Near-exact Distributions for the Block Equicorrelation and Equivariance Likelihood Ratio Test Statistic

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Abstract. In this paper the authors combine the equicorrelation and equivariance test introduced by Wilks [13] with the likelihood ratio test (l.r.t.) for independence of groups of variables to obtain the l.r.t. of block equicorrelation and equivariance. This test or its single block version may find applications in many areas as in psychology, education, medicine, genetics and they are important "in many tests of multivariate analysis, e.g. in MANOVA, Profile Analysis, Growth Curve analysis, etc" [12, 9]. By decomposing the overall hypothesis into the hypotheses of independence of groups of variables and the hypothesis of equicorrelation and equivariance we are able to obtain the expressions for the overall l.r.t. statistic and its moments. From these we obtain a suitable factorization of the characteristic function (c.f.) of the logarithm of the l.r.t. statistic, which enables us to develop highly manageable and precise near-exact distributions for the test statistic.

Keywords: Characteristic function, Equicorrelation, Equivariance, Independence test, Wilks test
AMS: 62H10, 62H15, 62E20

1. INTRODUCTION

The problem of testing whether a $p{\times}p$ covariance matrix has the equivariance and equicorrelation structure, that is, to test if it may be written as

$$\sigma^2(1-\rho)I_p+\sigma^2\rho E_p$$

where I_p is the identity matrix of order p and E_p is a $p{\times}p$ unitary matrix, was first addressed by Wilks [13], under the multivariate Normal setting.

Later on, Gleser and Olkin [4, Lemma 2.1] have shown that this is equivalent to test

$$H_0:\,\Sigma=\sigma^2(1-\rho)I_p+\sigma^2\rho E_p$$

or

$$H_0:\,\Gamma^T\Sigma\Gamma=diag\,(\sigma_1^2,\underbrace{\sigma_2^2,\dots,\sigma_2^2}_{p-1})$$

where $\sigma_1^2=\sigma^2+\sigma^2\rho(p-1)$, $\sigma_2^2=\sigma^2(1-\rho)$ and Γ is a Helmert orthonormal matrix of dimensions $p{\times}p$ whose first row is proportional to a vector of ones, and as such, only function of p and not of ρ.

This test may find applications in many areas, from psychology and medicine to genetics and it is important “in many tests of multivariate analysis, e.g. in MANOVA, Profile Analysis, Growth Curve analysis, etc”, as SenGupta [9] states.

The exact distribution of the likelihood ratio test (l.r.t.) statistic, under normality, was studied by many authors [13, 11, 7, 5, 8], but it still remains to obtain a manageable and very well-fitting approximation to its exact distribution, which is too elaborate to be used in practice.

In this paper the authors address the l.r.t. of the hypothesis

$$H_0:\,\Sigma=\begin{bmatrix}\Delta_1 & & & & \\ & \ddots & & 0 & \\ & & \Delta_k & & \\ & 0 & & \ddots & \\ & & & & \Delta_m\end{bmatrix} \qquad (1)$$

where

$$\Delta_k=\sigma_k^2\Big((1-\rho_k)I_{p_k}+\rho_k E_{p_k}\Big),\quad k=1,\dots,m, \qquad (2)$$

and develop very well-fitting and highly manageable approximations for the distribution of the test statistic, under the normality assumption.

2. SPLITTING THE NULL HYPOTHESIS

The hypothesis H_0 in (1) may be written as

$$H_{0b|a}\circ H_{0a} \qquad (3)$$

where ‘o’ is to be read as ‘after’ and where

$$H_{0a}:\,\Sigma=\begin{bmatrix}\Delta_1 & & & & \\ & \ddots & & 0 & \\ & & \Delta_k & & \\ & 0 & & \ddots & \\ & & & & \Delta_m\end{bmatrix}$$

where Δ_k is any $p_k{\times}p_k$ positive definite matrix $(k=1,\dots,m)$, is the null hypothesis of independence of the m groups of variables, the k-th of which has p_k variables, and

$$H_{0b|a} \equiv \bigwedge_{k=1}^{m} H_{0b_k|a}\,, \ \text{ assuming } H_{0a}$$

where

$$H_{0b_k|a}: \Delta_k = \sigma_k^2(1-\rho_k)I_{p_k} + \sigma_k^2\rho_k E_{p_k}\,, \qquad k=1,\ldots,m\,.$$

