\,\ell\,|\,\{z_2,\dots,z_{pm}\},r+k;\left\{\frac{n-2}{n},\dots,\frac{n-pm}{n}\right\},\lambda;pm\right)\frac{1}{\ell},\quad 0<\ell<1$$

and c.d.f.'s

$$F^*_\Lambda(w)=\sum_{k=0}^{m^*}p_k\left\{1-F^{GNIG}\left(-\log\,\ell\,|\,\{z_2,\dots,z_{pm}\},r+k;\left\{\frac{n-2}{n},\dots,\frac{n-pm}{n}\right\},\lambda;pm\right)\right\},\qquad 0<\ell<1\,.$$

As we may see, from the results of the numerical studies in the next section, these near-exact distributions yield very sharp approximations to the exact distribution, while remaining manageable, with closed form expressions for their p.d.f.'s and c.d.f.'s, and while exhibiting the flexibility of the choice of the number of exact moments that they match, which is the value of m^*. Of course, the larger m^* will be, the better will be the approximation obtained. Also, as we may see from the results of the numerical studies in the next section, the larger m^* is the clearer will be the asymptotic behaviors of these near-exact distributions, which besides being asymptotic for increasing sample sizes, will be also asymptotic for increasing values of p, the number of blocks of variables involved, which is expected given the expressions for the shape parameters z_j in (37) as increasing functions of p. Nevertheless, in what matters m, we expect the asymptotic behavior of the near-exact distributions to be not so clear, given that although for increasing values of m we will have a larger number of shape parameters z_j, we will also have a larger number of Logbeta distributions to approximate in $\Phi_{2,W}(t)$.

However, for the particular case $m=1$, even better approximations may be obtained if a more refined process of development of the near-exact distributions, which takes advantage of the particular structure of the Logbeta distributions in $\Phi_{2,W}(t)$, is undertaken, as in [14].

4.4. Numerical studies

In order to assess the closeness of the near-exact distributions to the exact distribution, we use the measure

$$\Delta = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left|\frac{\Phi_W(t)-\Phi_W^*(t)}{t}\right| dt\,, \tag{39}$$

which is related to the Berry-Esseen bound [2, 8, 11, 13] and which verifies the relation

$$\Delta \geq \max_{w>0}\Big|F_W(w)-F_W^*(w)\Big| = \max_{0<\ell<1}\Big|F_\Lambda(\ell)-F_\Lambda^*(\ell)\Big|\,,$$

where $\Phi_W(t)$ and $\Phi_W^*(t)$ represent respectively the exact and near-exact c.f.'s of W, $F_W(\,\cdot\,)$ and $F_W^*(\,\cdot\,)$ the exact and near-exact c.d.f.'s of W and $F_\Lambda(\,\cdot\,)$ and $F_\Lambda^*(\,\cdot\,)$ the exact and near-exact c.d.f.'s of Λ.

In Table 1 we may observe values of the measure Δ for the near-exact distributions which equate four, six and ten exact moments, for different values of m and p, and also for different sample sizes (n), which exceed mp by 2, 50, 100, 350 and 525.

Table 1. – Values of the measure Δ in (39) for different values of m, p and n.

