A note on the regularity of flows with shear-dependent viscosity

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Abstract

We consider a non Newtonian fluid governed by stationary, incompressible Navier-Stokes equations with shear-dependent viscosity. Using a fixed point argument in an appropriate functional setting, we establish the existence of a strong solution for small and suitably regular data. Uniqueness results are obtained under similar conditions.

Key words. Generalized Newtonian fluid, shear-dependent viscosity, strong solution.

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1 Introduction

The aim of this paper is to establish an existence and regularity result for solutions to the steady Navier-Stokes equations for flows with shear dependent viscosity, namely

$$\begin{cases} -\nabla \cdot (S(Du)) + u \cdot \nabla u + \nabla \pi = f & \text{in } \Omega, \\ \nabla \cdot u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where u is the velocity field, S the extra stress tensor, $Du = \frac{1}{2}\left(\nabla u + (\nabla u)^T\right)$ the symmetric part of the velocity gradient ∇u, π the pressure, f a given body force, and $\Omega \subset I\!R^n$ is a bounded domain. To simplify the redaction, we assume that S is a classical power law stress tensor of the form

$$S(\eta) = 2\nu \left(1 + |\eta|^2\right)^{\frac{\alpha-2}{2}} \eta \qquad \text{or} \qquad S(\eta) = 2\nu \left(1 + |\eta|\right)^{\alpha-2} \eta, \tag{1.2}$$

where ν and α positive constants with $\alpha > 1$. The system (1.1) is nowadays classical. If $\alpha = 2$, it is reduced to the Navier-Stokes system with the classical no-slip boundary conditions. The fluid is called shear thinning if $\alpha < 2$, and shear thickening if $\alpha > 2$. This class of fluids was first proposed by Ladyzhenskaya in [20], [21] and [22] as a modification of the Navier-Stokes system (the

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viscosity depending on the shear-rate), and was similarly suggested by Lions in [23]. Existence of weak solutions was proved by both authors using compactness arguments and the theory of monotone operators. Extensive work has been done since then and various existence and regularity properties have been established. We emphasize the work by Nečas *et al.* ([24] and [7]), and by Frehse, Málek, Steinhauer [14] and Růžička [26] who later established existence of weak solutions for exponents satisfying the less restrictive condition $\alpha > \frac{2n}{n+2}$.

Despite the fact that system (1.1) was widely studied there are still many open problems, especially concerning the regularity of weak solutions. Related to this aspect, and without any ambition of completeness, we refer to [19], [24], [25], [3], [5], [6], [12], [4], [8], [10], [13] and the references therein. Higher global regularity of solutions is difficult to obtain in general and there are only few such results known up to nowadays. For shear-thinning fluids, global regularity results up to the boundary have been obtained by Kaplický, Málek and Stará in the two-dimensional framework in [19] for $\frac{3}{2} < \alpha < 2$, without restrictions on the data. The authors prove that if $f \in L^{\frac{\alpha}{\alpha-1}}(\Omega)$, then there exists a solution $u \in W^{2,p}(\Omega)$ for some $p > 2$. In the three-dimensional case, the most significant result is due to Beirão da Veiga in [5]. In the case of the generalized Stokes systems (corresponding to (1.1) without the convective term), the author proves that if $\frac{3}{2} < \alpha < 2$ and $f \in L^{\frac{\alpha}{\alpha-1}}(\Omega)$, then a weak solution u belongs to $W^{1,4\alpha-2}(\Omega) \cap W^{2,\frac{4\alpha-2}{\alpha+1}}(\Omega)$. The same regularity is achieved in the case of generalized Navier-Stokes equations provided that $\alpha > \frac{20}{11}$. Concerning the shear-thickening fluids ($\alpha > 2$) we refer to [25] and [3] where global regularity results have been obtained for $n = 3$ without restricting the data. We also mention [6] where the authors improve and extend the results obtained in [4], establishing that $u \in W^{1,p} \cap W^{2,\frac{2p}{\alpha+p-2}}$ for any $p < +\infty$ if $n = 2$ and for $p = \frac{n\alpha+2-\alpha}{n-2}$ if $n \geq 3$.

In [10], Crispo and Grisanti consider problem (1.1) with the Lipschitz continuous extra stress tensor $(1.2)_2$ and with $1 < \alpha < 2$. They prove that if $f \in L^q(\Omega)$ with $q > n$, then there exists a $C^{1,\gamma}(\overline{\Omega})$ solution. Uniqueness is guaranteed if the solution belongs to $L^{\frac{2\alpha}{\alpha-1}}(\Omega)$ (which is the case if $\alpha \geq \frac{3n}{n+2}$) and existence of a $W^{2,2}(\Omega)$ solution is obtained if $q > 2n$. These results are achieved using the sucessive approximation method and applying a Hölder regularity result for solutions of elliptic systems due to Giaquinta and Modica [17]. The method is based on fixed point arguments and converges under restriction on the size of the data. The same technique has been used in [11] in the case of electrorheological fluids.

The aim of this paper is to establish similar results in both two-dimensional and three-dimensional case, for shear-thinning and shear-thickening flows. Assuming that the force belongs to $L^q(\Omega)$ with $q > n$, we prove existence of a strong solution $u \in W^{2,q}(\Omega)$ under precise conditions on the data. The method based on classical regularity results for the Stokes problem and on the Banach fixed-point technique, is much easier to handle . We consider both C^1 and Lipchitz continuous extra stress tensors. To deal with the supplementary difficulty induced by $|Du|$ in the second case, we introduce a family of approximate problems, establish existence of approximate strong solutions, derive corresponding uniform estimates and pass to the limit. Uniqueness of a solution is obtained for $\alpha \geq \frac{3n}{n+2}$.

This useful result is needed in many applications and particularly when studying optimal control problems of non-Newtonian fluids governed by this class of partial differential equations. It guarantees the boundedness of the velocity gradient appearing in the coefficients in the main part of the differential operator for the linearized and the adjoint equations, and is usually necessary to deal with the corresponding optimality conditions (see [29] and [27]).

The plan of the paper is as follows. Assumptions, notation and statement of the main results are given in Section 2. Section 3 is devoted to the existence of a strong solution in the case of the differentiable stress tensor $(1.2)_1$. Section 4 deals with the case of the Lipschitz continuous stress tensor $(1.2)_2$. An approximate family is considered, an existence result is obtained using a fixed point argument, precise estimates for the approximate solution are derived and convergence results established. In Section 5, we prove that under precise conditions on the data, the obtained strong solution coincide with weak solutions. Finally, in Section 6, we recall an auxiliary result related with the Stokes system and derive estimates related with the nonlinear terms appearing in our problems.

2 Notation and statement of the main result

In all that follows Ω is a bounded domain in $I\!R^n$ ($n = 2$ or $n = 3$). The boundary of Ω is denoted by $\partial\Omega$ and is of class C^2. Since many of the quantities occuring in the paper are vector-valued functions, the notation will be abreged for the sake of brevity and we will use the same notation of norms for scalar, vector and matrix-valued functions. For $x,\ y \in I\!R$, we set $(x,y)^+ = \max(x,y)$ and $x^+ = \max(x,0)$. For $\alpha > 1$, we define the following four constants that will be used throughout the paper

$$s_\alpha = (|\alpha-2|,2)^+\,, \qquad \bar{s}_\alpha = (|\alpha-2|,1)^+\,2^{(\alpha-3)^+}\,,$$

$$r_\alpha = \tfrac{1+(\alpha-3)^+-(\alpha-4)^+}{2}, \qquad \gamma_\alpha = \tfrac{((\alpha,3)^+-2)^{(\alpha,3)^+-2}}{((\alpha,3)^+-1)^{(\alpha,3)^+-1}}.$$

For $m \in I\!N$ and $1 < p < \infty$, the standard Sobolev spaces are denoted by $W^{m,p}(\Omega)$ and their norms by $\|\cdot\|_{m,p}$. We set $W^{0,p}(\Omega) \equiv L^p(\Omega)$ and $\|\cdot\|_{L^p} \equiv \|\cdot\|_p$, and we also define the space

$$V^{m,p} = \Big\{ v \in W^{1,p}_0(\Omega) \cap W^{m,p}(\Omega) \mid \nabla\cdot v = 0 \text{ in } \Omega \Big\}\,,$$

equipped with the usual norm $\|\cdot\|_{m,p}$. We finally denote by $C^{m,\gamma}(\overline{\Omega})$ ($m \in I\!N$ and $0 < \gamma < 1$) the subspace of functions $u \in C^m(\overline{\Omega})$ such that

$$\sum_{|h|=m} \sup_{\substack{x,y\in\overline{\Omega}\\ x\neq y}} \tfrac{|D^h u(x) - D^h u(y)|}{|x-y|^\gamma} < +\infty.$$

The first main result deals with existence of a $W^{2,q}$ strong solution for problem (1.1) in the case of the differentiable extra stress tensor $(1.2)_1$. The proof is achieved by rewriting the problem in an equivalent form and using a fixed point argument.

Theorem 2.1 *Let f be in $L^q(\Omega)$ with $q > n$ and let S be given by $S(\eta) = 2\nu(1 + |\eta|^2)^{\frac{\alpha-2}{2}}\eta$ with $\nu > 0$ and $\alpha > 1$. There exists a positive constant κ depending only on n, q and Ω such that, if*

$$\tfrac{\kappa^2\|f\|_q}{\nu^2} + \kappa s_\alpha \big(\tfrac{\kappa\|f\|_q}{\nu}\big)^{1+(\alpha-3)^+-(\alpha-4)^+}\big(1 + \tfrac{\kappa\|f\|_q}{\nu}\big)^{(\alpha-4)^+} < \tfrac{1}{4^{(\alpha-2,1)^+}} \tag{2.1}$$

then problem (1.1) *admits a solution $u \in V^{2,q}$. Moreover, the following estimate holds*

$$\|u\|_{2,q} \leq \tfrac{C^*\|f\|_q}{\nu} \tag{2.2}$$

with $C^* \equiv C^*(n, q, \Omega)$.