If we assume that Σ is the covariance matrix of

$$\underline{X} = \left[\underline{X}_1,\ldots,\underline{X}_k,\ldots,\underline{X}_m\right]' \sim N_p(\underline{\mu},\Sigma)\,, \tag{4}$$

where

$$\underline{X}_k \sim N_{p_k}(\underline{\mu}_k,\Delta_k)\,, \tag{5}$$

with $p=\sum_{k=1}^{m} p_k$, then the l.r.t. statistic to test H_{0a}, based on a sample of size n, is [1, Chap. 9]

$$\Lambda_a = \left(\frac{|A|}{\prod_{k=1}^{m}|A_k|}\right)^{n/2}\,,$$

where A is the maximum likelihood estimator (m.l.e.) of the covariance matrix of $\underline{X}$ in (4) and A_k is its k-th diagonal block, with (see [3, 6])

$$E\left(\Lambda_a^h\right) = \left\{\prod_{j=2}^{p}\left(\frac{n-j}{n}\right)^{r_j}\left(\frac{n-j}{n}+h\right)^{-r_j}\right\} \times\left(\frac{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{n-2}{2}+\frac{n}{2}h\right)}{\Gamma\left(\frac{n-2}{2}\right)\Gamma\left(\frac{n-1}{2}+\frac{n}{2}h\right)}\right)^{k^*} \tag{6}$$

where $k^*=\lfloor \ell/2\rfloor$, with ℓ denoting the number of $\underline{X}_k$'s with an odd number of variables, and

$$r_j = \begin{cases} 0\,, & j=2\\ h_{j-2}+(-1)^j k^*\,, & j=3,4\\ r_{j-2}+h_{j-2}\,, & j=5,\ldots,p\,, \end{cases} \tag{7}$$

where

$$h_j = (\#\text{ of } p_k\,(k=1,\ldots,m)\geq j)-1\,, \quad j=1,\ldots,p-2\,. \tag{8}$$

Then, following Lemma 2.1 in [4], the l.r.t. statistic to test $H_{0b_k|a}$, based on a sample of size n, may be written as

$$\Lambda_{b_k} = \left(\frac{|A_k|}{a_{1k}\left(tr\,\frac{A_{k2}}{p-1}\right)^{p-1}}\right)^{n/2}\,,$$

where A_k is the m.l.e. of Δ_k, being a_{1k} the element in the first row and first column of A_k and A_{k2} the diagonal block of dimension $(p_k-1)\times(p_k-1)$ that follows a_{1k}.

The h-th moment of Λ_{b_k} may then be written as

$$E\left(\Lambda_{b_k}^h\right) = \prod_{j=2}^{p_k}\frac{\Gamma\left(\frac{n-1}{2}+\frac{j-2}{p_k-1}\right)}{\Gamma\left(\frac{n-j}{2}\right)}\frac{\Gamma\left(\frac{n-j}{2}+\frac{n}{2}h\right)}{\Gamma\left(\frac{n-1}{2}+\frac{j-2}{p_k-1}+\frac{n}{2}h\right)}$$

which matches expression (1.10) in [13], and which, after some analytical work may be written as

$$\begin{aligned} E\left(\Lambda_{b_k}^h\right) = &\left\{\prod_{j=2}^{p_k}\left(\frac{n-j}{n}\right)^{r_{jk}}\left(\frac{n-j}{n}+h\right)^{-r_{jk}}\right\}\\ &\times\left\{\prod_{j=2}^{p_k}\frac{\Gamma\left(\frac{n-j}{2}+\frac{j-2}{p_k-1}+\frac{j-1}{2}\right)}{\Gamma\left(\frac{n-j}{2}+\left\lfloor\frac{j-2}{p_k-1}+\frac{j-1}{2}\right\rfloor\right)}\right.\\ &\left.\times\frac{\Gamma\left(\frac{n-j}{2}+\left\lfloor\frac{j-2}{p_k-1}+\frac{j-1}{2}\right\rfloor+\frac{n}{2}h\right)}{\Gamma\left(\frac{n-j}{2}+\frac{j-2}{p_k-1}+\frac{j-1}{2}+\frac{n}{2}h\right)}\right\}\end{aligned}$$

where

$$r_{jk} = \begin{cases} \left\lfloor\frac{p_k}{4}\right\rfloor, & j=2\\ \left\lfloor\frac{p_k-j+2}{2}\right\rfloor, & j=3,\ldots,p_k\,. \end{cases}$$

