On the Distribution of the Product of Independent Beta Random Variables – Applications

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Abstract

A first approach, based on recently obtained asymptotic expansions of ratios of gamma functions, enables the obtention of the distribution of the product of independent and identically distributed random variables in a much manageable form. However, for the general case, this approach leads to a form which although being much manageable and in line with some previous results, suffers from serious problems of precision and convergence, which have been completely overlooked by other authors and which in most cases prevent its practical use. Nevertheless, it is based on these first results that the authors, using the concept of near-exact distribution, are able to obtain highly manageable but extremely accurate approximations for all cases of the distribution of the product of independent beta random variables. These near-exact approximations, given their high manageability, accuracy and proximity to the exact distribution, may in practice be used instead of the exact distribution.

Keywords: Infinite mixtures of gamma distributions, mixtures of exponentiated gamma distributions, sums of gamma random variables, generalized near-integer gamma distribution.

1. Introduction

The distribution of the product of independent beta random variables is a distribution which plays a key role in Statistics. There are many likelihood ratio test statistics, namely in Multivariate Analysis, whose distribution has been shown to be that of the product of a number of independent beta random variables, as for example the likelihood ratio test statistics to test, under multivariate normality, and indeed under multivariate elliptically contoured or left orthogonal-invariant distributions, the independence of several sets of variables, the equality of several mean vectors, the equality of several variance-covariance matrices or sphericity of the covariance matrix (Anderson, 2003; Muirhead, 2005; Kshirsagar, 1972; Marques et al., 2011), the likelihood ratio test statistics to test similar hypotheses under the complex multivariate normal setting (Khatri, 1965a; Gupta, 1971; Pillai & Jouris, 1971; Coelho et al., 2011), the likelihood ratio test statistics to test multi-sample sphericity or block-scalar or block-matrix sphericity (Marques & Coelho, 2011a,b; Coelho & Marques, 2011a), the likelihood ratio test statistic to test reality of a covariance matrix in a complex Normal distribution (Khatri, 1965b) or the likelihood ratio test statistics to test circularity of the covariance matrix

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and stationarity (Olkin & Press, 1961), or even the likelihood ratio test statistic to test the equality of several exponential distributions, in the case of equal sample sizes (Coelho & Marques, 2011b).

Although along the years many authors have worked on this subject (Tukey & Wilks, 1946; Springer & Thompson, 1966, 1970; Nagarsenker & Das, 1975; Tretter & Walster, 1975; Carter & Springer, 1977; Springer, 1979; Nandi, 1980; Walster & Tretter, 1980; Bharbava & Khatri, 1981; Nagarsenker & Suniaga, 1983; Mathai, 1984; Tang & Gupta, 1984; Nagar et al., 1985; Pederzoli, 1985; Tang & Gupta, 1986), obtaining an explicit accurate and highly manageable expression for both the probability density function (p.d.f.) or the cumulative distribution function (c.d.f.) of this distribution has been a hard task and we are absolutely sure that there is still much room left for improvement.

Our aim is exactly to obtain explicit highly manageable expressions for both the p.d.f. and c.d.f. of extremely well-fitting approximations for this distribution, based on highly accurate and manageable mixture expansions.

One popular form to represent the exact p.d.f. and c.d.f. of the distribution of the product of independent beta random variables has been the use of either the Meijer G or Fox's H functions (Springer, 1979; Carter & Springer, 1977). However, although this is a very handy way to represent these distributions, these representations are not very adequate in practical terms since both Fox's H function and Meijer G function implementations have their drawbacks not only in terms of precision, but mainly in terms of computation time in all the commonly available softwares.

Other authors like Bharbava & Khatri (1981) and Pederzoli (1985) used a multiple series representation where each summation corresponds to a different beta random variable in the product, yielding a too much complicated structure for the distribution, mainly when the number of beta random variables involved is rather large.

Yet other authors like Nagarsenker & Das (1975), Tretter & Walster (1975), Walster & Tretter (1980), Nandi (1980), Nagarsenker & Suniaga (1983), Tang & Gupta (1984) and Mathai (1984) express the distribution of the product of independent beta random variables as infinite mixtures of beta distributions. However these representations have the drawback of not allowing for the development of further better performing approximations and are not able to take advantage from those cases where some of the beta random variables have the same distribution. Furthermore, the distributions in Tretter & Walster (1975), Nandi (1980), Walster & Tretter (1980), Tang & Gupta (1984) and Mathai (1984), depend on the particular ordering of the beta random variables considered, while the series in Nagarsenker & Das (1975) and Nagarsenker & Suniaga (1983) have highly complicated coefficients.

The papers by Tang & Gupta (1986), Nagar et al. (1985) and also Mathai (1984) present results very much in line with the results of what is our first approach to the problem, however without any reference to accuracy and convergence issues which indeed occur with the series obtained. Moreover, opposite to the results obtained in the present paper, in Nagar et al. (1985) the weights in the infinite mixture have a very complicated formulation, while in Tang & Gupta (1986) there is a parameter which does not have an *a priori* well-defined value and in Theorem 2 of Mathai (1984) the parameters in the distribution are not even explicitly obtained.

Furthermore, our approach enables the use of more adequate expansions for the cases where some of the beta random variables involved in the product have the same distribution. In a further step we will determine the weights in our mixture distributions by matching some of the first exact moments, what will enable us to obtain distributions which will converge on the whole support of the random variable, overcoming the problems mentioned in Lemma 3.1 and Remark 3.1 in Tang & Gupta (1986). We may note that this moment matching approach is supported by the well known fact that the product of independent beta random variables, having a bounded support, has its distribution determined by its moments.

In this paper the authors show how, based on asymptotic expansions of the ratio of two gamma functions, presented by Burić & Elezović (2011) but for which much simpler proofs are shown in Appendix A, it is possible to obtain a single mixture of exponentiated gamma distributions as an asymptotic approximation for the exact distribution of the product of any number of independent beta random variables, with easily computable coefficients. However, in practical terms this approach leads to almost impossible to handle difficulties, arising from the fact that the resulting series distributions are either very slowly convergent or even divergent.

To overcome these difficulties the authors, recover in Section 6 the concept of near-exact distribution, and based on this approach they develop near-exact approximations which although remaining highly manageable, lie very close to the exact distribution of the product of independent beta random variables, displaying much better performances than previous approaches proposed by different authors.

Many likelihood ratio test statistics have the same distribution as that of a product of independent beta random variables. In these cases, the first parameter in the distribution of these beta random variables is directly related with the sample sizes, while the second parameter is commonly directly related with the number of variables. The near-exact distributions developed by the authors, when applied to these settings, show very good performances even for situations in which the sample sizes are very small, that is, barely exceeding the number of variables, even when the number of variables involved is large.

2. Some results concerning ratios of gamma functions

Let

$$X_j \sim Beta(a_j, b_j) \quad \text{and} \quad Y_j = -\log X_j\,, \quad j = 1, \ldots, p\,, \tag{1}$$

be a set of p independent random variables and let

$$Z = \prod_{j=1}^{p} X_j\,, \quad W = -\log Z = -\sum_{j=1}^{p} \log X_j = \sum_{j=1}^{p} Y_j\,. \tag{2}$$

We will say that the random variable $Y = -\log X$ has a $Logbeta(a, b)$ distribution and we are interested in the distribution of Z, or, somehow equivalently, in the distribution of W.

At first sight we may think that a good idea to work around the difficulties of the representation of the distribution of Z might be to try to express the distribution of each X_j or Y_j as a mixture.

We might think about using expression following expression (18) together with expression (19) in Tricomi & Erdélyi (1951) or expression (1) in Fields (1966) which may be written as

$$\frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)} \approx \sum_{k=0}^{\infty} \frac{1}{k!}\,\frac{\Gamma(1+\alpha-\beta)}{\Gamma(1+\alpha-\beta-k)}\,B_k^{(1+\alpha-\beta)}(\alpha)\,z^{\alpha-\beta-k}\,, \quad (\text{as } z \to \infty,\ |arg(z+\alpha)| < \pi)\,, \tag{3}$$

where $B_k^{(1+\alpha-\beta)}(\,\cdot\,)$ is the generalized Bernoulli polynomial of degree k and order $1+\alpha-\beta$, in order to obtain an asymptotic expansion for the distribution of the negative logarithm of a beta distributed random variable, which hereafter we will call a logbeta random variable.

In fact, an application of expression (3), with $z = a - \mathrm{i}t$, $\alpha = 0$ and $\beta = b$, yields

$$\frac{\Gamma(a-\mathrm{i}t)}{\Gamma(a+b-\mathrm{i}t)} \approx \sum_{k=0}^{\infty} p_k(b)\,(a-\mathrm{i}t)^{-(b+k)} \quad (\text{as } a \to \infty) \tag{4}$$

with

$$p_k(b) = \frac{1}{k!}\,\frac{\Gamma(1-b)}{\Gamma(1-b-k)}\,B_k^{(1-b)}(0)$$

where $B_k^{(1-b)}(0)$ is the generalized Bernoulli number of degree k and order $1-b$.

The result in (4) enables us to represent asymptotically any *Logbeta*(a,b) distribution as an infinite mixture of *Gamma*$(b+k,a)$ distributions $(k=0,1,\dots)$ — see Appendix B for the notation used for the gamma distribution. Indeed, from (4) we may write the characteristic function (c.f.) of Y_j as

$$\begin{aligned}\Phi_{Y_j}(t) &= E\left(e^{\mathrm{i}tY_j}\right) = E\left(X_j^{-\mathrm{i}t}\right) = \frac{\Gamma(a_j+b_j)}{\Gamma(a_j)}\,\frac{\Gamma(a_j-\mathrm{i}t)}{\Gamma(a_j+b_j-\mathrm{i}t)}\\ &\approx \sum_{k=0}^{\infty}\underbrace{\frac{\Gamma(a_j+b_j)}{\Gamma(a_j)}\,\frac{p_k(b_j)}{a_j^{b_j+k}}}_{p_k^*(a_j,b_j)}\,a_j^{b_j+k}\,(a_j-\mathrm{i}t)^{-(b_j+k)}\,,\quad (\text{as } a_j\to\infty)\end{aligned}$$

which is the c.f. of an infinite mixture of *Gamma*(b_j+k,a_j) distributions, with weights $p_k^*(a_j,b_j)$ $(k=0,1,\dots)$.