The second main theorem is concerned with existence of a $W^{2,q}$ strong solution for problem (1.1) in the case of the extra stress tensor $(1.2)_2$. Unlike the first case where S depends on the differentiable term $|Du|^2$, we have to deal here with the merely Lipschitz continuous term $|Du|$. To overcome this difficulty, we introduce a family of approximate problems that fall into the differentiable case, prove existence of approximate solutions, establish uniform estimates and pass to the limit.

Theorem 2.2 *Let f be in $L^q(\Omega)$ with $q > n$ and let S be given by $S(\eta) = 2\nu(1 + |\eta|)^{\alpha-2}\eta$ with $\nu > 0$ and $\alpha > 1$. There exists a positive constant $\bar{\kappa}$ depending only on n, q and Ω such that, if*

$$\tfrac{\bar{\kappa}^2\|f\|_q}{\nu^2} + \bar{s}_\alpha \tfrac{\bar{\kappa}^2\|f\|_q}{\nu}\big(1 + \tfrac{\bar{\kappa}\|f\|_q}{\nu}\big)^{(\alpha-3)^+} < \tfrac{1}{4^{(\alpha-2,1)^+}} \tag{2.3}$$

then problem (1.1) *admits a solution $u \in V^{2,q}$. Moreover, the following estimate holds*

$$\|u\|_{2,q} \leq \tfrac{\bar{C}^*\|f\|_q}{\nu} \tag{2.4}$$

with $\bar{C}^* \equiv \bar{C}^*(n, q, \Omega)$.

The results stated in Theorem 2.1 and Theorem 2.2 establish existence of a strong solution in a certain ball, and do not imply that any weak solution is a $W^{2,q}$ solution. Nevertheless, as stated in the next two results, the weak solution coincides with the strong solution if the term $\frac{\|f\|_q}{\nu}$ is small enough.

Theorem 2.3 *Assume that $\alpha \geq 2$. Let v be a weak solution of problem* (1.1) *and let u be the strong solution given by Theorem* 2.1 (*or by Theorem* 2.2)*. If the following condition*

$$\tfrac{\|f\|_2}{\nu^2} < \tfrac{\sqrt{n^3}}{(n-1)^2|\Omega|^{\frac{1}{n-1}}} \tag{2.5}$$

holds, then u and v coincide.

Theorem 2.4 *Assume that* $\frac{3n}{n+2} \le \alpha < 2$. *Let* v *be a weak solution of problem* (1.1) *and let* u *be the strong solution given by Theorem* 2.1 (*or by Theorem* 2.2). *There exists a positive constant* $\kappa \equiv \kappa(n, q, \alpha, \Omega)$ *such that if the following condition*

$$\kappa_\alpha \left(1 + \frac{\|f\|_2}{\nu}\right)^{\frac{2-\alpha}{\alpha-1}} \frac{\|f\|_q}{\nu^2} < 1 \tag{2.6}$$

holds, then u *and* v *coincide.*

Remark 2.5 *Conditions* (2.1), (2.3), (2.5) *and* (2.6) *are fulfilled if the term* $\frac{\|f\|_q}{\nu}$ *is "small enough", and can be interpreted either as a constraint on the size of* $\|f\|_q$ (*small body force* f) *or as a restriction on the viscosity parameter* ν (*large viscosity parameter* ν).

Remark 2.6 *Due to compactness results on Sobolev spaces, we deduce that a* $W^{2,q}$ *strong solution belongs to* $C^{1,\gamma}(\overline{\Omega})$ *for every* $\gamma < 1 - \frac{n}{q}$. *Moreover, by taking into account* (2.2) *and* (2.4), *we have*

$$\|u\|_{C^{1,\gamma}(\overline{\Omega})} \le C \frac{\|f\|_q}{\nu}.$$

3 Proof of Theorem 2.1

Existence of a strong solution to system (1.1) is proved by applying a Banach fixed point theorem. Toward this aim, we first reformulate the original problem as

$$\begin{cases} -\nu\Delta u + \nabla\pi = f - u \cdot \nabla u + \nabla \cdot \left(2\nu\sigma\left(|Du|^2\right) Du\right) & \text{in } \Omega, \\ \nabla \cdot u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with $\sigma(x) = (1+x)^{\frac{\alpha-2}{2}} - 1$. Next, we define the mapping

$$\mathcal{A}: \ \zeta \ \longrightarrow \ u,$$

through the Stokes system

$$\begin{cases} -\nu\Delta u + \nabla\pi = f - \zeta \cdot \nabla\zeta + \nabla \cdot \left(2\nu\sigma\left(|D\zeta|^2\right) D\zeta\right) & \text{in } \Omega, \\ \nabla \cdot u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

and we look for the solution as a fixed point for $\mathcal{A}$. This approach was already used in [28] to study problems of Oldroyd and second-grade types in the Banach spaces $W^{2,q}$ ($q > n$). Here we carry out a careful analysis to obtain sharp estimates. This is particularly interesting to show the combined effect of the force f and the viscosity parameter ν appearing in the model.

For every $\delta > 0$, let $B(\delta)$ be the convex set defined by

$$B(\delta) = \left\{ \zeta \in V^{2,q} \mid C_E \|\nabla \zeta\|_{1,q} \leq \delta \right\}, \tag{3.2}$$

where C_E is the norm of the embedding of $W^{1,q}(\Omega)$ into $L^\infty(\Omega)$. Our aim is to prove that if $\frac{\|f\|_q}{\nu}$ is small enough, then $\mathcal{A}$ maps $B(\delta_0)$ into $B(\delta_0)$ for some $\delta_0 > 0$ and that it is a contraction.

Proposition 3.1 *Let $\alpha > 1$. There exists a positive constant $\kappa_1 \equiv \kappa_1(n, q, \Omega)$ such that if the following condition holds*

$$\tfrac{\kappa_1^2 \|f\|_q}{\nu^2} + \kappa_1 s_\alpha \left(\tfrac{\kappa_1 \|f\|_q}{\nu} \right)^{2r_\alpha} \left(1 + \tfrac{\kappa_1 \|f\|_q}{\nu} \right)^{(\alpha-4)^+} \leq \gamma_\alpha,$$

then the mapping $\mathcal{A}$ maps $B(\delta_0)$ into $B(\delta_0)$ for some $\delta_0 > \frac{\kappa_1 \|f\|_q}{\nu}$. Moreover, the following estimate holds

$$\delta_0 \leq \tfrac{2\kappa_1 \|f\|_q}{\nu}. \tag{3.3}$$

Proof. Let ζ be in $B(\delta)$ and denote by v its image by $\mathcal{A}$. Due to Lemma 6.1, $v \in V^{2,q}(\Omega)$ satisfies the estimate

$$\|\nabla v\|_{1,q} \leq \tfrac{C_{S,0}}{\nu} \left(\|f\|_q + 2\nu \left\| \nabla \cdot \left(\sigma\left(|D\zeta|^2 \right) D\zeta \right) \right\|_q + \|\zeta \cdot \nabla \zeta\|_q \right). \tag{3.4}$$

Standard arguments together with (6.2) show that

$$\left\| \nabla \cdot \left(\sigma\left(|D\zeta|^2 \right) D\zeta \right) \right\|_q \leq 2 s_\alpha \mathcal{F}\left(C_E \|\nabla \zeta\|_{1,q} \right) \|\nabla \zeta\|_{1,q} \leq \tfrac{2 s_\alpha}{C_E} \delta \mathcal{F}(\delta), \tag{3.5}$$

and

$$\begin{aligned} \|\zeta \cdot \nabla \zeta\|_q &\leq \|\zeta\|_q \|\nabla \zeta\|_\infty \leq C_E C_P \|\nabla \zeta\|_q \|\nabla \zeta\|_{1,q} \\ &\leq C_E C_P \|\nabla \zeta\|_{1,q}^2 \leq \tfrac{C_P}{C_E} \delta^2, \end{aligned} \tag{3.6}$$

where $\mathcal{F}(x) = x^{2r_\alpha} (1 + x)^{(\alpha-4)^+}$ and where $C_P \equiv C_P(n, q, \Omega)$ is the Poincaré constant. By combining (3.4), (3.5) and (3.6), we obtain

$$\|\nabla v\|_{1,q} \leq \tfrac{\kappa_1}{\nu} \left(\|f\|_q + \nu s_\alpha \, \delta \mathcal{F}(\delta) + \delta^2 \right),$$

where $\kappa_1 = C_{S,0} \max\left(1, \frac{4}{C_E}, \frac{C_P}{C_E} \right)$. To ensure that $\mathcal{A}(B(\delta) \subset B(\delta)$, it is sufficient that following condition

$$\tfrac{\kappa_1}{\nu} \left(\|f\|_q + \nu s_\alpha \, \delta \mathcal{F}(\delta) + \delta^2 \right) \leq \delta$$

holds. The conclusion follows from Proposition 6.4, by setting $A = \frac{\kappa_1}{\nu}$, $C = \kappa_1 s_\alpha$, $D = \frac{\kappa_1}{\nu} \|f\|_q$. ∎

Remark 3.2 *Notice that Proposition 6.4 implies that for every $\beta \in [1, 2]$ the following estimate holds*

$$\tfrac{\beta-1}{\beta} \delta_0 + \tfrac{2-\beta}{\beta} \tfrac{\kappa_1}{\nu} \delta_0^2 + \tfrac{2r_\alpha + 1 - \beta}{\beta} \kappa_1 s_\alpha \delta_0^{2r_\alpha + 1} (1 + \delta_0)^{(\alpha-4)^+} \leq \tfrac{\kappa_1 \|f\|_q}{\nu}.$$

It is then obvious that δ_0 satisfies (3.3).

Let us now prove that the mapping $\mathcal{A}$ is a contraction.