Then, the l.r.t. statistic to test $H_{0b|a}$ is,

$$\Lambda_{b|a} = \prod_{k=1}^{m}\Lambda_{b_k}\,,$$

where all Λ_{b_k} $(k=1,\ldots,m)$ are independent, and as such, with

$$\begin{aligned} E\left(\Lambda_{b|a}^h\right) &= \prod_{k=1}^{m} E\left(\Lambda_{b_k}^r\right)\\ &= \prod_{k=1}^{m}\left[\left\{\prod_{j=2}^{p_k}\left(\frac{n-j}{n}\right)^{r_{jk}}\left(\frac{n-j}{n}+h\right)^{-r_{jk}}\right\}\right.\\ &\quad\times\left\{\prod_{j=2}^{p_k}\frac{\Gamma\left(\frac{n-j}{2}+\frac{j-2}{p_k-1}+\frac{j-1}{2}\right)}{\Gamma\left(\frac{n-j}{2}+\left\lfloor\frac{j-2}{p_k-1}+\frac{j-1}{2}\right\rfloor\right)}\right.\\ &\quad\left.\left.\times\frac{\Gamma\left(\frac{n-j}{2}+\left\lfloor\frac{j-2}{p_k-1}+\frac{j-1}{2}\right\rfloor+\frac{n}{2}h\right)}{\Gamma\left(\frac{n-j}{2}+\frac{j-2}{p_k-1}+\frac{j-1}{2}+\frac{n}{2}h\right)}\right\}\right].\end{aligned} \tag{9}$$

The l.r.t. statistic to test H_0 in (1) is then

$$\Lambda = \Lambda_a\,\Lambda_{b|a}\,,$$

where, given the way H_0 was decomposed in (3), and given the independence between Λ_a and the matrices A_k $(k=1,\ldots,m)$, Λ_a and $\Lambda_{b|a}$ are independent. As such, we have

$$E\left(\Lambda^h\right) = E\left(\Lambda_a^h\right)E\left(\Lambda_{b|a}^h\right)\,, \tag{10}$$

and, as such, if we take $W=-\log\Lambda$, we have, from (6), (9) and (10), the c.f. of W given by

$$\Phi_W(t) = E\left(\Lambda^{-\mathrm{i}t}\right)$$
$$= \underbrace{\left\{\prod_{j=2}^{p}\left(\frac{n-j}{n}\right)^{r_j^*}\left(\frac{n-j}{n}-\mathrm{i}t\right)^{-r_j^*}\right\}}_{\Phi_1(t)}$$
$$\times \underbrace{\left(\frac{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{n-2}{2}-\frac{n}{2}\mathrm{i}t\right)}{\Gamma\left(\frac{n-2}{2}\right)\Gamma\left(\frac{n-1}{2}-\frac{n}{2}\mathrm{i}t\right)}\right)^{k^*+m}}_{\Phi_2(t)}$$
$$\times \underbrace{\prod_{k=1}^{m}\left\{\prod_{j=3}^{p_k}\frac{\Gamma\left(\frac{n-j}{2}+\frac{j-2}{p_k-1}+\frac{j-1}{2}\right)}{\Gamma\left(\frac{n-j}{2}+\left\lfloor\frac{j-2}{p_k-1}+\frac{j-1}{2}\right\rfloor\right)} \times \frac{\Gamma\left(\frac{n-j}{2}+\left\lfloor\frac{j-2}{p_k-1}+\frac{j-1}{2}\right\rfloor-\frac{n}{2}\mathrm{i}t\right)}{\Gamma\left(\frac{n-j}{2}+\frac{j-2}{p_k-1}+\frac{j-1}{2}-\frac{n}{2}\mathrm{i}t\right)}\right\},}_{\Phi_3(t)} \quad (11)$$