Then in order to obtain an asymptotic representation for the distribution of W we might think about convoluting p independent logbeta distributions in the form of mixtures of gamma distributions, in order to obtain, by exponentiation, the distribution of the corresponding product of independent beta random variables.

However, the problem is that in this case the asymptotic distribution of W would be an infinite mixture of sums of independent gamma random variables, with possibly different rate parameters a_j. Then, the distribution of each of these sums would have itself to be expressed in the form of a mixture, rendering the final expression for the whole distribution not manageable at all.

To overcome these difficulties, our aim is to approximate asymptotically the distribution of W by a single infinite mixture of gamma distributions. This may be achieved with the following approach.

Let, for the a_j in (1),

$$a=\min(a_1,a_2,\dots,a_p)\,, \tag{5}$$

and then, for

$$\alpha_j=a_j-a \quad \text{and} \quad \beta_j=a_j+b_j-a=\alpha_j+b_j\,, \quad (j=1,\dots,p)\,, \tag{6}$$

using (3) with $z=a-\mathrm{i}t$, write

$$\begin{aligned}\frac{\Gamma(a_j-\mathrm{i}t)}{\Gamma(a_j+b_j-\mathrm{i}t)}=\frac{\Gamma(a-\mathrm{i}t+\alpha_j)}{\Gamma(a-\mathrm{i}t+\beta_j)} &\approx \sum_{k=0}^{\infty}\frac{1}{k!}\,\frac{\Gamma(1-b_j)}{\Gamma(1-b_j-k)}\,B_k^{(1-b_j)}(\alpha_j)\,(a-\mathrm{i}t)^{-(b_j+k)} \quad (\text{as } a\to\infty)\\ &= \Gamma(1-b_j)\,(a-\mathrm{i}t)^{-b_j}\sum_{k=0}^{\infty}\frac{1}{k!\,\Gamma(1-b_j-k)}\,B_k^{(1-b_j)}(\alpha_j)\,(a-\mathrm{i}t)^{-k}\,.\end{aligned}$$

Then we may write

$$\begin{aligned}\Phi_W(t) &= \prod_{j=1}^{p}\frac{\Gamma(a_j+b_j)}{\Gamma(a_j)}\,\frac{\Gamma(a_j-\mathrm{i}t)}{\Gamma(a_j+b_j-\mathrm{i}t)} && (7)\\ &\approx C\,(a-\mathrm{i}t)^{-b}\prod_{j=1}^{p}\sum_{k=0}^{\infty}\underbrace{\frac{1}{k!\,\Gamma(1-b_j-k)}\,B_k^{(1-b_j)}(\alpha_j)}_{q_{jk}}\,(a-\mathrm{i}t)^{-k} && (8)\\ &= C\sum_{k=0}^{\infty}\frac{\omega_k}{a^{b+k}}\,a^{b+k}\,(a-\mathrm{i}t)^{-(b+k)} \qquad (\text{as } a\to\infty) && (9)\end{aligned}$$

where

$$C = \prod_{j=1}^{p} \frac{\Gamma(1-b_j)\,\Gamma(a_j+b_j)}{\Gamma(a_j)}\,, \qquad b = \sum_{j=1}^{p} b_j\,, \tag{10}$$

and

$$\omega_k = \sum_{r_1+r_2+\ldots+r_p=k} q_{1r_1} q_{2r_2} \ldots q_{pr_p}\,,$$

so that

$$\prod_{j=1}^{p} \sum_{k=0}^{\infty} q_{jk}\,(a-\mathrm{i}t)^{-k} = \sum_{k=0}^{\infty} \omega_k\,(a-\mathrm{i}t)^{-k}\,,$$

where each sequence $\{r_1, r_2, \ldots, r_p\}$ is a weak composition of the integer k into p parts, being the sum extended to the complete set of these compositions, which cardinality is $\binom{k+p-1}{p-1} = \binom{k+p-1}{k}$ (see for example Heuback & Mansour (2009)).

Expression (9) shows that for large enough a and for b in (10) the c.f. of W in (2) may be asymptotically approximated by the c.f. of an infinite mixture of $Gamma(b+k, a)$ $(k = 0, 1, \ldots)$ distributions, with weights $C\omega_k/a^{b+k}$.

However, in the more general case we may have each pair (a_j, b_j) in (7) repeated say m_j times. Although in this case we could still use the same approach as above, it happens that for each set of random variables with the same set of parameters (a_j, b_j) the computation of the weights in the mixture will be much more efficient if carried out through a slightly different way. In this case, instead of (3) we should use the more general expression

$$\left(\frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)}\right)^{m} \approx \sum_{k=0}^{\infty} \nu_{k,m}(\alpha,\beta)\, z^{m(\alpha-\beta)-k}\,, \quad (\text{as } z \to \infty,\ |arg(z+\alpha)| < \pi)\,, \tag{11}$$

with

$$\nu_{k,m}(\alpha,\beta) = \frac{1}{k} \sum_{j=1}^{k} j\,\delta_{j,m}(\alpha,\beta)\,\nu_{k-j,m}(\alpha,\beta)\,, \quad (k = 1, 2, \ldots), \quad \nu_{0,m}(\alpha,\beta) = 1\,, \tag{12}$$

where

$$\delta_{j,m}(\alpha,\beta) = (-1)^j m\, \frac{B_{j+1}(\beta) - B_{j+1}(\alpha)}{j\,(j+1)}\,, \tag{13}$$

and where $B_j(a)$ represents the value of the Bernoulli polynomial of degree j in a.

Expression (11) is expression (2.1) in Burić & Elezović (2011) and it may be quite easily obtained from Barnes expansion of the logarithm of the gamma function in (A.1). See Appendix A for a simpler and straightforward demonstration. The use of this expression is shown in the next section.

3. General results concerning the exact distribution of the product of independent beta random variables

3.1. The general case

In this 'general case' we will allow some of the beta random variables to be identically distributed. Let then

$$X_\ell \sim Beta(a_\ell, b_\ell) \qquad \ell = 1, \ldots, p$$

be a set of p independent random variables. Let then $q(\leq p)$ be the number of distinct pairs of parameters (a_ℓ, b_ℓ) among the p random variables X_ℓ and let (a_j, b_j) $(j = 1, \ldots, q)$ represent the set of distinct pairs (a_ℓ, b_ℓ) $(\ell = 1, \ldots, p)$. Let further m_j be the number of times the pair of parameters (a_j, b_j) appears among the p random variables X_ℓ, thus with

$$p = \sum_{j=1}^{q} m_j\,,$$

and of course with $q = p$ if all $m_j = 1$.

Since from (11), taking a as defined in (5), $z = a - \mathrm{i}t$ and α_j and β_j as in (6), we may write

$$\left(\frac{\Gamma(a_j - \mathrm{i}t)}{\Gamma(a_j + b_j - \mathrm{i}t)}\right)^{m_j} = \left(\frac{\Gamma(a - \mathrm{i}t + \alpha_j)}{\Gamma(a - \mathrm{i}t + \beta_j)}\right)^{m_j} \approx \sum_{k=0}^{\infty} \nu_{k,m_j}(\alpha_j, \beta_j)\,(a - \mathrm{i}t)^{-m_j b_j - k}$$

We may thus write, for W in (2),

$$\Phi_W(t) = \prod_{j=1}^{q} \left(\frac{\Gamma(a_j + b_j)}{\Gamma(a_j)}\right)^{m_j} \left(\frac{\Gamma(a_j - \mathrm{i}t)}{\Gamma(a_j + b_j - \mathrm{i}t)}\right)^{m_j} \tag{14}$$

$$\approx C\,(a - \mathrm{i}t)^{-b} \prod_{j=1}^{q} \sum_{k=0}^{\infty} \nu_{k,m_j}(\alpha_j, \beta_j)\,(a - \mathrm{i}t)^{-k} \quad (\text{as } a \to \infty) \tag{15}$$

where $\nu_{k,m_j}(\alpha_j, \beta_j)$ $(k = 0, 1, \ldots;\ j = 1, \ldots, q)$ are given by (12)-(13) and

$$C = \prod_{j=1}^{q} \left(\frac{\Gamma(a_j + b_j)}{\Gamma(a_j)}\right)^{m_j} \quad \text{and} \quad b = \sum_{j=1}^{q} m_j b_j\,. \tag{16}$$

In (15),

$$\prod_{j=1}^{q} \sum_{k=0}^{\infty} \nu_{k,m_j}(\alpha_j, \beta_j)\,(a - \mathrm{i}t)^{-k} = \sum_{k=0}^{\infty} \omega_k\,(a - \mathrm{i}t)^{-k}$$

with

$$\omega_k = \sum_{r_1 + r_2 + \ldots + r_q = k} \nu_{r_1,m_1}(\alpha_1, \beta_1)\,\nu_{r_2,m_2}(\alpha_2, \beta_2)\,\ldots\,\nu_{r_q,m_q}(\alpha_q, \beta_q)\,, \tag{17}$$

where, as in the previous section, each sequence $\{r_1, r_2, \ldots, r_q\}$ is a weak composition of the integer k into q parts, and the sum is extended to the complete set of these compositions, which cardinality is $\binom{k+q-1}{q-1} = \binom{k+q-1}{k}$.

From (15) and (17) we may write

$$\Phi_W(t) \approx C \sum_{k=0}^{\infty} \frac{\omega_k}{a^{b+k}}\, a^{b+k}\, (a - \mathrm{i}t)^{-(b+k)} \quad (\text{as } a \to \infty) \tag{18}$$

for a as in (5), C and b given by (16) and ω_k given by (17).

Expression (18) shows that for sufficiently large a the exact distribution of W is well approximated by an infinite mixture of $Gamma(b+k, a)$ distributions, with weights $C\omega_k/a^{b+k}$ $(k = 0, 1, \ldots)$, and as such, the exact distribution of $Z = e^{-W}$ will be also well approximated by the corresponding mixture, that is, with the same weights, of exponentiated $Gamma(b+k, a)$ distributions.