Proposition 3.3 *There exists a positive constant $\kappa_2 \equiv \kappa_2(n, q, \Omega)$ such that if the following condition is satisfied*

$$\kappa_2 \left(\frac{\kappa_1 \|f\|_q}{\nu^2} + s_\alpha \left(\frac{\kappa_1 \|f\|_q}{\nu} \right)^{2r_\alpha} \left(1 + \frac{\kappa_1 \|f\|_q}{\nu} \right)^{(\alpha-4)^+} \right) < \frac{1}{4^{(\alpha-2,1)^+}}, \tag{3.7}$$

then the mapping $\mathcal{A} : B(\delta_0) \longrightarrow B(\delta_0)$ is a contraction in $W^{1,q}(\Omega)$.

Proof. Let ζ and $\tilde{\zeta}$ be in $B(\delta_0)$ and let v and $\tilde{v}$ be their respective images by $\mathcal{A}$. Then

$$\begin{cases} -\nu\Delta(v-\tilde{v}) + \nabla(\pi - \tilde{\pi}) = \mathcal{R} & \text{in } \Omega, \\ \nabla \cdot (v - \tilde{v}) = 0 & \text{in } \Omega, \\ v - \tilde{v} = 0 & \text{on } \partial\Omega, \end{cases}$$

where

$$\mathcal{R} = \tilde{\zeta} \cdot \nabla\tilde{\zeta} - \zeta \cdot \nabla\zeta + 2\nu\nabla \cdot \left(\sigma\left(|D\zeta|^2\right) D\zeta - \sigma\big(|D\tilde{\zeta}|^2\big) D\tilde{\zeta} \right)$$

$$= \nabla \cdot \left(\tilde{\zeta} \otimes \tilde{\zeta} - \zeta \otimes \zeta + 2\nu \left(\sigma\big(|D\zeta|^2\big) D\zeta - \sigma\big(|D\tilde{\zeta}|^2\big) D\tilde{\zeta} \right) \right).$$

Lemma 6.1 and arguments similar to those used in the proof of Proposition 3.1 show that $v - \tilde{v}$ satisfies the following estimate

$$\big\|\nabla(v-\tilde{v})\big\|_q$$

$$\leq \frac{C_{S,-1}}{\nu} \left(\big\|\tilde{\zeta} \otimes \tilde{\zeta} - \zeta \otimes \zeta\big\|_q + 2\nu \big\|\sigma\left(|D\zeta|^2\right) D\zeta - \sigma\big(|D\tilde{\zeta}|^2\big) D\tilde{\zeta}\big\|_q \right). \tag{3.8}$$

On the other hand, standard arguments yield

$$\big\|\tilde{\zeta} \otimes \tilde{\zeta} - \zeta \otimes \zeta\big\|_q \leq \big\|(\tilde{\zeta} - \zeta) \otimes \tilde{\zeta}\big\|_q + \big\|\zeta \otimes (\tilde{\zeta} - \zeta)\big\|_q$$

$$\leq \big\|\tilde{\zeta} - \zeta\big\|_q \left(\|\zeta\|_\infty + \|\tilde{\zeta}\|_\infty \right) \leq C_E \big\|\tilde{\zeta} - \zeta\big\|_q \left(\|\zeta\|_{1,q} + \|\tilde{\zeta}\|_{1,q} \right)$$

$$\leq C_E C_P \left(C_P^q + 1\right)^{\frac{1}{q}} \big\|\nabla(\tilde{\zeta} - \zeta)\big\|_q \left(\|\nabla\zeta\|_q + \|\nabla\tilde{\zeta}\|_q \right)$$

$$\leq 2 C_P \left(C_P^q + 1\right)^{\frac{1}{q}} \delta_0 \big\|\nabla(\tilde{\zeta} - \zeta)\big\|_q$$

$$\leq C_P \left(C_P^q + 1\right)^{\frac{1}{q}} \frac{4\kappa_1}{\nu} \|f\|_q \big\|\nabla(\tilde{\zeta} - \zeta)\big\|_q. \tag{3.9}$$

(Estimate (3.3) was used in the last step.) Moreover, due to (6.3), we have

$$\big\|\sigma\left(|D\zeta|^2\right) D\zeta - \sigma\big(|D\tilde{\zeta}|^2\big) D\tilde{\zeta}\big\|_q \leq s_\alpha \mathcal{F}\left(\|D\zeta\|_\infty + \|D\tilde{\zeta}\|_\infty \right) \big\|\nabla(\zeta - \tilde{\zeta})\big\|_q$$

$$\leq s_\alpha \mathcal{F}\left(C_E \left(\|\nabla\zeta\|_{1,q} + \|\nabla\tilde{\zeta}\|_{1,q} \right) \right) \big\|\nabla(\zeta - \tilde{\zeta})\big\|_q$$

$$\leq s_\alpha \mathcal{F}\left(2\delta_0\right) \left\|\nabla(\zeta - \tilde{\zeta})\right\|_q \leq s_\alpha \mathcal{F}\left(\tfrac{4\kappa_1}{\nu}\|f\|_q\right) \left\|\nabla(\zeta - \tilde{\zeta})\right\|_q. \tag{3.10}$$

By combining (3.8), (3.9) and (3.10), we deduce that

$$\left\|\nabla(v - \tilde{v})\right\|_q \leq \kappa_2 \left(\tfrac{4\kappa_1}{\nu^2}\|f\|_q + s_\alpha \mathcal{F}\left(\tfrac{4\kappa_1}{\nu}\|f\|_q\right)\right) \left\|\nabla(\zeta - \tilde{\zeta})\right\|_q$$

$$\leq 4^{(\alpha-2,1)^+} \kappa_2 \left(\tfrac{\kappa_1}{\nu^2}\|f\|_q + s_\alpha \left(\tfrac{\kappa_1}{\nu}\|f\|_q\right)^{2r_\alpha} \left(1 + \tfrac{\kappa_1}{\nu}\|f\|_q\right)^{(\alpha-4)^+}\right) \left\|\nabla(\zeta - \tilde{\zeta})\right\|_q,$$

where $\kappa_2 = C_{S,-1} \max\left(2, C_P\left(C_P^q + 1\right)^{\frac{1}{q}}\right)$. It follows that $\mathcal{A}$ is a contraction if (3.7) is fulfilled. ∎

Notice finally that for $\alpha \leq 3$, $\gamma_\alpha = \frac{1}{4^{(\alpha-2,1)^+}} = \frac{1}{4}$ and that for $\alpha > 3$ we have $\gamma_\alpha > \frac{1}{4^{(\alpha-2,1)^+}}$. Hence, by setting $\kappa = (\kappa_1, \kappa_2)^+$, we see that the statement of Theorem 2.1 is a consequence of Proposition 3.1, Proposition 3.3 and the following version of the Banach fixed point theorem.

Theorem 3.4 *Let X and Y be Banach spaces such that X is reflexive and $X \hookrightarrow Y$. Let B be a non-empty, closed, convex and bounded subset of X and let $\mathcal{A} : B \longrightarrow B$ be a mapping such that*

$$\|\mathcal{A}(u) - \mathcal{A}(v)\|_Y \leq \kappa \|u - v\|_Y \qquad \text{for all } u, v \in B \qquad (0 < \kappa < 1),$$

then $\mathcal{A}$ has a unique fixed point in B.

4 Proof of Theorem 2.2

4.1 An approximate problem

For $0 < \varepsilon < 1$, consider the problem given by

$$\begin{cases} -\nabla \cdot \left(2\nu\left(1 + \sqrt{\varepsilon^2 + |Du|^2}\right)^{\alpha-2} Du\right) + u \cdot \nabla u + \nabla \pi = f & \text{in } \Omega, \\ \nabla \cdot u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.1}$$

The idea is to apply arguments similar to those used in the proof of Theorem 2.1 to establish existence of a strong solution for (4.1). Uniform estimates with respect to ε are then derived and convergence results obtained.

Problem (4.1) can be reformulated as

$$\begin{cases} -\nu\left(1 + \varepsilon\right)^{\alpha-2} \Delta u + \nabla \pi = f - u \cdot \nabla u + \nabla \cdot \left(2\nu\sigma_\varepsilon\left(|Du|^2\right) Du\right) & \text{in } \Omega, \\ \nabla \cdot u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with $\sigma_\varepsilon(x) = \left(1 + \sqrt{\varepsilon^2 + |Du|^2}\right)^{\alpha-2} - (1 + \varepsilon)^{\alpha-2}$. We define the mapping

$$\mathcal{A}_\varepsilon : \ \zeta \ \longrightarrow \ u_\varepsilon,$$

through the Stokes system

$$\begin{cases} -\nu(1+\varepsilon)^{\alpha-2}\Delta u_\varepsilon + \nabla\pi_\varepsilon = f - \zeta\cdot\nabla\zeta + \nabla\cdot\left(2\nu\sigma_\varepsilon\left(|D\zeta^2\right)D\zeta\right) & \text{in } \Omega, \\ \nabla\cdot u_\varepsilon = 0 & \text{in } \Omega, \\ u_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.2}$$

and we look for the solution as a fixed point for $\mathcal{A}_\varepsilon$.

Theorem 4.1 *Let f be in $L^q(\Omega)$ with $q > n$, $\alpha > 1$ and $0 < \varepsilon < 1$. There exists a positive constant $\bar{\kappa}$ depending only on n, q and Ω such that, if*

$$\tfrac{\bar{\kappa}^2\|f\|_q}{\nu^2} + \bar{s}_\alpha \tfrac{\bar{\kappa}^2\|f\|_q}{\nu}\left(1+\tfrac{\bar{\kappa}\|f\|_q}{\nu}\right)^{(\alpha-3)^+} < \tfrac{1}{4^{(\alpha-2,1)^+}}$$

then problem (4.1) admits a unique solution $u_\varepsilon \in V^{2,q}$. Moreover, the following estimate holds

$$\|u_\varepsilon\|_{2,q} \leq \tfrac{\bar{C}^*\|f\|_q}{\nu} \tag{4.3}$$

where $\bar{C}^* \equiv \bar{C}^*(n,q,\Omega)$ is independent of ε.