			near-exact distribution number of exact moments matched						near-exact distribution number of exact moments matched		
m	p	n	4	6	10	m	p	n	4	6	10
5	6	32	7.37×10^{-17}	1.42×10^{-21}	5.03×10^{-30}	15	6	92	1.44×10^{-17}	1.83×10^{-23}	5.42×10^{-35}
		80	1.50×10^{-17}	1.44×10^{-22}	1.87×10^{-31}			140	3.99×10^{-17}	7.85×10^{-23}	5.02×10^{-34}
		130	1.67×10^{-18}	6.62×10^{-24}	1.52×10^{-33}			190	1.30×10^{-17}	1.73×10^{-23}	4.89×10^{-35}
		380	9.47×10^{-21}	4.66×10^{-27}	1.68×10^{-38}			440	2.79×10^{-19}	8.86×10^{-26}	1.36×10^{-38}
		555	1.46×10^{-21}	3.40×10^{-28}	2.73×10^{-40}			615	5.48×10^{-20}	9.33×10^{-27}	4.06×10^{-40}
	10	52	3.08×10^{-19}	4.05×10^{-25}	2.34×10^{-35}		10	152	1.46×10^{-19}	2.75×10^{-26}	1.49×10^{-39}
		100	3.12×10^{-19}	3.72×10^{-25}	1.97×10^{-35}			200	7.97×10^{-19}	3.03×10^{-25}	6.23×10^{-38}
		150	6.07×10^{-20}	3.68×10^{-26}	5.29×10^{-37}			250	4.41×10^{-19}	1.39×10^{-25}	1.94×10^{-38}
		400	6.48×10^{-22}	6.20×10^{-29}	2.32×10^{-41}			500	2.41×10^{-20}	2.71×10^{-27}	4.50×10^{-41}
		575	1.12×10^{-22}	5.25×10^{-30}	4.78×10^{-43}			675	5.86×10^{-21}	3.89×10^{-28}	2.21×10^{-42}
	15	77	7.07×10^{-21}	3.66×10^{-28}	7.30×10^{-42}		15	227	4.40×10^{-21}	2.02×10^{-28}	5.90×10^{-43}
		125	1.80×10^{-20}	1.25×10^{-27}	3.92×10^{-41}			275	3.23×10^{-20}	3.37×10^{-27}	4.71×10^{-41}
		175	5.56×10^{-21}	2.38×10^{-28}	2.77×10^{-42}			325	2.51×10^{-20}	2.47×10^{-27}	3.05×10^{-41}
		425	1.13×10^{-22}	1.00×10^{-30}	4.70×10^{-46}			575	2.94×10^{-21}	1.41×10^{-28}	3.92×10^{-43}
		600	2.20×10^{-23}	1.01×10^{-31}	1.26×10^{-47}			750	8.80×10^{-22}	2.73×10^{-29}	3.14×10^{-44}
10	6	62	7.02×10^{-18}	1.05×10^{-23}	5.16×10^{-35}	20	6	122	7.58×10^{-19}	3.45×10^{-25}	1.46×10^{-37}
		110	5.91×10^{-18}	9.76×10^{-24}	5.30×10^{-35}			170	2.70×10^{-18}	2.21×10^{-24}	2.83×10^{-36}
		160	9.41×10^{-19}	8.66×10^{-25}	1.45×10^{-36}			220	1.02×10^{-18}	6.25×10^{-25}	4.47×10^{-37}
		410	4.91×10^{-21}	8.06×10^{-28}	4.30×10^{-41}			470	2.07×10^{-20}	3.63×10^{-27}	2.11×10^{-40}
		585	6.18×10^{-22}	5.11×10^{-29}	6.92×10^{-43}			645	3.49×10^{-21}	3.41×10^{-28}	6.14×10^{-42}
	10	102	6.55×10^{-20}	1.32×10^{-26}	1.02×10^{-39}		10	202	9.03×10^{-21}	6.24×10^{-28}	5.05×10^{-42}
		150	1.73×10^{-19}	5.73×10^{-26}	1.11×10^{-38}			250	5.61×10^{-20}	8.42×10^{-27}	3.05×10^{-40}
		200	5.31×10^{-20}	1.23×10^{-26}	1.16×10^{-39}			300	3.61×10^{-20}	4.87×10^{-27}	1.43×10^{-40}
		450	7.51×10^{-22}	4.38×10^{-29}	2.63×10^{-43}			550	2.31×10^{-21}	1.32×10^{-28}	6.96×10^{-43}
		625	1.16×10^{-22}	3.67×10^{-30}	6.45×10^{-45}			725	5.29×10^{-22}	1.88×10^{-29}	3.79×10^{-44}
	15	152	2.10×10^{-21}	9.89×10^{-29}	3.67×10^{-43}		15	302	3.38×10^{-22}	5.96×10^{-30}	2.78×10^{-45}
		200	9.92×10^{-21}	9.29×10^{-28}	1.29×10^{-41}			350	2.63×10^{-21}	1.08×10^{-28}	2.63×10^{-43}
		250	4.84×10^{-21}	3.71×10^{-28}	3.46×10^{-42}			400	2.33×10^{-21}	9.62×10^{-29}	2.35×10^{-43}
		500	1.66×10^{-22}	4.35×10^{-30}	4.76×10^{-45}			650	3.57×10^{-22}	8.44×10^{-30}	6.76×10^{-45}
		675	3.18×10^{-23}	4.89×10^{-31}	1.82×10^{-46}			825	1.09×10^{-22}	1.76×10^{-30}	6.56×10^{-46}