From (18) we may thus write

$$f_W(w) \approx C \sum_{k=0}^{\infty} \frac{\omega_k}{\Gamma(b+k)}\, e^{-aw}\, w^{b+k-1}\,, \quad (w > 0)\,,$$

and

$$F_W(w) \approx C \sum_{k=0}^{\infty} \omega_k\, \frac{\Gamma^*(b+k, aw)}{\Gamma(b+k)}\,, \quad (w > 0)\,,$$

where

$$\Gamma^*(b+k, aw) = \int_0^{aw} e^{-aw}\, w^{b+k-1}\, dw$$

is a version of the incomplete gamma function, and therefore also

$$f_Z(z) \approx C \sum_{k=0}^{\infty} \frac{\omega_k}{\Gamma(b+k)}\, z^{a-1}\, (-\log z)^{b+k-1}\,, \quad (0 < z \le 1)\,,$$

and

$$F_Z(z) \approx C \sum_{k=0}^{\infty} \omega_k\, \frac{\Gamma^*(b+k, -a\,\log z)}{\Gamma(b+k)}\,, \quad (0 < z \le 1)\,.$$

3.2. The particular case of all different sets of pairs of parameters (a_j, b_j)

In case all p random variables X_j have a different set of parameters (a_j, b_j), then all the m_j defined in the previous subsection will be equal to 1 and we will have $q = p$. In this case W will have an approximate c.f. still given by (15) and (18) in the previous subsection, for all $m_j = 1$, or, if we prefer, given by (8) or (9) in Section 2.

3.3. The particular case of all equal sets of pairs of parameters (a_j, b_j)

In case all p random variables X_j have the same distribution, although remaining independent, then we may either take $q = p$, with all $m_j = 1$ and then apply the results in Subsection 3.1, or take $q = 1$, with $m_1 = p$ and then apply the results in Subsection 3.1. From the derivations in Section 2 we are able to see that the second choice is far more efficient in terms of the computation of the weights ω_k.

However, using expression (7.1) in Burić & Elezović (2011), we may obtain a much faster converging series or mixture distribution. This expression is expression (A.9) in Appendix A, where we undertake a much simpler and straightforward demonstration path.

In this case, if we take $a_j = a$ and $b_j = b$ ($j = 1, \ldots, p$), and if we also take in (A.9), $z = a - \mathrm{i}t$, $\alpha = 0$ and $\beta = b$, we may write, for $\eta = (1 - b)/2$,

$$
\begin{aligned}
\Phi_W(t) &= \left(\frac{\Gamma(a+b)}{\Gamma(a)}\,\frac{\Gamma(a-\mathrm{i}t)}{\Gamma(a+b-\mathrm{i}t)}\right)^p \\
&\approx C\sum_{k=0}^{\infty}\nu_{2k,p}(\eta, b+\eta)\left(a-\mathrm{i}t+\frac{b-1}{2}\right)^{-pb-2k} \\
&= C\sum_{k=0}^{\infty}\frac{\nu_{2k,p}(\eta, b+\eta)}{\left(a+\frac{b-1}{2}\right)^{pb+2k}}\left(a+\frac{b-1}{2}\right)^{pb+2k}\left(a+\frac{b-1}{2}-\mathrm{i}t\right)^{-(pb+2k)}
\end{aligned}
$$

with $\nu_{2k,p}(\eta, b+\eta)$ given by (12)-(13) and

$$
C = \left(\frac{\Gamma(a+b)}{\Gamma(a)}\right)^p ,
$$

which shows that in this case the distribution of W is asymptotically approximated, for increasing $a+\frac{b-1}{2}$, by an infinite mixture of $Gamma\left(pb+2k, a+\frac{b-1}{2}\right)$ distributions with weights $C\nu_{2k,p}(\eta, b+\eta)\Big/\left(a+\frac{b-1}{2}\right)^{pb+2k}$ ($k = 0, 1, \ldots$).

3.4. The distribution of the product of powers of independent beta random variables

So far we only dealt with the distribution of the product of independent beta random variables as expressed in (2), that is, with the beta random variables raised to the power 1. Of course, the results obtained are easily extended to the case where all the beta random variables are raised to some common positive power. In this case one only has to consider the distribution of the product of the random variable $W = -\log Z$ considered so far, multiplied by that power or the distribution of the exponential of this random variable, if the distribution of Z is desired, being both very simple to obtain by simple transformation.

However, there may be cases where we are interested in the distribution of the random variable

$$
Z = \prod_{j=1}^{p} X_j^{c_j} \tag{19}
$$

where X_j are as in (2) and the c_j are positive reals, generally with different values for $j = 1, \ldots, p$. In this case the problem becomes much harder to tackle.

As in Subsection 3.1, let us suppose that among the p random variables X_ℓ ($\ell = 1, \ldots, p$) there are $q \le p$ of them with different first and second parameter or raised to a different power, that is, let us suppose that there are $q \le p$ different triplets (a_j, b_j, c_j) ($j = 1, \ldots, q$).

Let us further suppose that the triplet (a_j, b_j, c_j) appears m_j times ($j = 1, \ldots, q$), once again with $p = \sum_{j=1}^{q} m_j$. Then, for $a = \min(a_1, \ldots, a_q)$ and for α_j and β_j given by (6), we may write the c.f. of $W = -\log Z$

as

$$
\begin{aligned}
\Phi_W(t) &= \prod_{j=1}^{q} \left(\frac{\Gamma(a_j + b_j)}{\Gamma(a_j)} \right)^{m_j} \left(\frac{\Gamma(a_j - c_j \mathrm{i}t)}{\Gamma(a_j + b_j - c_j \mathrm{i}t)} \right)^{m_j} \\
&\approx C \prod_{j=1}^{q} \sum_{k=0}^{\infty} \nu_{k,m_j}(\alpha_j, \beta_j)\, (a - c_j \mathrm{i}t)^{-(m_j b_j + k)} \\
&= C \prod_{j=1}^{q} \sum_{k=0}^{\infty} \nu_{k,m_j}(\alpha_j, \beta_j) \left(\frac{a}{c_j} - \mathrm{i}t \right)^{-(m_j b_j + k)} (c_j)^{-(m_j b_j + k)} .
\end{aligned}
$$

The problem now is hard to tackle because the rate parameters a/c_j are now function of j, that is, their value varies now with j, and there is no simple way to work around this fact.

Of course we may take each set of beta random variables with parameters (a_j, b_j, c_j) individually and take the approach first outlined in Section 2, but this, as stated in that section, would lead us to an almost intractable sum of infinite mixtures, with the concomitant arduous problem related with the convergence of truncations of these distributions, when we try to use them in applications.

However, although there seems to be no really satisfactory formulation for the exact distribution of the random variable Z in this case, as we will see in Sections 6 and 8, the near-exact approach is able to handle this case with no big problems and extremely satisfactory results. See Subsection 6.2 for the theoretical developments and Subsection 8.3 for an application and some numerical results.

4. Particular cases with finite representation for the exact distribution

The contents relating to this section appear in a separate document.

5. The practical implementation of the exact distribution in Section 3 – another look at the exact distribution on the way to near-exact distributions

Since in any of the cases addressed in Section 3 the exact distribution always takes the form of an infinite mixture, in practice when using these distributions we will have to truncate them.

For a truncation corresponding to a c.f. of the form

$$
C \sum_{k=0}^{m} \frac{\omega_k}{a^{b+k}}\, a^{b+k} \,(a - \mathrm{i}t)^{-(b+k)} \tag{20}
$$

an upper-bound for the truncation error is

$$
1 - C \sum_{k=0}^{m} \frac{\omega_k}{a^{b+k}} .
$$

But, as we may see from the results in Section 7, we may do much better by using a truncation with completion of the weights in such a way that they add up to 1, that is, by using in (20) ω_k given by (17) and (12)-(13) for $k = 0, \ldots, m-1$, and

$$
\omega_m = \frac{a^{b+m}}{C} - \sum_{k=0}^{m-1} \omega_k \, a^{m-k} . \tag{21}
$$

Although for quite small values of k the computation of the weights ω_k does not pose any serious problem, that is not anymore the case for large values of k, where the weights ω_k, given the summation in (17), become quite heavy to compute.

Since we have indeed the exact c.f. of W at hand in the form in (14), we may think about determining the weights ω_k in (20), for $k = 0, \ldots, m-1$, in such a manner that the first m derivatives of the c.f. in (20) above, at $t = 0$, match the corresponding derivatives of $\Phi_W(t)$ in (14), setting then ω_m to the value given by (21).

This approach will indeed give much better results than the computation of the weights ω_k through their original expression. See the results in Section 7.

Yet, in order to better analyze the behavior of these approximations we will consider in Section 7 truncations where the first ω_k are computed from (17) and (12)-(13) and the remaining are computed by matching some of the exact moments, that is, we will use a c.f.

$$C \sum_{k=0}^{m^*} \frac{\omega_k}{a^{b+k}}\, a^{b+k}\, (a - \mathrm{i}t)^{-(b+k)} + \sum_{k=m^*+1}^{m^{**}} \pi_{k-m^*}\, a^{b+k}\, (a - \mathrm{i}t)^{-(b+k)} \tag{22}$$

where ω_k ($k = 0, \ldots, m^*$) are computed from (17) and (12)-(13) and π_{k-m^*} ($k = m^* + 1, \ldots, m^{**} - 1$) are computed by matching the first $m^{**} - 1 - m^*$ derivatives at $t = 0$ of $\Phi_W(t)$ and the c.f. in (22), with $\pi_{m^{**}-m^*} = 1 - C \sum_{k=0}^{m^*} \frac{\omega_k}{a^{b+k}} - \sum_{k=m^*+1}^{m^{**}-1} \pi_{k-m^*}$, so that all weights in the mixture add up to 1.

We should remark that although the notation used in this section is the one used in Subsection 3.1, in order to encompass the case treated in Subsection 3.3 one only has to consider $\omega_k = \nu_{2k,p}(\eta, b + \eta)$.