For $\delta > 0$, let $B(\delta)$ be the convex set given by (3.2). In order to prove Theorem 4.1, we first need the following result.

Proposition 4.2 *There exists a positive constant $\bar{\kappa}_1$ depending only on n, q and Ω such that if the following condition holds*

$$\tfrac{\bar{\kappa}_1^2\|f\|_q}{\nu} + \bar{s}_\alpha \tfrac{\bar{\kappa}_1^2\|f\|_q}{\nu}\left(1+\tfrac{\bar{\kappa}_1\|f\|_q}{\nu}\right)^{(\alpha-3)^+} \leq \gamma_\alpha,$$

then the mapping $\mathcal{A}_\varepsilon$ maps $B(\bar{\delta}_0)$ into $B(\bar{\delta}_0)$ for some $\bar{\delta}_0 > \frac{\bar{\kappa}_1\|f\|_q}{\nu}$ independent of ε. Moreover, the following estimate holds

$$\bar{\delta}_0 \leq \tfrac{2\bar{\kappa}_1}{\nu}\|f\|_q.$$

Proof. Let ζ be in $B(\delta)$ and denote by v_ε its image by $\mathcal{A}_\varepsilon$. Due to Lemma 6.1, $v_\varepsilon \in V^{2,q}(\Omega)$ satisfies the estimate

$$\begin{aligned} \|\nabla v_\varepsilon\|_{1,q} &\leq \tfrac{C_{S,0}}{(1+\varepsilon)^{\alpha-2}\nu}\left(\|f\|_q + 2\nu\left\|\nabla\cdot\left(\sigma_\varepsilon\left(|D\zeta|^2\right)D\zeta\right)\right\|_q + \|\zeta\cdot\nabla\zeta\|_q\right) \\ &\leq \tfrac{2C_{S,0}}{\nu}\left(\|f\|_q + 2\nu\left\|\nabla\cdot\left(\sigma_\varepsilon\left(|D\zeta|^2\right)D\zeta\right)\right\|_q + \|\zeta\cdot\nabla\zeta\|_q\right). \end{aligned} \tag{4.4}$$

Taking into account (4.4) and (6.9) and arguing as in the proof of Proposition 3.1, we deduce that

$$\|\nabla v_\varepsilon\|_{1,q} \leq \tfrac{\bar{\kappa}_1}{\nu}\left(\|f\|_q + \nu\bar{s}_\alpha\,\delta\mathcal{L}(\delta) + \delta^2\right),$$

where $\mathcal{L}(x) = x(1+x)^{(\alpha-3)^+}$ and $\bar{\kappa}_1 = 2C_{S,0}\max\left(1, \frac{8}{C_E}, \frac{C_P}{C_E}\right)$. To ensure that $\mathcal{A}_\varepsilon(B(\delta) \subset B(\delta)$, it is sufficient that following condition

$$\tfrac{\bar{\kappa}_1}{\nu}\left(\|f\|_q + \nu\bar{s}_\alpha\,\delta\mathcal{L}(\delta) + \delta^2\right) \leq \delta$$

holds. The conclusion follows from Proposition 6.5, by setting $A=\frac{\bar\kappa_1}{\nu}$, $C=\bar\kappa_1\bar s_\alpha$, $D=\frac{\bar\kappa_1}{\nu}\|f\|_q$. ∎

Remark 4.3 *Notice that Proposition* 6.5 *implies that for every* $\beta\in[1,2]$ *the following estimate holds*

$$\tfrac{\beta-1}{\beta}\bar\delta_0+\tfrac{2-\beta}{\beta}\tfrac{\bar\kappa_1}{\nu}\bar\delta_0^2+\tfrac{2-\beta}{\beta}\bar\kappa_1\bar s_\alpha\bar\delta_0^2\left(1+\bar\delta_0\right)^{(\alpha-3)^+}\le\tfrac{\bar\kappa_1}{\nu}\|f\|_q.$$

Obviously, the claimed inequality is obtained for $\beta=2$.

Let us now prove that the mapping $\mathcal{A}_\varepsilon$ is a contraction.

Proposition 4.4 *There exists a positive constant* $\bar\kappa_2$ *depending only on* n, q *and* Ω *such that if the following condition holds*

$$\bar\kappa_2\left(\tfrac{\bar\kappa_1\|f\|_q}{\nu^2}+\bar s_\alpha\tfrac{\bar\kappa_1\|f\|_q}{\nu}\left(1+\tfrac{\bar\kappa_1\|f\|_q}{\nu}\right)^{(\alpha-3)^+}\right)<\tfrac{1}{4^{(\alpha-2,1)^+}},\tag{4.5}$$

then the mapping $\mathcal{A}_\varepsilon: B(\bar\delta_0)\longrightarrow B(\bar\delta_0)$ *is a contraction in* $W^{1,q}(\Omega)$.

Proof. Let ζ and $\tilde\zeta$ be in $B(\bar\delta_0)$ and let v_ε and $\tilde v_\varepsilon$ be their respective images by $\mathcal{A}_\varepsilon$. Then

$$\begin{cases}-\nu\left(1+\varepsilon\right)^{\alpha-2}\Delta(v_\varepsilon-\tilde v_\varepsilon)+\nabla\left(\pi_\varepsilon-\tilde\pi_\varepsilon\right)=\mathcal{R}_\varepsilon & \text{in }\Omega,\\ \nabla\cdot(v_\varepsilon-\tilde v_\varepsilon)=0 & \text{in }\Omega,\\ v_\varepsilon-\tilde v_\varepsilon=0 & \text{on }\partial\Omega,\end{cases}$$

where

$$\begin{aligned}\mathcal{R}_\varepsilon&=\tilde\zeta\cdot\nabla\tilde\zeta-\zeta\cdot\nabla\zeta+2\nu\nabla\cdot\left(\sigma_\varepsilon\left(|D\zeta|^2\right)D\zeta-\sigma_\varepsilon\big(|D\tilde\zeta|^2\big)D\tilde\zeta\right)\\&=\nabla\cdot\left(\tilde\zeta\otimes\tilde\zeta-\zeta\otimes\zeta+2\nu\left(\sigma_\varepsilon\left(|D\zeta|^2\right)D\zeta-\sigma_\varepsilon\big(|D\tilde\zeta|^2\big)D\tilde\zeta\right)\right).\end{aligned}$$

Lemma 6.1 and arguments similar to those used in the proof of Proposition 4.2 show that $v_\varepsilon-\tilde v_\varepsilon$ satisfies the following estimate

$$\begin{aligned}&\big\|\nabla(v-\tilde v)\big\|_q\\ \le\ &\tfrac{2C_{S,-1}}{\nu}\left(\big\|\tilde\zeta\otimes\tilde\zeta-\zeta\otimes\zeta\big\|_q+2\nu\big\|\sigma_\varepsilon\left(|D\zeta|^2\right)D\zeta-\sigma_\varepsilon\big(|D\tilde\zeta|^2\big)D\tilde\zeta\big\|_q\right).\end{aligned}\tag{4.6}$$

Taking into account (4.6) and (6.10) and arguing as in the proof of Proposition 3.3, we obtain

$$\begin{aligned}\big\|\nabla(v_\varepsilon-\tilde v_\varepsilon)\big\|_q&\le\bar\kappa_2\left(\tfrac{4\bar\kappa_1}{\nu^2}\|f\|_q+\bar s_\alpha\,\mathcal{L}\left(\tfrac{4\bar\kappa_1}{\nu}\|f\|_q\right)\right)\big\|\nabla(\zeta-\tilde\zeta)\big\|_q\\ \le 4^{(\alpha-2,1)^+}\bar\kappa_2&\left(\tfrac{\bar\kappa_1}{\nu^2}\|f\|_q+\bar s_\alpha\tfrac{\bar\kappa_1}{\nu}\|f\|_q\left(1+\tfrac{\bar\kappa_1}{\nu}\|f\|_q\right)^{(\alpha-3)^+}\right)\big\|\nabla(\zeta-\tilde\zeta)\big\|_q,\end{aligned}$$

where $\bar\kappa_2=2C_{S,-1}\max\left(2,C_P\left(C_P^q+1\right)^{\frac1q}\right)$. It follows that $\mathcal{A}_\varepsilon$ is a contraction if (4.5) is fulfilled. ∎

Notice finally that for $\alpha\le3$, $\gamma_\alpha=\frac{1}{4^{(\alpha-2,1)^+}}=\frac14$ and that for $\alpha>3$ we have $\gamma_\alpha>\frac{1}{4^{(\alpha-2,1)^+}}$. Hence, by setting $\bar\kappa=(\bar\kappa_1,\bar\kappa_2)^+$, we see that the statement of Theorem 4.1 is a consequence of Proposition 4.2, Proposition 4.4 and Theorem 3.4.