where

$$r_j^* = \begin{cases} \sum_{k=1}^{m}\lfloor p_k/4\rfloor, & j=2 \\ r_j + \sum_{k=1}^{m} I_{\{p_k\geq j\}}\left\lfloor\frac{p_k-j+2}{2}\right\rfloor, & j=3,\ldots,\max p_k \\ r_j, & j=1+\max p_k,\ldots,p \end{cases}$$

for r_j given by (7) and (8) and $k^{**}=\lfloor \ell/2\rfloor+m$, with $I_{\{A\}}$ being the indicator function of the condition A, that is, a function that evaluates to 1 if the condition holds and to zero in the opposite case.

3. NEAR-EXACT DISTRIBUTIONS

In order to build near-exact distributions for W and Λ, we will then leave $\Phi_1(t)$ in (11) unchanged and we will replace $\Phi_2(t)\Phi_3(t)$, in (11), by

$$\Phi^*(t) = \sum_{\ell=0}^{m^*}\pi_\ell\,\lambda^{r+\ell}(\lambda-\mathrm{i}t)^{-(r+\ell)} \quad (12)$$

where, for k^* in (6),

$$r = \frac{m+k^*}{2} + \sum_{k=1}^{m}\sum_{j=3}^{p_k}\frac{j-2}{p_k-1}+\frac{j-1}{2}-\left\lfloor\frac{j-2}{p_k-1}+\frac{j-1}{2}\right\rfloor \quad (13)$$

which is the sum of all the second parameters of the Logbeta distributions in $\Phi_2(t)\Phi_3(t)$ in (11).

The choice of $\Phi^*(t)$ in (12) as an asymptotic replacement for $\Phi_2(t)\Phi_3(t)$ in (11) is based on the fact that $\Phi_2(t)\Phi_3(t)$ is the c.f. of a sum of k^*+m independent Logbeta r.v.'s with parameters $(n-2)/2$ and $1/2$, with another independent sum of $p=\sum_{k=1}^{m}p_k$ independent Logbeta r.v.'s with parameters $(n-j)/2+\lfloor(j-2)/(p_k-1)+(j-1)/2\rfloor$ and $(j-2)/(p_k-1)+(j-1)/2-\lfloor(j-2)/(p_k-1)+(j-1)/2\rfloor$ $(j=1,\ldots,p_k;k=1,\ldots,m)$ and the results from Tricomi and Erdélyi, in [10], which show that the c.f. of any *Logbeta*(a,b) r.v. may be asymptotically replaced by the c.f. of an infinite mixture of $\Gamma(b+\ell,a)$ $(\ell=0,1,\ldots)$ distributions.

The parameter λ in (12) is then taken as the rate parameter in

$$\Phi^{**}(t) = \theta\lambda^{s_1}(\lambda-\mathrm{i}t)^{-s_1}+(1-\theta)\lambda^{s_2}(\lambda-\mathrm{i}t)^{-s_2}$$

where θ, λ, s_1 and s_2 are determined in such a way that

$$\left.\frac{d(\Phi_2(t)\Phi_3(y))}{dt^h}\right|_{t=0} = \left.\frac{d\Phi^{**}(t)}{dt^h}\right|_{t=0} \quad \text{for} \quad h=1,\ldots,4.$$

The weights π_ℓ $(\ell=0,\ldots,m^*-1)$ in (12) will then be determined in such a way that

$$\left.\frac{d(\Phi_2(t)\Phi_3(t))}{dt^h}\right|_{t=0} = \left.\frac{d\Phi^{*}(t)}{dt^h}\right|_{t=0} \quad \text{for} \quad h=1,\ldots,m^*,$$

with $\pi_{m^*}=1-\sum_{\ell=0}^{m^*-1}\pi_\ell$.