We should also remark that the result obtained in (9) in Section 2 and further manipulated in Section 3 and in this section is very much in line with the results obtained by Tang & Gupta (1986), Mathai (1984) and Nagar et al. (1985), who used different approaches from the one used in the present paper. However, our approach, opposite to what happens in Mathai (1984), enabled us to obtain in a quite simple way the expressions for all coefficients involved in the mixture and also not only enabled us to obtain much simpler expressions for the weights than those in Nagar et al. (1985), as well as to not have any parameter left with a non well-defined value as it happens in Tang & Gupta (1986). Moreover, our approach will also enable us to better understand the problems that arise when we try to implement the use of these results. As we will see with the help of some numerical studies in Section 7, the practical implementation of these series representations and their truncations faces some problems which are not easy to solve and which are related with the facts that

i) while for sets of beta random variables with values of a_j which show a moderately large variability we may need an unsoundly large number of terms in the series to get a good approximation,

ii) on the other hand, the use of a larger number of terms may start to give worse approximations, namely for sets of beta random variables with values of a_j which show a small variability, while the use of a moderately large number of terms may still not give the desired precision.

These problems were completely overlooked by previous authors, whose results may be even more severely affected by the same problem, given that the distributions they obtain vary with the ordering of the beta random variables in the product.

The root of this problem sits indeed on the fact that the base expressions (3) and (11) and actually also Barnes expression (A.1) are non-convergent series, that is, they approximate the gamma ratios or the value of the gamma function on the left hand side of the corresponding expressions, with such approximations improving for increasing values of what in those expressions is the parameter z, that is, for a given number of terms in the summations the approximation is better for larger values of z and it also keeps improving till larger values of the number of terms in the summations for larger values of z, but, for any given value of z the approximations improve since very small numbers of terms in the summations till these numbers reach some threshold and then they become worse and worse as this number of terms in the summations goes above this threshold.

However, one thing we may notice is that, either when computing the weights through their original expression or by matching derivatives, that is, by matching exact moments, for a given number of terms used in (20), the approximations are always much better when the b_j's fall between zero and one. The ascertainment of this fact may lead us to a different approach in which we would 'extract' the integer part of the b_j's and somehow set it apart, whenever any of the b_j's exceeds the value of one. This is indeed the approach pursued in the next section.

6. Near-exact distributions for the product of independent beta random variables

6.1. Near-exact distributions for Z in (2)

In trying to 'extract' the integer part of the b_j's and set it apart, whenever any of the b_j's exceeds the value of one, this will indeed lead us to the use of what has been called 'near-exact distributions'.

In simple terms, near-exact distributions are distributions which keep intact a good part of the exact distribution and which approximate asymptotically the remaining part.

In order to try to keep intact a good part of the exact c.f. when at least one of the b_j's is greater than 1, we may work through the exact c.f. of W as it is done below, using the fact that for any real or complex a and positive integer n we may write

$$\frac{\Gamma(a+n)}{\Gamma(a)} = \prod_{\ell=0}^{n-1}(a+\ell)\,. \tag{23}$$

In fact, using (23), taking $b_j^* = \lfloor b_j \rfloor$ and $b_j^{**} = b_j - b_j^*$, we may write, for the general case of the distribution of W in Subsection 3.1,

$$\begin{aligned}
\Phi_W(t) &= \prod_{j=1}^{q}\left(\frac{\Gamma(a_j+b_j)}{\Gamma(a_j)}\right)^{m_j}\left(\frac{\Gamma(a_j-\mathrm{i}t)}{\Gamma(a_j+b_j-\mathrm{i}t)}\right)^{m_j}\\
&= \prod_{j=1}^{q}\left(\frac{\Gamma(a_j+b_j)}{\Gamma(a_j+b_j^*)}\,\frac{\Gamma(a_j+b_j^*)}{\Gamma(a_j)}\right)^{m_j}\left(\frac{\Gamma(a_j-\mathrm{i}t)}{\Gamma(a_j+b_j^*-\mathrm{i}t)}\,\frac{\Gamma(a_j+b_j^*-\mathrm{i}t)}{\Gamma(a_j+b_j-\mathrm{i}t)}\right)^{m_j}\\
&= \prod_{j=1}^{q}\left(\frac{\Gamma(a_j+b_j)}{\Gamma(a_j+b_j^*)}\,\frac{\Gamma(a_j+b_j^*-\mathrm{i}t)}{\Gamma(a_j+b_j-\mathrm{i}t)}\right)^{m_j}\prod_{\ell=0}^{b_j^*-1}(a_j+\ell)^{m_j}\,(a_j+\ell-\mathrm{i}t)^{-m_j}\\
&= \underbrace{\left\{\prod_{j=1}^{q}\prod_{\ell=0}^{b_j^*-1}(a_j+\ell)^{m_j}\,(a_j+\ell-\mathrm{i}t)^{-m_j}\right\}}_{\Phi_{1,W}(t)}\underbrace{\left\{\prod_{j=1}^{q}\left(\frac{\Gamma(a_j+b_j)}{\Gamma(a_j+b_j^*)}\,\frac{\Gamma(a_j+b_j^*-\mathrm{i}t)}{\Gamma(a_j+b_j-\mathrm{i}t)}\right)^{m_j}\right\}}_{\Phi_{2,W}(t)}
\end{aligned} \tag{24}$$

where $\Phi_{1,W}(t)$ is the c.f. of a sum of $\sum_{j=1}^{q} b_j^*$ independent Gamma random variables, with integer shape parameters m_j and rate parameters $a_j+\ell$ ($\ell = 0,\ldots,b_j^*-1; j = 1,\ldots,q$), which is a generalized integer gamma (GIG) distribution of depth $\sum_{j=1}^{q} b_j^*$, with rate parameters $a_j+\ell$ and shape parameters m_j ($j = 1,\ldots,q$;

$\ell = 0, \ldots, b_j^* - 1$) and $\Phi_{2,W}(t)$ is the c.f. of a sum of $p = \sum_{j=1}^{q} m_j$ independent logbeta random variables, m_j of which have the set of parameters $(a_j + b_j^*, b_j^{**})$ $(j = 1, \ldots, q)$.

We may note that if in (24) some b_j is smaller than 1, then the corresponding b_j^* equals zero and then the corresponding term in $\Phi_{1,W}(t)$ simply vanishes, i.e., equals 1.

Since, opposite to $\Phi_{2,W}(t)$, $\Phi_{1,W}(t)$ corresponds to a very manageable distribution, in building the near-exact distribution for W we will leave $\Phi_{1,W}(t)$ unchanged and we will approximate $\Phi_{2,W}(t)$ in exactly the same way we did approximate $\Phi_W(t)$ in Section 3. Taking

$$a = \min(a_1 + b_1^*, a_2 + b_2^*, \ldots, a_q + b_q^*), \quad \alpha_j = a_j + b_j^* - a, \quad \text{and} \quad \beta_j = \alpha_j + b_j^{**}, \tag{25}$$

we may write

$$\Phi_{2,W}(t) \approx C \sum_{k=0}^{\infty} \frac{\omega_k}{a^{b+k}} \, a^{b+k} \, (a - \mathrm{i}t)^{-(b+k)},$$

where

$$C = \prod_{j=1}^{q} \left(\frac{\Gamma(a_j + b_j)}{\Gamma(a_j + b_j^*)} \right)^{m_j} \qquad \text{and} \qquad b = \sum_{j=1}^{q} m_j b_j^{**}$$

and where ω_k is computed using (17), now with α_j and β_j given by (25), so that we will use as near-exact c.f. for W the c.f.

$$\Phi_W^*(t) = \Phi_{1,W}(t) \, C \sum_{k=0}^{m} \pi_k \, a^{b+k} \, (a - \mathrm{i}t)^{-(b+k)} \tag{26}$$

where the weights π_k $(k = 0, \ldots, m-1)$ will be determined in such a way that

$$\left. \frac{d^h}{dt^h} C \sum_{k=0}^{m} \pi_k \, a^{b+k} \, (a - \mathrm{i}t)^{-(b+k)} \right|_{t=0} = \left. \frac{d^h}{dt^h} \Phi_{2,W}(t) \right|_{t=0}, \qquad h = 1, \ldots, m$$

or, equivalently, in such a way that

$$\left. \frac{d^h}{dt^h} \Phi_W^*(t) \right|_{t=0} = \left. \frac{d^h}{dt^h} \Phi_W(t) \right|_{t=0}, \qquad h = 1, \ldots, m,$$

with $\pi_m = 1 - \sum_{k=0}^{m-1} \pi_k$. The reason to compute the weights π_k in this way is twofold: computed this way, instead of using expressions (17) and (12)-(13), the weights π_k, (i) will yield better approximations and (ii) will be easier to compute for moderately large values of k.

The distribution corresponding to the c.f. $\Phi_W^*(t)$ in (26) is a finite mixture of generalized near-integer gamma (GNIG) distributions (see Coelho (2004) for the definition of this distribution and Appendix B for a definition of its p.d.f. and c.d.f.) which in case all $a_j + \ell$ $(j = 1, \ldots, q;\ \ell = 0, \ldots, b_j^* - 1)$ are different will have depth $1 + \sum_{j=1}^{q} b_j^*$ with shape parameters

$$\underbrace{m_1, \ldots, m_1}_{b_1^* \text{ times}}, \ldots, \underbrace{m_j, \ldots, m_j}_{b_j^* \text{ times}}, \ldots, \underbrace{m_q, \ldots, m_q}_{b_q^* \text{ times}}, b + k$$

and rate parameters $\{a_j + 0, \ldots, a_j + b_j^* - 1 (j = 1, \ldots, q), a\}$.