4.2 Passage to the limit

Due to Theorem 4.1, the sequence $(u_\varepsilon)_\varepsilon$ is uniformly bounded in $W^{2,q}(\Omega)$. There then exists a subsequence, still indexed by ε, and some $u \in W^{2,q}(\Omega)$ such that $(u_\varepsilon)_\varepsilon$ weakly converges to u in $W^{2,q}(\Omega)$. Since $q > n$, by compactness results on Sobolev spaces, we deduce that u strongly converges to u in $C^{1,\gamma}(\overline{\Omega})$ with $\gamma < 1 - \frac{n}{q}$. By passing to the limit in the weak formulation

$$2\nu \left(\left(1 + \sqrt{\varepsilon^2 + |Du_\varepsilon|^2}\right)^{\alpha-2} Du_\varepsilon, D\varphi \right) + (u_\varepsilon \cdot \nabla u_\varepsilon, \varphi) = (f, \varphi) \quad \text{for all } \varphi \in V^{1,\alpha}$$

we obtain

$$2\nu \left(\left(1 + |Du|\right)^{\alpha-2} Du, D\varphi \right) + (u \cdot \nabla u, \varphi) = (f, \varphi) \qquad \text{for all } \varphi \in V^{1,\alpha}$$

i.e. u is a solution of (1.1). Estimate (2.2) is a direct consequence of (4.3). ■

5 Uniqueness results

5.1 Proof of Theorem 2.3

Let us first estimate $\|Dv\|_2$. Setting $\varphi = v$ in the weak formulation of (1.1) and using the Poincaré and the Korn inequalities yield

$$\begin{aligned} 2\nu \|Dv\|_2^2 &\leq (S(Dv), Dv) = (f, v) - (v \cdot \nabla v, v) = (f, v) \leq \|f\|_2 \, \|v\|_2 \\ &\leq \tfrac{(n-1)|\Omega|^{\frac{1}{n}}}{\sqrt{n}} \, \|f\|_2 \, \|\nabla v\|_2 = \tfrac{\sqrt{2}(n-1)|\Omega|^{\frac{1}{n}}}{\sqrt{n}} \, \|f\|_2 \, \|Dv\|_2 \end{aligned}$$

and thus

$$\|Dv\|_2 \leq \tfrac{(n-1)|\Omega|^{\frac{1}{n}}}{\sqrt{2n}} \, \tfrac{\|f\|_2}{\nu}. \tag{5.1}$$

(See for example [16], Chapter 2 for the expression of the Poincaré constant.) Similarly, by substituing in the weak formulation of (1.1), setting $\varphi = u - v$ and taking into account Lemma 1.1, Chapter 8 in [16], the Korn inequality and estimate (5.1), we obtain

$$\begin{aligned} &(S(Du) - S(Dv), D(u - v)) \\ &= -(u \cdot \nabla(u - v), u - v) - ((u - v) \cdot \nabla v, u - v) = -((u - v) \cdot \nabla v, u - v) \\ &\leq \tfrac{(n-1)|\Omega|^{\frac{1}{n(n-1)}}}{n} \, \|\nabla(u - v)\|_2^2 \, \|\nabla v\|_2 = \tfrac{2^{\frac{3}{2}}(n-1)|\Omega|^{\frac{1}{n(n-1)}}}{n} \, \|D(u - v)\|_2^2 \, \|Dv\|_2 \\ &\leq \tfrac{2(n-1)^2|\Omega|^{\frac{1}{n-1}}}{\sqrt{n^3}} \, \tfrac{\|f\|_2}{\nu} \, \|D(u - v)\|_2^2 . \end{aligned} \tag{5.2}$$

On the other hand, the coercivity condition

$$(S(\eta) - S(\zeta)) : (\eta - \zeta) \geq 2\nu \, |\eta - \zeta|^2$$

yields

$$2\nu \left\| D(u-v) \right\|_2^2 \leq \left(S(Du) - S(Dv), D(u-v) \right). \tag{5.3}$$

Combining (5.2) and (5.3), we deduce that

$$\left(2\nu - \frac{2(n-1)^2 |\Omega|^{\frac{1}{n-1}}}{\sqrt{n^3}} \frac{\|f\|_2}{\nu} \right) \left\| D(u-v) \right\|_2^2 \leq 0$$

and thus $u \equiv v$ if condition (2.5) is fulfilled. ∎

5.2 Proof of Theorem 2.4

The proof is split into three steps and is given for the extra stress tensor $(1.2)_1$. The proof corresponding to the extra stress tensor $(1.2)_2$ is obtained with very minor changes (see for example [10] where similar arguments are developed).

Step 1. Let w be a weak solution of (1.1). Standard arguments show that

$$\begin{aligned} \|Dw\|_\alpha^\alpha &= \int_{\{x \mid |Dw(x)| \geq 1\}} |Dw(x)|^\alpha \, dx + \int_{\{x \mid |Dw(x)| < 1\}} |Dw(x)|^\alpha \, dx \\ &\leq 2^{\frac{2-\alpha}{2}} \int_{\{x \mid |Dw(x)| \geq 1\}} \tfrac{1}{2\nu} S(Dw(x)) : Dw(x) \, dx + |\Omega| \\ &\leq \frac{2^{\frac{-\alpha}{2}}}{\nu} \left(S(Dw), Dw \right) + |\Omega| \leq \left(\tfrac{f}{\nu}, w \right) + |\Omega|. \end{aligned} \tag{5.4}$$

Since $\alpha > \frac{2n}{n+2}$ we have $L^2(\Omega) \hookrightarrow W_0^{1,\alpha}(\Omega)$ and by using the Hölder, the Sobolev and the Korn inequalities, we deduce that

$$\begin{aligned} \left| \left(\tfrac{f}{\nu}, w \right) \right| &\leq \tfrac{1}{\nu} \|f\|_{\frac{n\alpha}{(n+1)\alpha - n}} \|w\|_{\frac{n\alpha}{n-\alpha}} \leq \frac{\alpha(n-1)}{2(n-\alpha)\sqrt{n}} \frac{\|f\|_2}{\nu} \|w\|_{\frac{n\alpha}{n-\alpha}} \\ &\leq \frac{\alpha(n-1)|\Omega|^{\frac{(n+2)\alpha-2n}{2\alpha n}}}{2(n-\alpha)\sqrt{n}} \frac{\|f\|_2}{\nu} \|\nabla w\|_\alpha \leq \kappa_\alpha \frac{\|f\|_2}{\nu} \|Dw\|_\alpha \end{aligned}$$

with $\kappa_{1,\alpha} = \frac{\alpha(n-1)|\Omega|^{\frac{(n+2)\alpha-2n}{2\alpha n}}}{2(n-\alpha)\sqrt{n} C_{K,\alpha}}$ and where $C_{K,\alpha}$ is the constant of Korn. (See [16], Chapter 2 for the expression of the Poincaré and the Sobolev constants.) This inequality together with (5.4) imply

$$\|Dw\|_\alpha^\alpha \leq \kappa_{1,\alpha} \frac{\|f\|_2}{\nu} \|Dw\|_\alpha + |\Omega|. \tag{5.5}$$

On the other hand, due to the Young inequality

$$\kappa_{1,\alpha} \frac{\|f\|_2}{\nu} \|Dw\|_\alpha \leq \frac{\left(\kappa_{1,\alpha} \frac{\|f\|_2}{\nu} \right)^{\alpha'}}{\alpha'} + \frac{\|Dw\|_\alpha^\alpha}{\alpha}. \tag{5.6}$$

Combining (5.5) and (5.6), we deduce that

$$\|Dw\|_\alpha^\alpha \leq \left(\kappa_{1,\alpha} \frac{\|f\|_2}{\nu} \right)^{\alpha'} + |\Omega|. \tag{5.7}$$

Step 2. Let w_1 and w_2 be two weak solutions of (1.1). By taking into account the monotonicity condition

$$(S(\eta) - S(\zeta)) : (\eta - \zeta) \geq 2\nu \left(1 + |\eta^2| + |\zeta|^2\right)^{\frac{\alpha-2}{2}} |\eta - \zeta|^2$$

we deduce that the following inequality holds a.e.

$$|D(w_1 - w_2)|^2$$
$$\leq \tfrac{1}{2\nu} \left(S(Dw_1) - S(Dw_2)\right) : D(w_1 - w_2) \left(1 + |Dw_1|^2 + |Dw_2|^2\right)^{\frac{2-\alpha}{2}} .$$

Integrating and using the Hölder inequality, we obtain

$$\|D(w_1 - w_2)\|_\alpha^2$$
$$\leq \left\|(1 + |Dw_1|^2 + |Dw_2|^2)^{\frac{2-\alpha}{2}}\right\|_{\frac{\alpha}{2-\alpha}} \left\|\tfrac{1}{2\nu} \left(S(Dw_1) - S(Dw_2)\right) : D(w_1 - w_2)\right\|_1$$
$$\leq \tfrac{1}{2\nu} \left(|\Omega| + \|Dw_1\|_\alpha^\alpha + \|Dw_2\|_\alpha^\alpha\right)^{\frac{2-\alpha}{\alpha}} \left(S(Dw_1) - S(Dw_2), D(w_1 - w_2)\right)$$

which gives

$$\left(S(Dw_1) - S(Dw_2), D(w_1 - w_2)\right) \geq \tfrac{2\nu\|D(w_1-w_2)\|_\alpha^2}{(|\Omega|+\|Dw_1\|_\alpha^\alpha+\|Dw_2\|_\alpha^\alpha)^{\frac{2-\alpha}{\alpha}}}.$$

Setting $\varphi = w_1 - w_2$ in the corresponding weak formulation and taking into account the previous inequality, we obtain

$$\tfrac{2\nu\|D(w_1-w_2)\|_\alpha^2}{\left(|\Omega|+\|Dw_1\|_\alpha^\alpha+\|Dw_2\|_\alpha^\alpha\right)^{\frac{2-\alpha}{\alpha}}}$$
$$\leq \left(S(Dw_1) - S(Dw_2), D\left(w_1 - w_2\right)\right) = -\left(\left(w_1 - w_2\right) \cdot \nabla w_2, w_1 - w_2\right). \tag{5.8}$$