As expected, the near-exact distributions that equate a larger number of exact moments display a much better performance, with much lower values of the measure Δ, and a more pronounced asymptotic behavior, translated into a faster decrease in the values of the measure Δ for increasing values of n and p, although as the value of mp becomes larger, the asymptotic behavior in terms of increasing values of n becomes only visible for larger values of n.

Although for increasing values of m the asymptotic behavior of the near-exact distributions developed may be not too clear, we are still able to see a steady tendency for a decrease in the value of the measure Δ in (39) for increasing values of m.

Remarkable are the facts that the near-exact distributions yield extremely good approximations to the exact distribution in all cases, even for very small sample sizes and that they exhibit a much clear asymptotic behavior for increasing values of p, the number of sub-sets of variables in $\underline{X}$, or, equivalently, the number of row or column blocks in Σ, or yet, the number of variables in the non-block case $m = 1$.

4.5. Bridging to other tests

In Figure 1 we have a schematic representation of the four main block-structures we are interested in. These are, (i) the actually non-structured situation of Σ, the covariance matrix of $\underline{X}$ being only a positive-definite matrix, in Figure 1 represented as $\Sigma > 0$ for short, (ii) the block-circular structure, (iii) the block-equicorrelation and equivariance structure and (iv) the block-spherical structure. In Figure 1, each one of the six tests depicted is indicated by an arrow, which has its left hand origin at the structure pertaining to H_1 and its right hand tip at the structure pertaining to H_0. So far we have just addressed test #1. In this section we will give some indications concerning tests #4, 5 and 6, although not addressing their l.r.t. statistics, neither the exact or approximate distributions of these statistics in detail, in order to keep the manuscript within a reasonable dimension. Tests #2 and 3 are addressed respectively in [7] and [6].

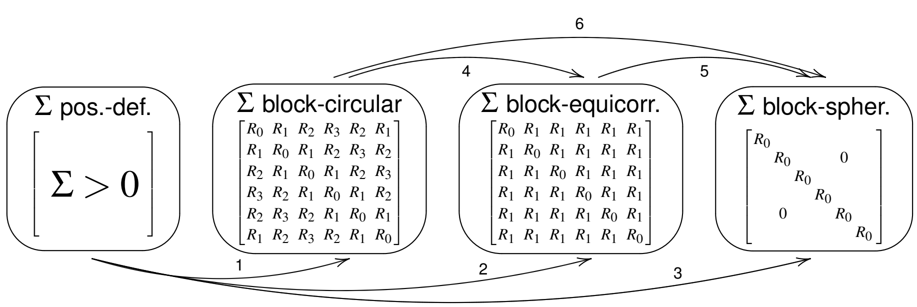

Figure 1. – Relations between the block-circularity structure and other block-structures:
$\Sigma > 0$ represents the absence of structure, with Σ being only positive-definite;
for each of the 6 possible tests of hypothesis depicted,
the structure on the left hand side of the arrow represents H_1,
while the structure on the right hand side of the arrow represents H_0.

The block-decomposition or construct in (29) or (30) will reveal itself extremely useful in obtaining the l.r.t. statistics and their moments for tests #4, 5 and 6.

4.5.1. Testing between block-circularity and block-equicorrelation and equivariance or compound symmetry

The block-equicorrelation and equivariance or compound symmetry structure is, say, similar to the block-circularity structure, but with $R_1 = \cdots = R_{\lfloor \frac{p}{2} \rfloor}$, together with the relation in (14) (see Figure 1), which through (26) and (27) implies

$$\Lambda_1 = R_0 + (p-1)R_1 \qquad \text{and} \qquad \Lambda_2 = \cdots = \Lambda_p = R_0 + R_1 \sum_{\ell=1}^{\lfloor p/2 \rfloor} \lambda_{\alpha\ell} = R_0 - R_1$$

given that, for $\alpha = 2, \ldots, p$, we have $\sum_{\ell=1}^{\lfloor p/2 \rfloor} \lambda_{\alpha\ell} = -1$.