For the general case, where some of the $a_j + \ell$ may have the same value, let

$$\left\{ a_\nu^*;\ \nu = 1, \ldots, q^* \leq \textstyle\sum_{j=1}^{q} b_j^* \right\} = \{\{a_j + \ell;\ \ell = 0, \ldots, b_j^* - 1;\ j = 1, \ldots, q\}\}$$

represent the set of all different rate parameters $a_j+\ell$, that is, the set of all different values of $a_j+\ell$ ($\ell=0,\dots,b_j^*-1;\, j=1,\dots,q$) and let

$$\left\{m_\nu^*;\ \nu=1,\dots,q^*\le\sum_{j=1}^q b_j^*\right\}=\{\{m_{j\ell}=m_j;\ \ell=0,\dots,b_j^*-1; j=1,\dots,q\}\}_{\{a_j+\ell\}}$$

be the set of the corresponding shape parameters, where the shape parameter m_ν^* is the shape parameter corresponding to a_ν^*, that is, it is the sum of all $m_{j\ell}=m_j$ corresponding to the ν-th distinct value $a_j+\ell$. Then the p.d.f. and c.d.f. corresponding to the c.f. in (26), using the notation in Appendix B for the GNIG distribution are respectively

$$f_W^*(w)=\sum_{k=0}^m \pi_k f^{GNIG}\left(w\,|\,\{m_\nu^*\}_{\nu=1:q^*}, b+k; \{a_\nu^*\}_{\nu=1:q^*}, a; q^*+1\right)$$

and

$$F_W^*(w)=\sum_{k=0}^m \pi_k F^{GNIG}\left(w\,|\,\{m_\nu^*\}_{\nu=1:q^*}, b+k; \{a_\nu^*\}_{\nu=1:q^*}, a; q^*+1\right)$$

and the corresponding p.d.f. and c.d.f. for $Z=e^{-W}$ are

$$f_Z^*(z)=\sum_{k=0}^m \pi_k f^{GNIG}\left(-\log\, z\,|\,\{m_\nu^*\}_{\nu=1:q^*}, b+k; \{a_\nu^*\}_{\nu=1:q^*}, a; q^*+1\right)\frac{1}{z}$$

and

$$F_Z^*(z)=(m+1)-\sum_{k=0}^m \pi_k F^{GNIG}\left(-\log\, z\,|\,\{m_\nu^*\}_{\nu=1:q^*}, b+k; \{a_\nu^*\}_{\nu=1:q^*}, a; q^*+1\right).$$

Built in this way, the near-exact distribution which c.f. is in (26) will provide extremely good approximations to the exact distribution, while remaining quite manageable.

Some of the several vast advantages of the near-exact distributions built in this way are that:

i) they provide the exact distribution in cases where all b_j are integer,

ii) the approximations provided will be even better for cases where the b_j are larger,

iii) they have a much stable performance for a wide range of values of the parameters a_j, b_j and m_j,

iv) they will perform even slightly better in the situations where the other approximations show severe difficulties.

Actually, since the weights π_k in (26) are not going to be computed from (17) and (12)-(13) and we got a bit far from the original formulation, we may also think about a couple of alternative ways to compute the parameter a in (26), based on a somewhat heuristic approach. These may be to take a in (26) as one of the following choices:

$$\begin{array}{l}
\text{i)}\quad a=\min_{j=1,\dots,q}(a_j+b_j^*) \qquad \text{ii)}\quad a=\frac{1}{q}\sum_{j=1}^q(a_j+b_j^*) \qquad \text{iii)}\quad a=\frac{\sum_{j=1}^q(b_j-b_j^*)(a_j+b_j^*)}{\sum_{j=1}^q(b_j-b_j^*)}\\
\text{iv)}\quad a=\frac{\sum_{j=1}^q m_j(a_j+b_j^*)}{\sum_{j=1}^q m_j} \qquad \text{v)}\quad a=\frac{\sum_{j=1}^q m_j(b_j-b_j^*)(a_j+b_j^*)}{\sum_{j=1}^q m_j(b_j-b_j^*)}\\
\text{vi)}\quad \text{the rate parameter } a \text{ in } \Phi^{**}(t)=a^s(a-\mathrm{i}t)^{-s},\ \text{where}\\
\qquad \frac{\partial^h}{\partial t^h}\Phi^{**}(t)\Big|_{t=0}=\frac{\partial^h}{\partial t^h}\Phi_{2,W}(t)\Big|_{t=0},\ h=1,2\\
\text{vii)}\quad \text{the rate parameter } a \text{ in } \Phi^{***}(t)=\pi a^{s_1}(a-\mathrm{i}t)^{-s_1}+(1-\pi)a^{s_2}(a-\mathrm{i}t)^{-s_2},\ \text{where}\\
\qquad \frac{\partial^h}{\partial t^h}\Phi^{***}(t)\Big|_{t=0}=\frac{\partial^h}{\partial t^h}\Phi_{2,W}(t)\Big|_{t=0},\ h=1,\dots,4.
\end{array} \tag{27}$$

Details on the performance of such near-exact approximations may be analyzed in the next section and also in Section 8.

Mathematica modules to compute these as well as all the other approximations developed in this manuscript are available, on demand, from the authors.

These different strategies for the choice of a will be used only for approximations which correspond to the situations studied in Subsections 3.1 and 3.2, since for the situation addressed in Subsection 3.3 there is no doubt in the choice of a, having only to consider that in this case the parameter a in (26) has to be taken as $a + \frac{b-1}{2}$, where b is the common value of the second parameter for all the beta random variables involved.

In cases where all the b_j are smaller than 1, the near-exact distribution will then provide an asymptotic distribution of the same kind of the ones in Section 3. Anyway, for these cases the distributions in Section 3, which are indeed asymptotic distributions, work very well, as we may see from the results in Section 7. Furthermore, these are not the most common cases that occur, mainly when these distributions refer to likelihood ratio test statistics, and, as we may also see from the results in Section 7, by determining some of the weights through equating some of the first exact moments we may obtain approximations which lie very close to the exact distribution.

Actually, in these latter cases where all b_j are smaller than 1, as it may be seen from the results analyzed in Section 7 and reported in Appendix C, a good balance between a number of weights determined through (17) and (12)-(13) and a number of weights determined by equating some of the first exact moments leads usually to the best results and although this approach might also be applied to the general near-exact distributions, for reasons of simplicity and extent of the manuscript we decided to use in the near-exact distributions only weights determined by equating exact moments. The adaptation of the approach followed to this other one of determining some of the weights through their formulation in (17) and (12)-(13) is quite easy and straightforward, although the computation of the weights through (17) and (12)-(13), namely for higher orders, may require more computation than their determination by equating exact moments.

6.2. Near-exact distributions for the product of independent beta random variables raised to different powers

However, in case we plan to take the near-exact approach, it is interesting to note that this case of different powers c_j poses indeed no big problems when we adopt the near-exact approach. It indeed poses no problem at all in what concerns the handling of the part of the c.f. $\Phi_W(t)$ which will be left unchanged, that is, $\Phi_{1,W}(t)$, anyway posing a similar problem to the one described above for $\Phi_{2,W}(t)$, the part of the c.f. $\Phi_W(t)$ to be asymptotically approximated, since in the present case, following the same lines as in (24) we have

$$
\begin{aligned}
\Phi_W(t) &= \left\{ \prod_{j=1}^{q} \prod_{\ell=0}^{b_j^*-1} (a_j+\ell)^{m_j} \, (a_j+\ell-c_j\mathrm{i}t)^{-m_j} \right\} \left\{ \prod_{j=1}^{q} \left(\frac{\Gamma(a_j+b_j)}{\Gamma(a_j+b_j^*)} \, \frac{\Gamma(a_j+b_j^*-c_j\mathrm{i}t)}{\Gamma(a_j+b_j-c_j\mathrm{i}t)} \right)^{m_j} \right\} \\
&= \underbrace{\left\{ \prod_{j=1}^{q} \prod_{\ell=0}^{b_j^*-1} \left(\frac{a_j+\ell}{c_j} \right)^{m_j} \left(\frac{a_j+\ell}{c_j} - \mathrm{i}t \right)^{-m_j} \right\}}_{\Phi_{1,W}(t)} \underbrace{\left\{ \prod_{j=1}^{q} \left(\frac{\Gamma(a_j+b_j)}{\Gamma(a_j+b_j^*)} \, \frac{\Gamma(a_j+b_j^*-c_j\mathrm{i}t)}{\Gamma(a_j+b_j-c_j\mathrm{i}t)} \right)^{m_j} \right\}}_{\Phi_{2,W}(t)}
\end{aligned}
$$

where now $\Phi_{1,W}(t)$ is the c.f. of a sum of $\sum_{j=1}^{q} b_j^*$ independent Gamma random variables, with integer shape parameters m_j and rate parameters $(a_j+\ell)/c_j$ ($\ell = 0, \ldots, b_j^*-1$; $j = 1, \ldots, q$), which is a GIG distribution of depth $\sum_{j=1}^{q} b_j^*$, with rate parameters $(a_j+\ell)/c_j$ and shape parameters m_j ($j = 1, \ldots, q$; $\ell = 0, \ldots, b_j^*-1$) and

$\Phi_{2,W}(t)$ is the c.f. of a sum of $p = \sum_{j=1}^{q} m_j$ independent logbeta random variables, m_j of which have the set of parameters $(a_j + b_j^*, b_j^{**})$ and is multiplied by c_j $(j = 1, \ldots, q)$.

Then, based on a heuristic approach, in building the near-exact distribution for W we will leave $\Phi_{1,W}(t)$ unchanged and we will now approximate $\Phi_{2,W}(t)$ in exactly the same way we did in the previous subsection, now taking a in (26) as one of the following choices:

$$\begin{aligned}
&\text{i)}\quad a = \min_{j=1,\ldots,q}\left(\frac{a_j+b_j^*}{c_j}\right) \qquad \text{ii)}\quad a = \frac{1}{q}\sum_{j=1}^{q}\frac{a_j+b_j^*}{c_j} \qquad \text{iii)}\quad a = \frac{\sum_{j=1}^{q}(b_j-b_j^*)\frac{a_j+b_j^*}{c_j}}{\sum_{j=1}^{q}(b_j-b_j^*)}\\
&\text{iv)}\quad a = \frac{\sum_{j=1}^{q} m_j\frac{a_j+b_j^*}{c_j}}{\sum_{j=1}^{q} m_j} \qquad \text{v)}\quad a = \frac{\sum_{j=1}^{q} m_j(b_j-b_j^*)\frac{a_j+b_j^*}{c_j}}{\sum_{j=1}^{q} m_j(b_j-b_j^*)}\\
&\text{vi)}\quad \text{the rate parameter } a \text{ in } \Phi^{**}(t) = a^s(a-\mathrm{i}t)^{-s}, \text{ where}\\
&\qquad \frac{\partial^h}{\partial t^h}\Phi^{**}(t)\bigg|_{t=0} = \frac{\partial^h}{\partial t^h}\Phi_{2,W}(t)\bigg|_{t=0},\ h=1,2\\
&\text{vii)}\quad \text{the rate parameter } a \text{ in } \Phi^{***}(t) = \pi a^{s_1}(a-\mathrm{i}t)^{-s_1} + (1-\pi)a^{s_2}(a-\mathrm{i}t)^{-s_2}, \text{ where}\\
&\qquad \frac{\partial^h}{\partial t^h}\Phi^{***}(t)\bigg|_{t=0} = \frac{\partial^h}{\partial t^h}\Phi_{2,W}(t)\bigg|_{t=0},\ h=1,\ldots,4\,.
\end{aligned} \tag{28}$$

As it happened in the previous subsection, these different strategies for the choice of a will be used only for approximations which correspond to the situations studied in Subsections 3.1 and 3.2, since for the situation addressed in Subsection 3.3, once again there is no doubt in the choice of a, having now to consider that in this case not only the first and second parameters of all the beta random variables have to be equal but also all the powers to which they are raised have also to be the same. Then we may either take the parameter a in (26) as $\left(a + \frac{b-1}{2}\right)/c$, where a and b are respectively the common values of the first and second parameters for all the beta random variables involved and c the common power, or just take the transformation procedure outlined in Subsection 3.4.