This procedure yields near-exact distributions for W which have c.f.

$$\Phi_1(t)\Phi^*(t),$$

with $\Phi_1(t)$ given by (11) and $\Phi^*(t)$ by (12), where r, given by (13) is always either an integer or a half-integer, since, for $p_k>1$,

$$\sum_{j=3}^{p_k}\left(\frac{j-2}{p_k-1}+\frac{j-1}{2}-\left\lfloor\frac{j-2}{p_k-1}+\frac{j-1}{2}\right\rfloor\right) = \frac{p_k-3}{2}+\frac{1}{2}\left\lfloor\frac{Mod(p_k,4)}{2}\right\rfloor.$$

As such, the near-exact distributions developed yield, for W, distributions which, for non-integer r, are mixtures, with weights p_k $(k=0,\ldots,m^*)$, of m^*+1 Generalized Near-Integer Gamma (GNIG) distributions of depth p with integer shape parameters r_j^* $(j=2,\ldots,p)$ and real shape parameter r and corresponding rate parameters $(n-j)/n$ $(j=2,\ldots,p)$ and λ, and which, for integer r, are similar mixtures but of Generalized Integer Gamma (GIG) distributions, with the same shape and rate parameters (see [2, 3] and Appendix A for further details on the GIG and GNIG distributions and their probability density and cumulative distribution functions).

Using the notation in Appendix A, the near-exact distributions obtained for W, for the case of non-integer r, will have probability density and cumulative distribution functions respectively of the form

$$f_W^*(w) = \sum_{\ell=0}^{m^*}\pi_\ell\, f^{GNIG}\left(w \mid r_2^*,\ldots,r_p^*;r+\ell;\ \frac{n-2}{n},\ldots,\frac{n-p}{n};\lambda;p\right), \quad w>0$$

and

$$F^*_W(w) = \sum_{\ell=0}^{m^*} \pi_\ell \, F^{GNIG}\Big(w \,|\, r_2^*, \dots, r_p^*; r+\ell; \frac{n-2}{n}, \dots, \frac{n-p}{n}; \lambda; p\Big), \quad w>0,$$

while the near-exact probability density and cumulative distribution functions of Λ are respectively given by

$$f^*_\Lambda(z) = \sum_{\ell=0}^{m^*} \pi_\ell \, f^{GNIG}\Big(-\log z \,|\, r_2^*, \dots, r_p^*; r+\ell; \frac{n-2}{n}, \dots, \frac{n-p}{n}; \lambda; p\Big)\frac{1}{z}, \quad 0<z<1$$

and

$$F^*_\Lambda(z) = \sum_{\ell=0}^{m^*} \pi_\ell \Big(1 - F^{GNIG}\Big(-\log z \,|\, r_2^*, \dots, r_p^*; r+\ell; \frac{n-2}{n}, \dots, \frac{n-p}{n}; \lambda; p\Big)\Big), \quad 0<z<1 .$$

For integer r, all we have to do is to replace the GNIG probability density and cumulative distribution functions by their GIG counterparts.

4. NUMERICAL STUDIES

In order to assess the performance of the near-exact distributions developed we will use

$$\Delta = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left| \frac{\Phi_W(t) - \Phi_1(t)\Phi^*(t)}{t} \right| dt$$

with

$$\Delta \geq \max_w \left| F_W(w) - F^*_W(w) \right| ,$$

as a measure of proximity between the exact and the near-exact distributions, where $\Phi_W(t)$ is the exact c.f. of W in (11) and $F_W(\cdot)$ and $F^*_W(\cdot)$ represent respectively the exact and near-exact cumulative distribution functions of W, corresponding respectively to $\Phi_W(t)$ and $\Phi_1(t)\Phi^*(t)$.

In Table 1 we may analyze values of Δ for different combinations of p_k and different sample sizes. Smaller values of Δ indicate a closer agreement with the exact distribution and as such, a better performance. Anyway, even for very small sample sizes, that is, for sample sizes hardly exceeding the total number of variables involved, the near-exact distributions provide very sharp approximations to the exact distribution, with upper bounds on the difference between the exact and near-exact c.d.f.'s of the order of 10^{-16} or smaller.