Thus, once considered the construct or decomposition in (29) or (30), the test #4 may be seen as the test of

$$\begin{aligned} &H_0 : \Sigma^* = \textit{block-diagonal}\big(\Lambda_1, \underbrace{\Lambda_2, \Lambda_2, \ldots, \Lambda_2, \Lambda_2, \ldots, \Lambda_2, \Lambda_2}_{p-1}\big), \\ &\qquad\qquad \text{once assumed } \Sigma^* = \textit{block-diagonal}\big(\Lambda_1, \underbrace{\Lambda_2, \Lambda_3, \ldots, \Lambda_{\lfloor \frac{p}{2} \rfloor + 1}, \Lambda_{\lfloor \frac{p}{2} \rfloor + 1}, \ldots, \Lambda_3, \Lambda_2}_{p-1}\big) \\ &\text{vs.} \\ &H_1 : \Sigma^* = \textit{block-diagonal}\big(\Lambda_1, \underbrace{\Lambda_2, \Lambda_3, \ldots, \Lambda_{\lfloor \frac{p}{2} \rfloor + 1}, \Lambda_{\lfloor \frac{p}{2} \rfloor + 1}, \ldots, \Lambda_3, \Lambda_2}_{p-1}\big). \end{aligned} \tag{40}$$

This allows us to see this test as a test of equality of $\lfloor \frac{p}{2} \rfloor$ covariance matrices of dimension $m \times m$. Indeed H_0 in (40) may be re-written as

$$\begin{aligned} &H_0 : \Lambda_2 = \Lambda_3 = \cdots = \Lambda_{\lfloor \frac{p}{2} \rfloor + 1} \\ &\qquad \text{assuming } \Lambda_{p-k+2} = \Lambda_k \quad (k = 2, \ldots, \lfloor \tfrac{p}{2} \rfloor + 1)\,. \end{aligned} \tag{41}$$

For odd p, this will be a test of equality of $(p-1)/2$ covariance matrices, the m.l.e. of Λ_k ($k = 2, \ldots, (p+1)/2$) being $\frac{1}{2}(A_k + A_{p-k+2})$ (where A_k is the k-th $m \times m$ diagonal block of A, the m.l.e. of Σ^*), which, under H_0 in (40) or (41), is a Wishart $W_m(2(n-1), \frac{1}{2}\Lambda_2)$ matrix, while for even p it will be a test of equality of $\frac{p}{2}$ covariance matrices, the m.l.e.'s of $\frac{p}{2} - 1$ of them being Wishart $W_m(2(n-1), \frac{1}{2}\Lambda_2)$ matrices and the m.l.e. of one of them, $\Lambda_{1+p/2}$, being a Wishart $W_m(n-1, \Lambda_2)$ matrix.

4.5.2. Testing between block-equicorrelation and equivariance or compound symmetry and block-sphericity

The block-sphericity structure may be considered equivalent to the block-circularity structure, but with $R_1 = \cdots = R_{\lfloor \frac{p}{2} \rfloor} = 0_{m \times m}$ (see Figure 1), which implies

$$\Lambda_1 = \Lambda_2 = \cdots = \Lambda_p = R_0\,,$$

so that, once considered the construct or decomposition in (29) or (30), the test #5 may be seen as the test of

$$
\begin{aligned}
H_0 : \Sigma^* &= \textit{block-diagonal}\big(\Lambda_1, \underbrace{\Lambda_1, \Lambda_1, \dots, \Lambda_1, \Lambda_1, \dots, \Lambda_1, \Lambda_1}_{p-1}\big),\\
&\text{once assumed } \Sigma^* = \textit{block-diagonal}\big(\Lambda_1, \underbrace{\Lambda_2, \Lambda_2, \dots, \Lambda_2, \Lambda_2, \dots, \Lambda_2, \Lambda_2}_{p-1}\big)
\end{aligned}
\tag{42}
$$

vs.