From the results in the next section as well as in Section 8, Subsection 8.3, we may see that this approach really works well in practice.

7. Some evidence better analyzed from numerical studies

While we clearly expect the 'completed' truncation to perform much better than the simple truncation, and the version with the weights computed from the moments to perform even much better, we also expect the near-exact distributions to display an even much better performance than any of these. The results coming out of a few numerical studies may indeed help us in better figuring out the fine behavior of each of these approximations and also to better see that the near-exact approximations are indeed the only ones with a remarkable outstanding performance in all situations, that is, for any combination of values of the parameters involved.

In order to try to keep things not too long, while still giving a good appraisal of the behavior of the several approximations across a number of different situations we considered a set of thirteen scenarios. The differences among the first nine of them are the span of the values of the a_j, the magnitude of the b_j and the number of beta random variables involved in the product, since these are the factors which most affect the performance of the truncations. By doing so we are able to obtain at least a couple of combinations of parameters for which the truncations perform more or less well and a number of other for which they perform quite badly. The thirteen scenarios chosen were:

– scenario I: $a_j = \{5.6, 7.8, 4.5\}$, $b_j = \{2.3, 1.5, 3.4\}$, $m_j = \{2, 1, 2\}$
– scenario II: $a_j = \{5.7, 6.2, 6.0\}$, $b_j = \{2.3, 1.5, 3.4\}$, $m_j = \{2, 1, 2\}$
– scenario III: $a_j = \{5.8, 7.3, 5.5, 7.9, 4.5\}$, $b_j = \{2.6, 1.7, 3.4, 1.4, 3.1\}$, $m_j = \{2, 1, 2, 2, 3\}$
– scenario IV: $a_j = \{5.8, 6.3, 6.0, 6.1, 5.7\}$, $b_j = \{2.6, 1.7, 3.4, 1.4, 3.1\}$, $m_j = \{2, 1, 2, 2, 3\}$
– scenario V: $a_j = \{5.8, 6.3, 6.0, 6.1, 5.7\}$, $b_j = \{3.6, 2.7, 4.4, 2.4, 4.1\}$, $m_j = \{2, 1, 2, 2, 3\}$
– scenario VI: $a_j = \{5.6, 7.8, 4.5\}$, $b_j = \{0.3, 0.5, 0.4\}$, $m_j = \{2, 1, 2\}$
– scenario VII: $a_j = \{5.7, 6.2, 6.0\}$, $b_j = \{0.3, 0.5, 0.4\}$, $m_j = \{2, 1, 2\}$
– scenario VIII: $a_j = \{5.8, 7.3, 5.5, 7.9, 4.5\}$, $b_j = \{0.6, 0.7, 0.4, 0.4, 0.1\}$, $m_j = \{2, 1, 2, 2, 3\}$
– scenario IX: $a_j = \{5.8, 6.3, 6.0, 6.1, 5.7\}$, $b_j = \{0.6, 0.7, 0.4, 0.4, 0.1\}$, $m_j = \{2, 1, 2, 2, 3\}$
– scenario X: $a_j = \{5.6, 7.8, 4.5\}$, $b_j = \{2.3, 1.5, 3.4\}$, $m_j = \{2, 1, 2\}$, $c_j = \{1.6, 2.2, 3.4\}$
– scenario XI: $a_j = \{5.7, 6.2, 6.0\}$, $b_j = \{2.3, 1.5, 3.4\}$, $m_j = \{2, 1, 2\}$, $c_j = \{1.6, 2.2, 3.4\}$
– scenario XII: $a_j = \{5.8, 7.3, 5.5, 7.9, 4.5\}$, $b_j = \{2.6, 1.7, 3.4, 1.4, 3.1\}$, $m_j = \{2, 1, 2, 2, 3\}$, $c_j = \{1.6, 2.2, 3.4, 3.6, 2.1\}$
– scenario XIII: $a_j = \{5.8, 6.3, 6.0, 6.1, 5.7\}$, $b_j = \{2.6, 1.7, 3.4, 1.4, 3.1\}$, $m_j = \{2, 1, 2, 2, 3\}$, $c_j = \{1.6, 2.2, 3.4, 3.6, 2.1\}$.

Scenarios I and II, on one hand, and scenarios III and IV on the other hand, differ only in the span of the a_j. Scenarios I and III have the larger but similar spans and scenarios II and IV have smaller but also similar spans for the a_j. Scenarios I and II have $p = 5$ and $q = 3$, that is, 5 independent beta random variables with 3 different sets of parameters, while scenarios III and IV both have $p = 10$ and $q = 5$. Scenario V is the same as scenario IV, except that all b_j's were increased by 1. Scenarios VI-IX correspond respectively to scenarios I-IV, now with all b_j's smaller than 1, actually taken as the non-integer part of the b_j's in scenarios I-IV, respectively. Finally, scenarios X-XIII refer to situations where we consider the distribution of the product of different powers of independent beta random variables. These scenarios have respectively the same a_j, b_j and m_j as scenarios I-IV. Scenarios X and XI have the same set of power parameters, as also scenarios XII and XIII do.

In the tables in Appendix C we use the following descriptors for the approximations studied:

– for the truncations:

BT – for the truncated series corresponding to the c.f. in (20) (acronym for 'Barnes truncated')

BC – for the truncated series corresponding to the c.f. in (20), with 'completion' of the weights in order to add up to 1 (acronym for 'Barnes completed')

BWFM – for the truncated series corresponding to the c.f. in (20), with all weights but the last, computed by equating exact moments, with the last one defined in such a way that all weights add up to 1 (acronym for 'Barnes with weights from moments')

BWFM2 – for the truncated series corresponding to the c.f. in (22), with the first $m^* + 1$ weights computed using (17) and the remaining computed by equating exact moments and having the last one defined in such a way that all weights add up to 1 (acronym for 'Barnes with weights from moments – 2')

– for the near-exact distributions:

NE-1 through NE-7 for the near-exact distributions which, correspondingly, use a given by i) through vii) in (27) or (28).

From (20) and (22), taking into account that BWFM corresponds to the c.f. in (20) but with the weights computed by matching exact moments, we may easily see that the approach BWFM is indeed equivalent to BWFM2 with $m^* = -1$ and $m^{**} = m$.

For scenarios VI-IX we do not consider the near-exact approximations, since, as remarked in Section 6, given that all b_j are smaller than one, they would correspond to the approximation BWFM2 with $m^* = -1$, or to the approximation BWFM. On the other hand, according to what is exposed in Subsection 3.4, for scenarios X-XIII we only consider near-exact distributions, denoted NE-1 through NE-7, using the methodology described in Subsection 6.2.

In order to assess the proximity between the exact distributions of W and Z and the approximations suggested in this manuscript we use the measure

$$\Delta = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left|\frac{\Phi_W(t)-\Phi^*(t)}{t}\right|dt \tag{29}$$

with

$$\max_{w\in S_W}\left|F_W(w)-F_W^*(w)\right| = \max_{z\in S_Z}\left|F_Z(z)-F_Z^*(z)\right| \leq \Delta\,,$$

where $\Phi_W(t)$ represents the exact c.f. of W given by (7) and $\Phi^*(t)$ the approximate c.f. corresponding to the approximation being used, and where $F_W(w)$ and $F_W^*(w)$ represent the c.d.f.'s corresponding to $\Phi_W(t)$ and $\Phi^*(t)$, and $F_Z(z)$ and $F_Z^*(z)$ the corresponding c.d.f.'s of Z.

The measure in (29) may be seen as related with the Berry-Esseen bound Berry (1941); Esseen (1945); Hsu (1945); Hwang (1998) and has been used in several studies as a measure of proximity between distributions Coelho et al. (2010, 2011); Marques & Coelho (2011a); Marques et al. (2011).

From the results in Tables 1-23 in Appendix C, we may see how both approaches BT and BC only give somewhat satisfactory results for the scenarios with b_j's smaller than 1, with completely nonsense results for the scenarios with larger spans of the a_j, with values of Δ larger than 1, indicating that indeed the supposedly approximating c.f.'s are not indeed true c.f.'s, that is, they do not even correspond to true distributions. Anyway the behavior of the approach BC is in all cases much better than that of BT and the approach BWFM has always a much better performance than both BC and BT, with quite satisfactory results for all scenarios. However, all these approaches suffer from the problem of starting to yield worse approximations when m goes above some threshold for which it is not easy to obtain an 'a priori' guess. This problem is more pronounced for the BT approach and much less for the BWFM and it also starts at much lower values of m for BT than for BC, and for this approach much earlier than for BWFM.

The approach BWFM2 is much useful in studying the practical adequacy of having some of the first weights computed through (17) and (12)-(13) and the remaining ones by matching some of the first exact moments, versus having all of the weights computed by matching exact moments, as in BWFM. The results show that in general there is no advantage of having some of the first weight computed through their exact expressions, that is, they show that it is indeed more advantageous to have all moments computed by matching exact moments. For the BWFM2 approach we may compare the results for

- $m^*=0$ and $m^{**}-m^*=20$ with the results for $m^* = 15;\ m^{**} - m^* = 5$
$m^* = 10;\ m^{**} - m^* = 10$ and
$m^* = 5;\ m^{**} - m^* = 15$;
- $m^* = 0$ and $m^{**} - m^* = 15$ with the results for $m^* = 10;\ m^{**} - m^* = 5$ and
$m^* = 5;\ m^{**} - m^* = 10$;
- $m^* = 0$ and $m^{**} - m^* = 10$ with the results for $m^* = 5;\ m^{**} - m^* = 5$.