The Hölder, the Sobolev and the Korn inequalities together with classical embedding results show that if $\frac{2\alpha}{\alpha-1} \leq \frac{n\alpha}{n-\alpha}$ (and thus $\alpha \geq \frac{3n}{n+2}$), then

$$|((w_1 - w_2) \cdot \nabla w_2, w_1 - w_2)|$$
$$\leq \|w_1 - w_2\|_{\frac{2\alpha}{\alpha-1}}^2 \|\nabla w_2\|_\alpha \leq |\Omega|^{\frac{(n+2)\alpha-3n}{n\alpha}} \|w_1 - w_2\|_{\frac{n\alpha}{n-\alpha}}^2 \|\nabla w_2\|_\alpha$$
$$\leq \tfrac{(\alpha(n-1))^2|\Omega|^{\frac{(n+2)\alpha-3n}{n\alpha}}}{4n(n-\alpha)^2} \|\nabla(w_1 - w_2)\|_\alpha^2 \|\nabla w_2\|_\alpha$$
$$\leq \kappa_{2,\alpha} \|D(w_1 - w_2)\|_\alpha^2 \|Dw_2\|_\alpha , \tag{5.9}$$

where $\kappa_{2,\alpha} = \frac{(\alpha(n-1))^2|\Omega|^{\frac{(n+2)\alpha-3n}{n\alpha}}}{4n(n-\alpha)^2C_{K,\alpha}^3}$. Combining (5.7), (5.8) and (5.9), we obtain

$$2\nu \|D(w_1 - w_2)\|_\alpha^2$$
$$\leq \kappa_{2,\alpha} \left(|\Omega| + \|Dw_1\|_\alpha^\alpha + \|Dw_2\|_\alpha^\alpha\right)^{\frac{2-\alpha}{\alpha}} \|Dw_2\|_\alpha \|D(w_1 - w_2)\|_\alpha^2$$

$$\leq \kappa_{2,\alpha}\left(3|\Omega|+2\left(\kappa_{1,\alpha}\frac{\|f\|_2}{\nu}\right)^{\alpha'}\right)^{\frac{2-\alpha}{\alpha}}\|Dw_2\|_\alpha\,\|D(w_1-w_2)\|_\alpha^2$$

$$\leq \kappa_{3,\alpha}\left(1+\left(\tfrac{\|f\|_2}{\nu}\right)^{\alpha'}\right)^{\frac{2-\alpha}{\alpha}}\|Dw_2\|_\alpha\,\|D(w_1-w_2)\|_\alpha^2$$

$$\leq 2\kappa_{3,\alpha}\left(1+\frac{\|f\|_2}{\nu}\right)^{\frac{2-\alpha}{\alpha-1}}\|Dw_2\|_\alpha\,\|D(w_1-w_2)\|_\alpha^2 \tag{5.10}$$

with $\kappa_{3,\alpha}=\kappa_{2,\alpha}\left(3|\Omega|+2\left(\kappa_{1,\alpha}\right)^{\alpha'}\right)^{\frac{2-\alpha}{\alpha}}$.

Step 3. Let u be the $W^{2,q}$ solution of problem (1.1) given by Theorem 2.1 and let $v\in V_\alpha$ be a weak solution. Due to (5.10) and (2.2), we have

$$\|D(v-u)\|_\alpha^2 \leq \kappa_{3,\alpha}\left(1+\frac{\|f\|_2}{\nu}\right)^{\frac{2-\alpha}{\alpha-1}}\frac{\|Du\|_\alpha}{\nu}\|D(v-u)\|_\alpha^2$$

$$\leq \kappa_{3,\alpha}|\Omega|^{\frac{1}{\alpha}-\frac{1}{q}}\left(1+\frac{\|f\|_2}{\nu}\right)^{\frac{2-\alpha}{\alpha-1}}\frac{\|\nabla u\|_q}{\nu}\|D(v-u)\|_\alpha^2$$

$$\leq C^*\kappa_{3,\alpha}|\Omega|^{\frac{1}{\alpha}-\frac{1}{q}}\left(1+\frac{\|f\|_2}{\nu}\right)^{\frac{2-\alpha}{\alpha-1}}\frac{\|f\|_q}{\nu^2}\|D(v-u)\|_\alpha^2\,.$$

It follows that $u\equiv v$ if (2.6) is fulfilled with $\kappa_\alpha=C^*\kappa_{3,\alpha}|\Omega|^{\frac{1}{\alpha}-\frac{1}{q}}$. ∎

6 Appendix

We first recall a classical result concerning the existence and uniqueness of solutions to the Stokes system. The complete treatment of this problem in the case of a bounded domain in $I\!R^3$ is due to Cattabriga [9]. (See also [16] where the proof, based on similar arguments, is developed for bidimensional and tridimensional domains.)

Lemma 6.1 *Let $m\geq -1$ be an integer and let Ω be a bounded domain in $I\!R^n$ $(n=2,3)$ with boundary $\partial\Omega$ of class C^k with $k=\max(m+2,2)$. Then for any $\tau\in W^{m+1,q}(\Omega)$, the following system*

$$\begin{cases} -\Delta u+\nabla\pi=\nabla\cdot\tau & \text{in } \Omega,\\ \nabla\cdot u=0 & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega,\end{cases}$$

admits a unique solution $(u,\pi)\in W^{m+2,q}(\Omega)\times\left(W^{m+1,q}(\Omega)\cap L^q_0(\Omega)\right)$. Moreover, the following estimate holds

$$\|\nabla u\|_{m+1,q}+\|\pi\|_{m+1,q}\leq C_{S,m}\|\tau\|_{m+1,q}, \tag{6.1}$$

where $C_{S,m}\equiv C_{S,m}(n,q,m,\Omega)$ is a positive constant.

In the next two propositions, we establish some useful estimates related with the shear-dependent viscosity term appearing in system (3.1) and (4.2).

Proposition 6.2 *Assume that $\alpha > 1$ and let ζ, $\tilde{\zeta}$ be in $W^{2,q}(\Omega)$ with $q > n$. Then the following estimates hold*

$$\left\|\nabla \cdot \left(\sigma\left(|D\zeta|^2\right) D\zeta\right)\right\|_q \leq 2s_\alpha\, \mathcal{F}\left(\|D\zeta\|_\infty\right) \|\nabla\zeta\|_{1,q} \tag{6.2}$$

$$\left\|\sigma\left(|D\zeta|^2\right)D\zeta - \sigma\left(|D\tilde{\zeta}|^2\right)D\tilde{\zeta}\right\|_q \leq s_\alpha\, \mathcal{F}\left(\|D\zeta\|_\infty + \|D\tilde{\zeta}\|_\infty\right)\left\|\nabla(\zeta - \tilde{\zeta})\right\|_q \tag{6.3}$$

where $\mathcal{F}(x) = x^{2r_\alpha}\,(1+x)^{(\alpha-4)^+}$ *with* $r_\alpha = \frac{1}{2}\left(1 + (\alpha-3)^+ - (\alpha-4)^+\right)$, *and* $s_\alpha = (|\alpha-2|, 2)^+$.

Proof. Let us first recall that for every $x, y \in I\!R^+$, we have

$$2|\sigma\left(x^2\right)| \leq s_\alpha\, \mathcal{F}(x), \qquad\qquad 2|\sigma'(x^2)|x^2 \leq s_\alpha\, \mathcal{F}(x), \tag{6.4}$$

$$\left|\sigma(x^2) - \sigma(y^2)\right| \leq s_\alpha\,(x+y)^{2r_\alpha - 1}\,(1+x+y)^{(\alpha-4)^+}\,|x-y|, \tag{6.5}$$

where s_α and $\mathcal{F}$ as defined in the statement (see [2] for the corresponding proofs). Standard calculations show that the following estimate holds a.e.

$$\begin{aligned}
&\left|\nabla \cdot \left(\sigma(|D\zeta|^2)D\zeta\right)\right| \\
&\leq \left|\sigma\left(|D\zeta|^2\right) \nabla \cdot D\zeta\right| + \left|\sigma'\left(|D\zeta|^2\right) D\zeta \cdot \nabla|D\zeta|^2\right| \\
&\leq \left|\sigma\left(|D\zeta|^2\right)\right| |\nabla \cdot D\zeta| + 2\left|\sigma'\left(|D\zeta|^2\right)|D\zeta|^2\left|\nabla^2\zeta\right|\right| \\
&\leq \left(\sqrt{n}\left|\sigma\left(|D\zeta|^2\right)\right| + 2\left|\sigma'\left(|D\zeta|^2\right)|D\zeta|^2\right|\right)\left|\nabla^2\zeta\right| \\
&\leq 2\left(\left|\sigma\left(|D\zeta|^2\right)\right| + \left|\sigma'\left(|D\zeta|^2\right)|D\zeta|^2\right|\right)\left|\nabla^2\zeta\right|
\end{aligned}$$

and consequently

$$\begin{aligned}
&\left\|\nabla \cdot \left(\sigma(|D\zeta|^2)D\zeta\right)\right\|_q \\
&\leq 2\left(\left\|\sigma\left(|D\zeta|^2\right)\right\|_\infty + \left\|\sigma'\left(|D\zeta|^2\right)|D\zeta|^2\right\|_\infty\right)\left\|\nabla^2\zeta\right\|_q \\
&\leq 2\left(\left\|\sigma\left(|D\zeta|^2\right)\right\|_\infty + \left\|\sigma'\left(|D\zeta|^2\right)|D\zeta|^2\right\|_\infty\right)\|\nabla\zeta\|_{1,q}\,.
\end{aligned} \tag{6.6}$$

Taking into account (6.4), and the fact that $\mathcal{F}$ is a nondecreasing function we deduce that

$$2\left\|\sigma(|D\zeta|^2)\right\|_\infty \leq s_\alpha\, \|\mathcal{F}\left(|D\zeta|\right)\|_\infty \leq s_\alpha \mathcal{F}\left(\|D\zeta\|_\infty\right), \tag{6.7}$$

and

$$2\left\|\sigma'\left(|D\zeta|^2\right)|D\zeta|^2\right\|_\infty \leq s_\alpha \mathcal{F}\left(\|D\zeta\|_\infty\right). \tag{6.8}$$