$$
H_1 : \Sigma^* = \textit{block-diagonal}\big(\Lambda_1, \underbrace{\Lambda_2, \Lambda_2, \dots, \Lambda_2, \Lambda_2, \dots, \Lambda_2, \Lambda_2}_{p-1}\big).
$$

This allows us to see this test as a test of equality of two covariance matrices of dimension $m{\times}m$, since H_0 in (42) may be re-written as

$$
\begin{aligned}
H_0 : \Lambda_1 = \Lambda_2 = \cdots = \Lambda_p\\
\text{assuming } \Lambda_2 = \cdots = \Lambda_p\,,
\end{aligned}
\tag{43}
$$

and as such this will be a test of equality of two covariance matrices, the m.l.e. of Λ_1 being $A_1 \sim W_m(n-1, \Lambda_1)$, while the m.l.e. of Λ_2 is $\frac{1}{p-1}\sum_{k=2}^{p} A_k$ (where A_k is the k-th $m{\times}m$ diagonal block of A, the m.l.e. of Σ^*), which, under H_0 in (42) or (43), is a Wishart $W_m((p-1)(n-1), \frac{1}{p-1}\Lambda_1)$ matrix.

4.5.3. Testing between block-circularity and block-sphericity

Considering the construct or decomposition in (29) or (30), the test #6 is then a test of

$$
\begin{aligned}
H_0 : \Sigma^* &= \textit{block-diagonal}\big(\Lambda_1, \underbrace{\Lambda_1, \Lambda_1, \dots, \Lambda_1, \Lambda_1, \dots, \Lambda_1, \Lambda_1}_{p-1}\big),\\
&\text{once assumed } \Sigma^* = \textit{block-diagonal}\big(\Lambda_1, \underbrace{\Lambda_2, \Lambda_3, \dots, \Lambda_{\lfloor \frac{p}{2} \rfloor + 1}, \Lambda_{\lfloor \frac{p}{2} \rfloor + 1}, \dots, \Lambda_3, \Lambda_2}_{p-1}\big)
\end{aligned}
\tag{44}
$$

vs.

$$
H_1 : \Sigma^* = \textit{block-diagonal}\big(\Lambda_1, \underbrace{\Lambda_2, \Lambda_3, \dots, \Lambda_{\lfloor \frac{p}{2} \rfloor + 1}, \Lambda_{\lfloor \frac{p}{2} \rfloor + 1}, \dots, \Lambda_3, \Lambda_2}_{p-1}\big).
$$

This allows us to see this test as a test of equality of $1 + \lfloor \frac{p}{2} \rfloor$ covariance matrices of dimension $m{\times}m$. Indeed H_0 in (40) may be re-written as

$$
\begin{aligned}
H_0 : \Lambda_1 = \Lambda_2 = \Lambda_3 = \cdots = \Lambda_{\lfloor \frac{p}{2} \rfloor + 1}\\
\text{assuming } \Lambda_{p-k+2} = \Lambda_k \quad (k = 2, \dots, \lfloor \tfrac{p}{2} \rfloor + 1)\,.
\end{aligned}
\tag{45}
$$

For odd p, this will be a test of equality of $1 + (p-1)/2$ covariance matrices, the m.l.e. of Λ_k $(k = 2, \dots, (p+1)/2)$ being $\frac{1}{2}(A_k + A_{p-k+2})$ (where A_k is the k-th $m{\times}m$ diagonal block of A, the m.l.e. of Σ^*), which, under H_0 in (44) or (45), is a Wishart $W_m(2(n-1), \frac{1}{2}\Lambda_1)$ matrix, and the m.l.e. of Λ_1 being $A_1 \sim W_m(n-1), \Lambda_1)$. For even p it will be a test of equality of $1 + \frac{p}{2}$ covariance matrices, the m.l.e.'s of $\frac{p}{2} - 1$ of them being Wishart $W_m(2(n-1), \frac{1}{2}\Lambda_2)$ matrices and the m.l.e. of two of them, Λ_1 and $\Lambda_{1+\frac{p}{2}}$, being Wishart $W_m(n-1, \Lambda_2)$ matrices.