However for the scenarios with b_j's smaller than 1 some combinations of moderate values of m^* with somehow larger values of $m^{**} - m^*$ may give somewhat better results than those obtained when all the weights are obtained by equating the first exact moments.

This may show that this approach might be useful to be implemented for the near-exact approximations, where the remaining part of the c.f. left to be asymptotically approximated corresponds indeed to a product of independent beta random variables, all with b_j parameters smaller than 1. However, mostly for reasons related with the already large length of this manuscript, such approaches were not explored, although they may be easily implemented from the approaches developed.

For scenarios I-IX the near-exact approximations all yield quite similar results which are by far much better than any of the other approaches. For the same number of exact moments matched, and for the same number of terms in the mixtures, that is, comparing the near-exact approximations with the BWFM2 approximation for $m^* = -1$, or the BWFM approximation, the near-exact distributions distributions yield approximations which exhibit values of Δ which are several millions of times smaller than the values of Δ for the corresponding BWFM or BWFM2 approximations. Moreover, the near-exact distributions show a much stable behavior across all scenarios, actually tending to show slightly better performances in those scenarios where the other approaches tend to behave worse, which are the scenarios where the a_j show larger spans, scenarios where the b_j have larger values, or scenarios where there are more beta random variables involved, showing this way their great ability and adequacy to handle more complicated situations. See scenarios III and V.

For scenarios X-XIII, according to what is exposed in Subsections 3.4 and 6.2, we only have the chance to use the near-exact approximations. The results in Tables 24-27 show that although for these situations, where we consider the distribution of the product of different powers of independent beta random variables, the near-exact approximations exhibit rather larger values of the measure Δ than what they do for the scenarios where no powers are considered, they still have very good performances, with the near-exact approximation NE-1 being the one with a less good performance and the near-exact approximations NE-6 and NE-7 being the ones with the best performance, namely the latter one, except for scenario XIII, and with NE-5 having a similar performance to NE-6 in scenario XI. This also shows that there are good reasons to consider the different choices for a in (28), although introduced then in a somewhat heuristic way. As we will also see in Subsection 8.3, these near-exact approximations also perform very well when applied to the distribution of likelihood ratio test statistics.

8. Applications

In this section we have chosen the likelihood ratio test statistics to test i) the independence of several sets of variables, ii) sphericity and iii) equality of several covariance matrices, under multivariate normal or elliptically contoured distributions to illustrate the performance of the near-exact distributions suggested in Section 6 and the different cases that may appear, which were addressed in Sections 3 and 6.

We have also decided, for each type of near-exact distribution studied, to equate 4, 6, 10 or 15 exact moments in order to be easier to compare the results obtained with the ones published elsewhere for other near-exact distributions which equated exactly the same numbers of exact moments. As it may be seen from the numerical results, some of the near-exact distributions suggested in this paper, for the general case of the product of independent beta random variables, yield even slightly better results than the best performing near-exact distributions earlier expressly developed for each of the statistics used as examples.

8.1. The distribution of the likelihood ratio test statistic to test independence of sets of variables

According to Marques et al. (2011), the distribution of Λ_1, the likelihood ratio test statistic to test the independence of m groups of variables, the k-th of them having p_k variables, based on a sample of size n

from a multivariate normal or elliptically contoured distribution is the same as the distribution of

$$\prod_{k=1}^{m-1}\prod_{j=1}^{p_k}\left(Y_{jk}\right)^{n/2} \qquad \text{where} \qquad Y_{jk}\sim Beta\left(\frac{n-q_k-j}{2},\frac{q_k}{2}\right) \tag{30}$$

or the distribution of

$$\left\{\prod_{j=3}^{p} e^{-Z_j}\right\}\left\{\prod_{j=1}^{k^*}\left(Y_j^*\right)^{n/2}\right\} \qquad \text{where} \qquad Z_j\sim\Gamma\left(r_j,\frac{n-j}{n}\right) \quad \text{and} \quad Y_j^*\sim Beta\left(\frac{n-2}{2},\frac{1}{2}\right) \tag{31}$$

with

$$p=\sum_{k=1}^{m}p_k\,, \quad q_k=p_{k+1}+\ldots+p_m \quad \text{and} \quad k^*=\left\lfloor\frac{\ell}{2}\right\rfloor$$

where ℓ is the number of sets of variables with an odd number of variables and where r_j $(j=3,\ldots,p)$ are given by (A.2)-(A.3) in Marques et al. (2011).

Given the evidence that the near-exact distributions give much better approximations, although remaining much manageable, we will in this section use only the near-exact approximations developed in Section 6.

We may note that the integer part of the second parameters of the beta random variables in (31) is always either zero or 1/2, thus rendering the near-exact approach for the distribution in (30) quite simple and much adequate, while in case $k^*=0$, that is, when at most one of the sets has an odd number of variables, it is clear from (31) above, that we have in this case the exact distribution of Λ_1 as an exponentiated GIG distribution.

While the formulation in (30) corresponds to a less elaborate formulation of the distribution of Λ_1, (31) corresponds to a more elaborate formulation.

The way the distribution of Λ_1 is presented in (31), although requiring a more elaborate work on the c.f. of $-\log\,\Lambda_1$ is more convenient since it allows for better approximations. The c.f. corresponding to $\prod_{j=3}^{p}e^{Z_j}$ in (31) plays the role of $\Phi^*_{1,W}(t)$ in (24), while the c.f. corresponding to $\prod_{j=1}^{k^*}\left(Y_j^*\right)^{n/2}$ in (31) plays the role of $\Phi_{2,W}(t)$ in (24). In this case, in order to approximate $\Phi_{2,W}(t)$ we will use the approach outlined in Section 6, corresponding to the case studied in Subsection 3.3.

While the first approach, leading to (30) will give us a simpler way to approximate the distribution of Λ_1, the second approach, leading to the distribution in (31) will give us much more precise approximations than the ones obtained so far.

In Table 28 in Appendix D we have values of the measure Δ in (29) for near-exact distributions for Λ_1 for different numbers of sets of variables, different numbers of variables in each set and different sample sizes. In these tables, NE-1, NE-2, NE-6 and NE-7 refer to the corresponding different choices for a in (27) for the near-exact distributions for the representation of the distribution of Λ_1 in (30) and NE-II refers to the near-exact distribution for the representation of the distribution of Λ_1 corresponding to (31), in which case, according to what is described at the end of Subsection 6.1 and in Subsection 3.3, we will use a mixture of gamma distributions, all with rate parameters equal to $\frac{n-2}{2}-\frac{1}{4}$ and shape parameters $\frac{k^*}{2}+2k$, for $k=0,\ldots,m$. We should note that given the fact that for the representation of the distribution of Λ_1 in (30) all $b_j^*=\lfloor b_j\rfloor$ are equal to either zero or 1/2 and all m_j are equal to 1, then the near-exact distributions NE-2 through NE-5, corresponding to the choices for a in ii) through v) in (27) yield exactly the same near-exact distribution.

It is interesting to note that although all the near-exact distributions always give very good results, with values for the measure Δ in (29) extremely low, as expected, the form of the distribution in (31) allows for much better approximations. Actually, the results obtained for this form of the distribution by using the approach described in Section 6 and Subsection 3.3 allows for much better approximations than any of the ones obtained before for this statistic Coelho et al. (2010); Marques et al. (2011).

Some more detailed comments on the performance of the several near-exact distributions are:

- as expected, all the near-exact distributions show a clear asymptotic behavior both for increasing values of m, the number of exact moments matched, and n, the sample size,
- for increasing values of $p = \sum_{k=1}^{\nu} p_k$, keeping $n-p$ fixed, even for very small values of $n-p$ and reduced values of m (the number of exact moments matched), the performance of the near-exact approximations NE-1 through NE-7 does not worsen while the performance of NE-II even improves, what may be seen as a much desirable feature,
- NE-1 is the worse performing near-exact approximation, what amply justifies the choices for the rate parameter a in ii)-vii) of (27) in Section 6, which then seemed to be a somewhat heuristic approach,
- the near-exact approximation NE-6, with the rate parameter a based on only two moments, performs always better than NE-7, with a rate parameter a based on four moments, and this latter one better than NE-2,
- NE-II is by far the best performing near-exact distribution, with a performance even much better than the near-exact distributions in Coelho et al. (2010) and Marques et al. (2011).

8.2. The distribution of the likelihood ratio test statistic to test sphericity

Once again according to Marques et al. (2011), the distribution of Λ_2, the likelihood ratio test statistic to test sphericity in a p-multivariate normal or elliptically contoured distribution, based on a sample of size n has the same distribution as

$$\prod_{j=2}^{p}\left(Y_j\right)^{n/2} \qquad \text{where} \qquad Y_j \sim Beta\left(\frac{n-j}{2}, \frac{j-1}{p}+\frac{j-1}{2}\right) \tag{32}$$

or

$$\left\{\prod_{j=2}^{p} e^{-Z_j}\right\}\left\{\prod_{j=2}^{p-k^*}\left(Y_j^*\right)^{n/2}\right\}\left\{\prod_{j=p-k^*+1}^{p}\left(Y_j^{**}\right)^{n/2}\right\} \tag{33}$$

where $k^* = \lfloor p/2 \rfloor$, and

$$Z_j \sim Gamma\left(r_j, \frac{n-j}{n}\right), \quad Y_j^* \sim Beta\left(\frac{n-1}{2}, \frac{j-1}{p}\right) \quad \text{and} \quad Y_j^{**} \sim Beta\left(\frac{n}{2}, \frac{j-1}{p}-\frac{1}{2}\right)$$

with

$$r_j = \left\lfloor \frac{p-j+2}{2} \right\rfloor, \quad j = 2, \ldots, p\,.$$

As in the previous subsection, we will only focus on the near-exact approximations. Since all beta random variables in either (32) or (33) are all different, with different first and second parameters, in this case all the m_j will be equal to 1 and the most of the seven possible choices for a suggested in (27) will yield different near-exact distributions. More precisely, only the near-exact distributions corresponding to the choices of a in ii) and iii) of (27), on one side, and the ones corresponding to iv) and v) in (27) will yield the same result.