Combining (6.6), (6.7) and (6.8) we obtain (6.2). Similarly, we have

$$\left|\sigma\left(|D\zeta|^2\right)D\zeta - \sigma\left(|D\tilde{\zeta}|^2\right)D\tilde{\zeta}\right|$$

$$\leq \left|\sigma\left(|D\zeta|^2\right)\right|\left|D(\zeta-\tilde{\zeta})\right| + \left|\sigma\left(|D\zeta|^2\right) - \sigma\big(|D\tilde{\zeta}|^2\big)\right|\left|D\tilde{\zeta}\right|$$

$$\leq \tfrac{s_\alpha}{2}|D\zeta|^{2r_\alpha}\left(1+|D\zeta|\right)^{(\alpha-4)^+}\left|D(\zeta-\tilde{\zeta})\right|$$

$$+s_\alpha\big(|D\zeta|+|D\tilde{\zeta}|\big)^{2r_\alpha-1}\left|D\tilde{\zeta}\right|\big(1+|D\zeta|+|D\tilde{\zeta}|\big)^{(\alpha-4)^+}\left|\,|D\zeta|-\left|D\tilde{\zeta}\right|\right|$$

$$\leq s_\alpha \mathcal{F}\big(|D\zeta|+|D\tilde{\zeta}|\big)\left|D\zeta - D\tilde{\zeta}\right|,$$

and thus

$$\left\|\sigma\left(|D\zeta|^2\right)D\zeta - \sigma\big(|D\tilde{\zeta}|^2\big)D\tilde{\zeta}\right\|_q$$

$$\leq s_\alpha\left\|\mathcal{F}\big(|D\zeta|+|D\tilde{\zeta}|\big)\right\|_\infty\left\|D\big(\zeta-\tilde{\zeta}\big)\right\|_q$$

$$\leq s_\alpha\mathcal{F}\big(\|D\zeta\|_\infty + \|D\tilde{\zeta}\|_\infty\big)\left\|D\big(\zeta-\tilde{\zeta}\big)\right\|_q.$$

The claimed result is then proven. ∎

Proposition 6.3 *Assume that $0<\varepsilon<1$, $\alpha>1$ and let ζ, $\tilde{\zeta}$ be in $W^{2,q}(\Omega)$ with $q>n$. Then the following estimates hold*

$$\left\|\nabla\cdot\big(\sigma_\varepsilon(|D\zeta|^2)D\zeta\big)\right\|_q \leq 4\bar{s}_\alpha\,\mathcal{L}\left(\|D\zeta\|_\infty\right)\|\nabla\zeta\|_{1,q} \tag{6.9}$$

$$\left\|\sigma_\varepsilon(|D\zeta|^2)D\zeta - \sigma_\varepsilon(|D\tilde{\zeta}|^2)D\tilde{\zeta}\right\|_q \leq \bar{s}_\alpha\,\mathcal{L}\big(\|D\zeta\|_\infty+\|D\tilde{\zeta}\|_\infty\big)\left\|\nabla(\zeta-\tilde{\zeta})\right\|_q \tag{6.10}$$

with $\mathcal{L}(x) = x\,(1+x)^{(\alpha-3)^+}$ and $\bar{s}_\alpha = (|\alpha-2|,1)^+2^{(\alpha-3)^+}$.

Proof. The proof is split into two steps

Step 1. Let us first prove that for all $x,y\in I\!R^+$, the following estimates hold

$$\left|\sigma_\varepsilon(x^2)\right| \leq \bar{s}_\alpha\mathcal{L}(x) \qquad \text{and} \qquad \left|\sigma'_\varepsilon\left(x^2\right)\right|x^2 \leq \bar{s}_\alpha\mathcal{L}(x), \tag{6.11}$$

$$\left|\left(1+\sqrt{\varepsilon+x^2}\right)^{\alpha-2} - \left(1+\sqrt{\varepsilon+y^2}\right)^{\alpha-2}\right|$$

$$\leq \bar{s}_\alpha\left(1+x+y\right)^{(\alpha-3)^+}|x-y|\,, \tag{6.12}$$

where $\mathcal{L}$ and $\bar{s}_\alpha$ are as in the statement.

• Suppose that $\alpha\in[2,3]$. Then $|\sigma_\varepsilon(x)| = \sigma_\varepsilon(x)$. Consider the function $f : I\!R^+ \longrightarrow I\!R$ defined by

$$f(x) = |\sigma_\varepsilon(x)| - \sqrt{x} = \left(1+\sqrt{\varepsilon^2+x}\right)^{\alpha-2} - (1+\varepsilon)^{\alpha-2} - \sqrt{x}.$$

For $x>0$, we have

$$\begin{aligned} f'(x) &= \tfrac{\alpha-2}{2\sqrt{\varepsilon^2+x}}\left(1+\sqrt{\varepsilon^2+x}\right)^{\alpha-3} - \tfrac{1}{2\sqrt{x}} \\ &= \tfrac{1}{2\sqrt{x}}\left(\tfrac{(\alpha-2)\sqrt{x}}{\sqrt{\varepsilon^2+x}}\left(1+\sqrt{\varepsilon^2+x}\right)^{\alpha-3} - 1\right) \leq \tfrac{\alpha-3}{2\sqrt{x}} \leq 0. \end{aligned}$$

Therefore, f is a decreasing function and $f(x) \leq f(0) = 0$ for all $x \in I\!R^+$, i.e.

$$|\sigma_\varepsilon(x)| \leq \sqrt{x} \qquad \text{for all } x \in I\!R^+. \tag{6.13}$$

• Suppose that $\alpha \in]1,2]$. Then

$$|\sigma_\varepsilon(x)| = -\sigma_\varepsilon(x) = \frac{\left(1+\sqrt{\varepsilon^2+x}\right)^{2-\alpha} - (1+\varepsilon)^{2-\alpha}}{(1+\varepsilon)^{2-\alpha}\left(1+\sqrt{\varepsilon^2+x}\right)^{2-\alpha}} \leq \left(1+\sqrt{\varepsilon^2+x}\right)^{2-\alpha} - (1+\varepsilon)^{2-\alpha}$$

and using arguments similar to the previous case, we deduce that

$$|\sigma_\varepsilon(x)| \leq \sqrt{x} \qquad \text{for all } x \in I\!R^+. \tag{6.14}$$

• Suppose that $\alpha > 3$. Then $|\sigma_\varepsilon(x)| = \sigma_\varepsilon(x)$. Consider the function $g : I\!R^+ \longrightarrow I\!R$ defined by

$$\begin{aligned} g(x) &= |\sigma_\varepsilon(x)| - (\alpha-2)\sqrt{x}\left(1+\sqrt{\varepsilon^2+x}\right)^{\alpha-3} \\ &= \left(1+\sqrt{\varepsilon^2+x}\right)^{\alpha-2} - (1+\varepsilon)^{\alpha-2} - (\alpha-2)\sqrt{x}\left(1+\sqrt{\varepsilon^2+x}\right)^{\alpha-3}. \end{aligned}$$

For $x > 0$, we have

$$g'(x) = \frac{(\alpha-2)}{2\sqrt{\varepsilon^2+x}}\left(1+\sqrt{\varepsilon^2+x}\right)^{\alpha-4}\left(1+\sqrt{\varepsilon^2+x} - \sqrt{x} - \frac{\sqrt{\varepsilon^2+x}}{\sqrt{x}}\left(1+\sqrt{\varepsilon^2+x}\right)\right).$$

Observing that

$$1+\sqrt{\varepsilon^2+x} - \sqrt{x} \leq 1+\varepsilon \qquad \text{and} \qquad \frac{\sqrt{\varepsilon^2+x}}{\sqrt{x}}\left(1+\sqrt{\varepsilon^2+x}\right) \geq 1+\varepsilon$$

we deduce that g is a decreasing function and $g(x) \leq g(0) = 0$ for all $x \in I\!R^+$. Therefore,

$$\begin{aligned} |\sigma_\varepsilon(x)| &\leq (\alpha-2)\sqrt{x}\left(1+\sqrt{\varepsilon^2+x}\right)^{\alpha-3} \leq (\alpha-2)\sqrt{x}\left(1+\varepsilon+\sqrt{x}\right)^{\alpha-3} \\ &\leq (\alpha-2)2^{\alpha-3}\sqrt{x}\left(1+\sqrt{x}\right)^{\alpha-3}. \end{aligned} \tag{6.15}$$

Taking into account (6.13), (6.14) and (6.15), we obtain $(6.11)_1$. On the other hand, simple calculation show that

$$|\sigma'_\varepsilon(x)|\, x = \frac{|\alpha-2|\sqrt{x}}{2\sqrt{\varepsilon^2+x}}\sqrt{x}\left(1+\sqrt{\varepsilon^2+x}\right)^{\alpha-3}$$

$$\leq |\alpha-2|\sqrt{x}\left(1+\sqrt{\varepsilon^2+x}\right)^{\alpha-3} \leq |\alpha-2|\sqrt{x}\left(1+\sqrt{\varepsilon^2+x}\right)^{(\alpha-3)^+}$$

$$\leq |\alpha-2|\sqrt{x}\left(1+\varepsilon+\sqrt{x}\right)^{(\alpha-3)^+} \leq |\alpha-2|2^{(\alpha-3)^+}\sqrt{x}\left(1+\sqrt{x}\right)^{(\alpha-3)^+}$$

which gives $(6.11)_2$.

• Let us recall (see [2]) that for every $a, b \in I\!R^+$, we have

$$\left|(1+a)^{\alpha-2} - (1+b)^{\alpha-2}\right| \leq (|\alpha-2|, 1)^+ \, |a-b| \left(1+(a,b)^+\right)^{(\alpha-2)^+-1}.$$

Setting $a = \sqrt{\varepsilon^2 + x^2}$ and $b = \sqrt{\varepsilon^2 + y^2}$, we deduce that

$$\begin{aligned}
&\left|\left(1+\sqrt{\varepsilon^2+x^2}\right)^{\alpha-2} - \left(1+\sqrt{\varepsilon^2+y^2}\right)^{\alpha-2}\right| \\
&\leq (|\alpha-2|,1)^+ \left|\sqrt{\varepsilon^2+x^2} - \sqrt{\varepsilon^2+y^2}\right| \left(1+\sqrt{\varepsilon^2+((x,y)^+)^2}\right)^{(\alpha-2)^+-1} \\
&= (|\alpha-2|,1)^+ \, |x-y| \, \tfrac{x+y}{\sqrt{\varepsilon^2+x^2}+\sqrt{\varepsilon^2+y^2}} \left(1+\sqrt{\varepsilon^2+((x,y)^+)^2}\right)^{(\alpha-2)^+-1} \\
&\leq (|\alpha-2|,1)^+ \, |x-y| \left(1+\sqrt{\varepsilon^2+((x,y)^+)^2}\right)^{(\alpha-2)^+-1} \\
&\leq (|\alpha-2|,1)^+ \, |x-y| \left(1+\sqrt{\varepsilon^2+((x,y)^+)^2}\right)^{(\alpha-3)^+} \\
&\leq (|\alpha-2|,1)^+ \, |x-y| \left(1+\varepsilon+(x,y)^+\right)^{(\alpha-3)^+} \\
&\leq (|\alpha-2|,1)^+ \, 2^{(\alpha-3)^+} \, |x-y| \left(1+(x,y)^+\right)^{(\alpha-3)^+}
\end{aligned}$$

which gives estimate (6.12).