5. Conclusions

Eigenblock and eigenmatrix constructs or decompositions of the type in (2) and (3) or (23) and (24), where the matrices U_α are not function of the elements in A, are conceptually extremely useful in helping us in (i) the derivation of l.r.t.'s and corresponding statistics, (ii) the study of the distributions of such statistics, (iii) the design of bridging tests to other related block-structures and (iv) the development and implementation of the same l.r.t. tests for complex random variables, since by using such construct or decomposition we may avoid the need for elaborate estimation procedures for patterned covariance matrices in order to be able to build the l.r.t's, thus rendering all the above tasks much simpler, and, not less important, by enabling us to view the tests under study as compositions of other, much simpler and well-known, tests. This is exactly what happens with the proposed example of application, the block-circularity test, whose l.r.t. statistic, by using the eigenblock-eigenmatrix construct, was shown to be possible to be expressed as a product of well known statistics, whose distributions in case $\underline{X}$ is assumed to have a complex multivariate normal distribution, are also possible to be obtained in the form of products of independent Beta r.v.'s by using the results in [12, 9].

Furthermore, the use of the method proposed still often enables an immediate extension of the results obtained both in terms of the exact as well as the near-exact distributions developed to the cases where $\underline{X}$ has an elliptically contoured distribution. This is indeed what happens with the block-circularity test, but this was only made clear through the use of the eigenblock and eigenmatrix decomposition, since by using this decomposition it was possible to express the likelihood ratio statistic of this test as the product of independent l.r.t. statistics to test the independence of blocks of variables and equality of covariance matrices, whose distributions under H_0 remain the same for elliptically contoured distributions as well as for the multivariate normal distribution (see Chapters 8–10 of [1]), this way widening in a great deal the results obtained.

Although the use of such constructs is indeed not completely new (see for example [16]), its use in deriving the l.r.t. statistics themselves and its use in obtaining a useful decomposition of the null hypothesis, which induces a concomitant factorization of the c.f. of the negative logarithm of the l.r.t. statistic, much useful in obtaining very precise near-exact approximations for the distributions of the l.r.t. statistics, was never done before, while, on the other hand, in order to be able to fully use and enjoy the entire potential of this method, some systematization was necessary, allowing for a more embracing view of all the capabilities of the method.

An application to the development and study of the l.r.t. statistic to test for block-circularity illustrates well the usefulness and capabilities of this type of constructs or decompositions, further enabling for an almost easy development of very well-fitting near-exact approximations for the distribution of this statistic. At the same time, this application will also be much useful in studying the bridging l.r.t. statistics in subsection 4.5 and obtaining near-exact approximations for their distributions.

These near-exact approximations obtained exhibit a good asymptotic behavior for increasing sample sizes, which for larger values of p and m becomes only evident for larger samples, together with very good performances for very small sample sizes and a marked asymptotic behavior for increasing values of p, the number of blocks of variables, or the number of variables for the non-block version of the test. Although for increasing values of the common dimension of the blocks, m, the near-exact distributions developed are not able to show a so marked asymptotic behavior, they show an outstanding behavior for very small sample sizes for all combinations of values of m and p, behavior that gets even better for larger values of either one of these two parameters, as it is shown for example by the value of the upper-bound on the difference between the exact and the near-exact c.d.f. for the near-exact distribution that matches ten exact moments, for $m = 20$, $p = 15$ and $n = 302$, which is of the order of $2.78{\times}10^{-45}$. To have a sense of the magnitude of this value, we may say that if we commit an error of this magnitude when measuring the diameter of the Milky Way, which, when including the tidal streams, is about 120,000 light-years (with 1 light-year$\approx 9.46{\times}10^{12}$ kilometers), we would commit an error of about $3.16{\times}10^{-15}$ nanometers, which is about $1.43{\times}10^{-14}$ of the diameter of a carbon atom, taking this diameter as being about 0.22 nanometers.

Acknowledgments

This research was partially financially supported by CMA/FCT/UNL, under project PEst-OE/MAT/UI0297/2011.

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