TABLE 1. Values of Δ for different combinations of p_k values and different sample sizes

p_k	p	n	Δ	
			$m^*=4$	$m^*=6$
{3,5,9,6}	23	25	1.47×10^{-16}	7.33×10^{-20}
		125	3.59×10^{-18}	6.02×10^{-23}
		225	2.33×10^{-19}	1.12×10^{-24}
{3,5,4,5,6}	23	25	6.76×10^{-16}	9.42×10^{-20}
		125	3.29×10^{-18}	8.52×10^{-23}
		225	1.80×10^{-19}	1.59×10^{-24}
{8,10,14,11}	43	45	4.42×10^{-19}	4.42×10^{-24}
		145	1.90×10^{-19}	1.92×10^{-25}
		245	1.96×10^{-20}	6.64×10^{-27}
{7,6,8,9,13}	43	45	1.24×10^{-18}	6.32×10^{-24}
		145	2.54×10^{-19}	2.58×10^{-25}
		245	2.51×10^{-20}	8.85×10^{-27}
{18,20,24,23}	85	87	6.44×10^{-21}	3.47×10^{-28}
		187	3.76×10^{-21}	1.73×10^{-28}
		287	4.83×10^{-22}	9.87×10^{-30}
{12,14,16,19,24}	85	87	4.51×10^{-21}	1.51×10^{-28}
		187	1.96×10^{-21}	4.11×10^{-30}
		287	1.88×10^{-22}	4.42×10^{-30}

5. CONCLUSIONS

From the results of numerical studies carried out we see that the near-exact distributions developed show an interesting set of nice features. They not only have a good asymptotic behavior for increasing sample sizes but also an extraordinary performance for very small sample sizes, as for example for sample sizes exceeding only by 2 the overall number of variables. Furthermore, these near-exact distributions also display a marked asymptotic behavior for increasing values of p_k, and consequently also of p and similar behaviors for different numbers of sets of variables, for a given value of p, with a somewhat slight asymptotic behavior for increasing numbers of sets of variables, for the larger values of p, which is another interesting feature. All these features add up to make the near-exact approximations developed the best choice for practical applications of the test studied.

A similar procedure to the one used may be applied to the case where the random vector $\underline{X}$ has a complex multivariate Normal distribution.

For $m = 1$, the present test reduces to the equivariance-equicorrelation Wilks [13] test.

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A. THE GIG AND GNIG DISTRIBUTIONS

We will say that a r.v. Y has a GIG (Generalized Integer Gamma) distribution of depth p, with integer shape parameters r_j and rate parameters λ_j ($j = 1,\ldots,p$), if

$$Y = \sum_{j=1}^{p} Y_j$$

where

$$Y_j \sim \Gamma(r_j,\lambda_j), \quad r_j \in I\!N,\ \lambda_j > 0,\ j = 1,\ldots,p$$

are p independent integer Gamma or Erlang r.v.'s, with $\lambda_j \neq \lambda_{j'}$ for all $j \neq j'$, with $j, j' \in \{1,\ldots,p\}$ [2].

The r.v. Y has p.d.f. and c.d.f. given by (see [2]),

$$f^{GIG}(y;r_j,\lambda_j;p) = K\sum_{j=1}^{p} P_j(y)\,e^{-\lambda_j y}, \quad (y>0)$$

and

$$F^{GIG}(y;r_j,\lambda_j;p) = 1-K\sum_{j=1}^{p} P_j^*(y)\,e^{-\lambda_j y}, \quad (y>0)$$

where $K = \prod_{j=1}^{p}\lambda_j^{r_j}$,

$$P_j(y)=\sum_{k=1}^{r_j} c_{j,k}\,y^{k-1}\,, \quad P_j^*(y)=\sum_{k=1}^{r_j} c_{j,k}\sum_{i=0}^{k-1}\frac{y^i\,(k-1)!}{i!\,\lambda_j^{k-i}}$$

with

$$c_{j,r_j} = \frac{1}{(r_j-1)!}\prod_{i=1,i\neq j}^{p}(\lambda_i-\lambda_j)^{-r_i}\,, \qquad j=1,\ldots,p,$$

and, for $k = 1,\ldots,r_j-1$; $j = 1,\ldots,p$,

$$c_{j,r_j-k} = \frac{1}{k}\sum_{i=1}^{k}\frac{(r_j-k+i-1)!}{(r_j-k-1)!}R(i,j,p)\,c_{j,r_j-(k-i)}\,,$$

where

$$R(i,j,p) = \sum_{k=1,k\neq j}^{p} r_k\left(\lambda_j-\lambda_k\right)^{-i} \quad (i=1,\ldots,r_j-1).$$