• Estimate (6.9) (respect. estimate (6.10)) can be obtained by taking into account (6.11) (respect. (6.12)) and following step by step the proof of estimate (6.2) (respect. estimate (6.3)). ∎

We finally derive conditions that guarantee existence of a real root for auxiliary functions involved in the proof of our main results.

Proposition 6.4 *Let* $\gamma_\alpha = \frac{((\alpha,3)^+-2)^{(\alpha,3)^+-2}}{((\alpha,3)^+-1)^{(\alpha,3)^+-1}}$ *and let* $F : I\!R^+ \longrightarrow I\!R$ *be the function defined by*

$$F(\delta) = A\delta^2 - \delta + C\,\delta\mathcal{F}(\delta) + D$$

where A*,* C*,* D *are positive constants and where* $\mathcal{F}$ *is defined in Proposition* 6.2*. If the following assertion holds*

$$AD + CD^{2r_\alpha}\,(1+D)^{(\alpha-4)^+} \leq \gamma_\alpha,$$

then F *possesses at least one root* δ_0*. Moreover,* $\delta_0 > D$ *and for every* $\beta \in [1,2]$ *the following estimate holds*

$$\tfrac{\beta-1}{\beta}\delta_0 + \tfrac{2-\beta}{\beta}A\delta_0^2 + \tfrac{2r_\alpha+1-\beta}{\beta}C\delta_0\mathcal{F}(\delta_0) + \tfrac{C(\alpha-4)^+}{\beta}\delta_0^{2r_\alpha+2}\,(1+\delta_0)^{(\alpha-4)^+-1} \leq D.$$

Proof. Since the result is direct for $\alpha \in]1,3]$, we assume in the rest of the proof that $\alpha > 3$. Let us first notice that for every $\lambda \in [0,1]$, we have

$$F(\lambda D) = A\lambda^2 D^2 + C\lambda D\mathcal{F}(\lambda D) + (1-\lambda)\,D > 0$$

and consequently, a root of F should be of the form $\lambda_0 D$ with $\lambda_0 > 1$. For $\lambda \geq 1$, consider the functions G and H defined by

$$G(\lambda) = F(\lambda D) = A(\lambda D)^2 + C(\lambda D)^{2r_\alpha+1} (1+\lambda D)^{(\alpha-4)^+} - \lambda D + D,$$

$$H(\lambda) = \left(AD^2 + CD^{2r_\alpha+1} (1+D)^{(\alpha-4)^+}\right) \lambda^{\alpha-1} - \lambda D + D.$$

Let us first compare these two functions and their derivatives. Straightforward calculation shows that

$$H(\lambda) - G(\lambda)$$

$$= AD^2 \left(\lambda^{\alpha-1} - \lambda^2\right) + CD^{2r_\alpha+1} \left(\lambda^{\alpha-1} (1+D)^{(\alpha-4)^+} - \lambda^{2r_\alpha+1} (1+\lambda D)^{(\alpha-4)^+}\right).$$

Since for $\lambda \geq 1$ we have $\lambda^2 \leq \lambda^{\alpha-1}$ and

$$\lambda^{2r_\alpha+1} (1+\lambda D)^{(\alpha-4)^+} \leq \lambda^{2r_\alpha+1} (\lambda+\lambda D)^{(\alpha-4)^+} = \lambda^{\alpha-1} (1+D)^{(\alpha-4)^+},$$

we deduce that

$$G(\lambda) \leq H(\lambda) \qquad \text{for all } \lambda \geq 1. \tag{6.16}$$

Similarly,

$$H'(\lambda) - G'(\lambda)$$

$$= AD^2 \left((\alpha-1)\lambda^{\alpha-2} - 2\lambda\right) + CD^{2r_\alpha+1}(\alpha-1) (1+D)^{(\alpha-4)^+} \lambda^{\alpha-2}$$

$$-CD^{2r_\alpha+1}\lambda^{2r_\alpha} (1+\lambda D)^{(\alpha-4)^+-1} (2r_\alpha + 1 + (\alpha-1)D\lambda).$$

Since for $\lambda \geq 1$, we have $2\lambda \leq (\alpha-1)\lambda^{\alpha-2}$ and

$$\lambda^{2r_\alpha} (1+\lambda D)^{(\alpha-4)^+-1} (2r_\alpha + 1 + (\alpha-1)D\lambda)$$

$$\leq (\alpha-1)\lambda^{2r_\alpha} (1+\lambda D)^{(\alpha-4)^+} \leq (\alpha-1)\lambda^{\alpha-2} (1+D)^{(\alpha-4)^+},$$

it follows that

$$G'(\lambda) \leq H'(\lambda) \qquad \text{for all } \lambda \geq 1. \tag{6.17}$$

On the other hand, the function H admits a critical point

$$\lambda_* = \frac{1}{\left((\alpha-1)\left(AD + CD^{2r_\alpha}(1+D)^{(\alpha-4)^+}\right)\right)^{\frac{1}{\alpha-2}}}$$

and obviously

$$\lambda_* > 1 \qquad \text{if} \quad AD + CD^{2r_\alpha} (1+D)^{(\alpha-4)^+} < \tfrac{1}{\alpha-1}. \tag{6.18}$$

Moreover, if

$$H(\lambda_*) \leq 0 \quad \iff \quad AD + CD^{2r_\alpha} (1+D)^{(\alpha-4)^+} \leq \tfrac{(\alpha-2)^{\alpha-2}}{(\alpha-1)^{\alpha-1}}, \tag{6.19}$$

then H admits a zero $\lambda_H \in]1, \lambda_*]$ such that $H'(\lambda_H) \leq 0$. If condition (6.19) is fulfilled, then (6.18) holds. By using (6.16), we deduce that $G(\lambda_H) \leq H(\lambda_H) = 0$ and

$$G(1)G(\lambda_H) \leq 0.$$

There then exists $1 < \lambda_G \leq \lambda_H$ such that

$$G(\lambda_G) = 0 \quad \iff \quad F(\lambda_G D) = 0$$

and thus $\delta_0 = \lambda_G D$ is a zero for F. Moreover, (6.17) together with the fact that H' is nondecreasing yields

$$F'(\delta_0) = \tfrac{1}{D} G'(\lambda_G) \leq \tfrac{1}{D} H'(\lambda_G) \leq \tfrac{1}{D} H'(\lambda_H) \leq 0.$$

Therefore, for $\beta \in [1, 2]$, we have

$$\begin{aligned}
\delta_0 &= A\delta_0^2 + C\delta_0^{2r_\alpha+1}\left(1+\delta_0\right)^{(\alpha-4)^+} + D \\
&= \tfrac{1}{\beta}F'(\delta_0)\delta_0 + \tfrac{1}{\beta}\delta_0 - \tfrac{2-\beta}{\beta}A\delta_0^2 - \tfrac{2r_\alpha+1-\beta}{\beta}C\delta_0^{2r_\alpha+1}\left(1+\delta_0\right)^{(\alpha-4)^+} \\
&\quad - \tfrac{C(\alpha-4)^+}{\beta}\delta_0^{2r_\alpha+2}\left(1+\delta_0\right)^{(\alpha-4)^+-1} + D \\
&\leq \tfrac{1}{\beta}\delta_0 - \tfrac{2-\beta}{\beta}A\delta_0^2 - \tfrac{2r_\alpha+1-\beta}{\beta}C\delta_0^{2r_\alpha+1}\left(1+\delta_0\right)^{(\alpha-4)^+} \\
&\quad - \tfrac{C(\alpha-4)^+}{\beta}\delta_0^{2r_\alpha+2}\left(1+\delta_0\right)^{(\alpha-4)^+-1} + D
\end{aligned}$$

which implies the estimate and completes the proof. ∎

Proposition 6.5 *Let* $\gamma_\alpha = \frac{((\alpha,3)^+-2)^{(\alpha,3)^+-2}}{((\alpha,3)^+-1)^{(\alpha,3)^+-1}}$ *and let* $L : I\!R^+ \longrightarrow I\!R$ *be the function defined by*

$$L(\delta) = A\delta^2 - \delta + C\,\delta\mathcal{L}(\delta) + D$$

where A, C, D *are positive constants and where* $\mathcal{L}$ *is defined in Proposition* 6.3. *If the following assertion holds*

$$AD + CD\left(1+D\right)^{(\alpha-3)^+} \leq \gamma_\alpha,$$

then L *possesses at least one root* $\bar{\delta}_0$. *Moreover,* $\bar{\delta}_0 > D$ *and for every* $\beta \in [1, 2]$ *the following estimate holds*

$$\tfrac{\beta-1}{\beta}\bar{\delta}_0 + \tfrac{2-\beta}{\beta}A\bar{\delta}_0^2 + \tfrac{2-\beta}{\beta}C\bar{\delta}_0\mathcal{L}(\bar{\delta}_0) + \tfrac{C(\alpha-3)^+}{\beta}\bar{\delta}_0^3\left(1+\bar{\delta}_0\right)^{(\alpha-3)^+-1} \leq D.$$

Proof. The claimed result is obtained by direct adaptation of the proof of Proposition 6.4. ∎

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