If Y_p has a Gamma distribution with a non-integer shape parameter r_p, then we will say that the r.v. Y has a GNIG (Generalized Near-Integer Gamma) distribution of depth p. The p.d.f. and c.d.f. of Y are, for $y>0$, respectively given by [3]

$$f^{GNIG}(y\,|\,r_1,\ldots,r_{p-1};r_p;\,\lambda_1,\ldots,\lambda_{p-1};\lambda_p;p) = K\lambda_p^{r_p}\sum_{j=1}^{p-1}e^{-\lambda_j y}\sum_{k=1}^{r_j}\left\{c_{j,k}\frac{\Gamma(k)}{\Gamma(k+r)}y^{k+r_p-1}\;{}_1F_1(r_p,k+r_p,-(\lambda_p-\lambda_j)y)\right\},$$

and

$$F^{GNIG}(y\,|\,r_1,\ldots,r_{p-1};r_p;\,\lambda_1,\ldots,\lambda_{p-1};\lambda_p;p) = \frac{\lambda_p^{r_p}\,z^{r_p}}{\Gamma(r_p+1)}\,{}_1F_1(r_p,r_p+1,-\lambda_p z) - K\lambda^r\sum_{j=1}^{p-1}e^{-\lambda_j y}\sum_{k=1}^{r_j}\frac{c_{j,k}\Gamma(k)}{\lambda_j^k}\sum_{i=0}^{k-1}\frac{z^{r_p+i}\lambda_j^i}{\Gamma(r_p+1+i)}\;{}_1F_1(r_p,r_p+1+i,-(\lambda_p-\lambda_j)y)\,,$$

with $K = \prod_{j=1}^{p-1}\lambda_j^{r_j}$.

REFERENCES

1. T. W. Anderson, *An Introduction to Multivariate Statistical Analysis*, 3rd ed., J. Wiley & Sons, New York, 2003, pp. 381-384.
2. C. A. Coelho, The Generalized Integer Gamma distribution – a basis for distributions in Multivariate Statistics. *Journal of Multivariate Analysis*, **64**, 86–102 (1998).
3. C. A. Coelho, The generalized near-integer Gamma distribution: a basis for 'near-exact' approximations to the distribution of statistics which are the product of an odd number of independent Beta random variables. *J. Multivariate Anal.*, **89**, 191-218 (2004).
4. L. J. Gleser and L. Olkin, Testing for equality of means, equality of variances and equality of covariances under restrictions upon the parameter space. *Ann. Inst. Statist. Math.*, **21**, 33-48 (1969).
5. C. G. Khatri and M. S. Srivastava, On the exact non-null distribution of likelihood ratio criteria for covariance matrices. *Ann. Inst. Statist. Math.*, **25**, 345-354 (1973).
6. F. J. Marques, C. A. Coelho and B. C. Arnold, A general near-exact distribution theory for the most common likelihood ratio test statistics used in Multivariate Analysis. *Test*, **20**, 180-203 (2011).
7. B. N. Nagarsenker, Percentage points of Wilks' L_{vc} criterion. *Comm. Statist.*, **4**, 629-641 (1975).
8. K. C. S. Pillai and A. Singh, On the exact non-null distribution of Wilks' L_{vc} criterion and power studies. *Ann. Inst. Statist. Math.*, **33**, 45-55 (1981).
9. A. Sen Gupta, On tests for equicorrelation coefficient and the generalized variance of a standard symmetric multivariate Normal distribution. *Technical Report No. 181, Department of Statistics, Stanford University*, U.S.A. (1982).
10. F. G. Tricomi, A. Erdélyi, The asymptotic expansion of a ratio of Gamma functions. *Pacific J. Math.* **1**, 133-142 (1951).
11. K. B. Varma, On the exact distribution of Wilks' L_{mvc} and L_{vc} criteria. *Proc. Inst. Int. Statist. Conf.*, 181-214 (1951).
12. D. F. Votaw, Testing compound symmetry in a Normal multivariate distribution. *Ann. Math. Statist.*, **19**, 447-473 (1948).
13. S. S. Wilks, Sample criteria for testing equality of means, equality of variances, and equality of covariances in a Normal multivariate distribution. *Ann. Math. Statist.* **17**, 257-281 (1946).