As such, in Tables 29-31, in Subsection D.2 of Appendix D, we refer to the near-exact distributions for the distribution of Λ_2 as depicted in (32) by NE-1 through NE-7, not using NE-3 neither NE-5, and by NE2-1 through NE2-7 for the distribution in (33), not using NE2-3 neither NE2-5.

For this statistic,

- as it also happens with the statistic Λ_1 in the previous section, and as it was expected, all near-exact approximations show a marked asymptotic behavior for increasing values of n, the sample size, and m, the number of exact moments matched,
- generally, the near-exact distributions based on the second approach show a better performance than the ones based on the first approach, although the gain may be rather slim and tends to reduce for larger values of m,
- for the first approach, the NE-2 near-exact distribution presents the worse performance, while NE-1 is comparable to NE-4, with NE-6 and NE-7, which are comparable, exhibiting the best performance,
- for the second approach, NE2-6 and NE2-7 are comparable, showing a better performance than NE2-2 and NE2-4, which are comparable,
- also for the second approach, and for the larger values of m, NE2-1 generally performs better than NE2-6 and NE2-7 for smaller values of n, and the other way around for larger values of n,
- an important characteristic of all the near-exact approximations for Λ_2 is that, for all values of m, even the smaller ones, they perform even better for larger values of p, when keeping $n-p$ constant, even for very small values of $n-p$.

As the near-exact distributions based on the first approach, based on (32), need less work on the exact c.f. and the gains with the second approach, based on (33), tend to be rather slim, the near-exact distributions based on the first approach may represent a good choice.

8.3. The distribution of the likelihood ratio test statistic to test equality of covariance matrices

Once again according to Marques et al. (2011), the distribution of Λ_3, the likelihood ratio test statistic to test the equality of q covariance matrices, based on q independent samples, all with size n, from as many p-multivariate normal or elliptically contoured distributions, has the same distribution as

$$\prod_{\substack{j=1\\ \text{except } j=k=1}}^{p}\prod_{k=1}^{q}\left(Y_{jk}\right)^{n/2} \qquad \text{where} \qquad Y_{jk}\sim Beta\left(\frac{n-j}{2},\frac{j(q-1)+2k-1-q}{2q}\right), \tag{34}$$

or

$$\left\{\prod_{j=2}^{p}e^{-Z_j}\right\}\left\{\prod_{j=1}^{\lfloor p/2\rfloor}\prod_{k=1}^{q}\left(Y^*_{jk}\right)^{n}\right\}\left\{\prod_{k=1}^{q}\left(Y^*_{k}\right)^{n/2}\right\}^{p\perp\!\!\!\perp 2} \tag{35}$$

where $p \perp\!\!\!\perp 2$ is the remainder of the integer division of p by 2, or equivalently, the indicator function of p being odd,

$$Z_j\sim Gamma\left(r_j,\frac{n-j}{n}\right),\quad Y^*_{jk}\sim Beta\left(a_j+b^*_{jk},b_{jk}-b^*_{jk}\right)\quad \text{and}\quad Y^{**}_j\sim Beta\left(a_p+b^*_{pk},b_{pk}-b^*_{pk}\right)$$

with

$$a_j=n-2j,\quad b_{jk}=2j-1+\frac{k-2j}{q},\quad b^*_{jk}=\lfloor b_{jk}\rfloor,$$

$$a_p=\frac{n-p}{2},\quad b_{pk}=\frac{pq-q-p+2k-1}{2q},\quad b^*_{pk}=\lfloor b_{pk}\rfloor,$$

and r_j given by (A.12)-(A.16) in Marques et al. (2011).

Once again we will only focus on the near-exact approximations. As it happened in the previous subsection, also in this case all beta random variables in either (34) or (35) are all different, with different first and second parameters, all the m_j will be once again equal to 1 and, as it happened in the previous subsection, the near-exact distributions corresponding to the choices of a in ii) and iii) of (27), on one side, and the ones corresponding to iv) and v) in (27) will yield the same result.

As such, in Tables 32-34 in Appendix D, similar to what happened in Tables 29-31, we refer to the near-exact distributions for the distribution of Λ_3 as depicted in (32) by NE-1 through NE-7, not using NE-3 neither NE-5, and by NE2-1 through NE2-7 for the distribution in (33), not using NE2-3 neither NE2-5.

We should also note that for odd p, when using the representation of the distribution of Λ_3 in (35), we will have two products of beta random variables where in each product the beta random variables involved are raised to a different power. As such, when using this representation of the distribution for odd p, we will have to use the approach outlined in Subsection 6.2.

As overall comments to the values in Tables 32-34 we may see that

– once again, and as expected, all near-exact approximations show a marked asymptotic behavior for increasing values of n, the sample size, and m, the number of exact moments matched,

– while NE-1 and NE2-1, which are generally the least performing near-exact approximations, seem to worsen a bit with the increase in q, NE-2, NE-4, NE2-2 and NE2-4 seem to be quite insensitive, but NE-6, NE-7, NE2-6 and NE2-7, which are the best performing approximations, even seem to improve quite a bit with the increase in q, what is a token in favor of these near-exact approximations, or, if we want, in favor of the choice of the computation of the parameter a as in vi) and vii) of (27) and (28), and, as in subsection 8.1, it shows that what might have seemed then as a mostly heuristic choices for the computation of the parameter a, are indeed well justified, moreover since this behavior even seems to be more accentuated for larger values of m and n,

– generally the second approach, based on (35) yields somewhat better results, although the difference to the first approach is rather slim mainly for NE-6 and NE-7, what shows that these two near-exact approximations, based on a simpler and more crude approach may indeed constitute good alternatives to NE2-6 and NE2-7,

– very good approximations are obtained, namely when using NE2-6 and NE2-7, but also NE-6 and NE-7, even for small values $n - p$, which is a much desirable feature.

9. Conclusions

Although the asymptotic distributions, which are finite mixtures of gamma distributions, are simpler and as such easier to compute, in most situations they do not provide the necessary precision, with simple truncations of these series yielding in many situations completely inadequate approximations. However, opposite to these, the near-exact approximations provide high quality approximations for all situations, even though remaining much manageable. Mainly in situations where the parameters b_j have large values, the a_j present large spans and/or the number of beta r.v's involved in the product is moderately large or large, the near-exact distributions, which are finite mixtures of GNIG distributions, provide the sensible approach and the adequate answer to the problem of approximating the distribution of the product of independent beta random variables with high quality but manageable approximations. These approximations also show very stable performances for all kinds of situations, that is values of the measure Δ which seem to be not much affected for changes in the number of beta variables in the product, or the span of the values of the a_j, actually giving even better approximations for situations where the number of beta random variables involved in the product increase or the values of the parameters b_j increase, and this in clear contrast with all other asymptotic approximations.

Among all near-exact distributions, the ones with the parameter a defined by vi) or vii) in either (27) or (28) are usually the best performing ones. The fact that NE-II was by far the best performing near-exact approximation for Λ_1 and that the near-exact distributions corresponding to the 'second approaches' for both Λ_2 and Λ_3 yield in general the best approaches, indicates that specific further work on the exact c.f.'s in order to develop near-exact distributions which may lead to the ability of keeping as much as possible of the original c.f. under its exact form, usually pays off. However, for both Λ_1 and Λ_2 this gain is rather slim, indicating that the application of the near-exact approach and near-exact distributions to a more crude approach obtained by working directly on the original product of independent beta random variables also leads to very accurate approximations, with c.d.f.'s which may easily lie apart from the original by much less than a hundredth of a millionth part, which would never be possible to attain with any other approach.

In fact, the results in this paper have an even much wider scope than the one analyzed so far. Actually, the distribution of W in (14) is not only the distribution of the negative logarithm of a large number of likelihood ratio test statistics but it is also the distribution of a linear combination of independent gamma random variables.

Indeed, although in the setting that expression (14) is taken in Section 3, all m_j were there taken as positive integers, as stated in Appendix A, m in (11) does not have to be an integer. In fact, if in (14) we take $b_j = 1$ and m_j as positive reals ($j = 1, \dots, q$), the c.f. of W may be written as

$$\Phi_W(t) = \prod_{j=1}^{q} (a_j)^{m_j} (a_j - \mathrm{i}t)^{-m_j} \tag{36}$$

which is the c.f. of a sum of independent $Gamma(m_j, a_j)$ random variables, which is also the c.f. of a linear combination of independent gamma distributed random variables. If in (36) we take $m_j = k_j/2$ for some positive integers m_j ($j = 1, \dots, q$), we may write

$$\Phi_W(t) = \prod_{j=1}^{q} (a_j)^{k_j/2} (a_j - \mathrm{i}t)^{-k_j/2} = \prod_{j=1}^{q} \left(\frac{1}{2}\right)^{k_j/2} \left(\frac{1}{2} - \frac{1}{2a_j}\mathrm{i}t\right)^{-k_j/2},$$

which is the c.f. of a linear combination of independent $\chi^2_{k_j}$ random variables with coefficients $\frac{1}{2a_j}$ ($j = 1, \dots, q$). This distribution is of key importance since it is intimately related with the distribution of quadratic forms, more precisely, quadratic forms in normal variables (Imhof, 1961; Shah, 1963; Kotz et al., 1967a,b; Baksalary et al., 1994) which arise in many estimation and testing problems related with Gaussian processes and normal models, or which arise as limiting distributions in non-normal processes (Jensen & Solomon, 1972; Khatri, 1980).

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Appendix A. Simple proofs for expressions (11) and (A.9)

The asymptotic expansions in (A.3) and (A.9) were first proved by Burić & Elezović (2011). We propose here simpler and straightforward demonstrations, while presenting highly manageable expressions which provide simple numerical calculations.

Barnes (1899) established an asymptotic expansion for the logarithm of the gamma function in the form

$$\log\,\Gamma(z+h) \approx \log\,\sqrt{2\pi} + \left(z+h-\frac{1}{2}\right)\log\,z - z - \sum_{r=1}^{\infty}(-1)^r\frac{B_{r+1}(h)}{r(r+1)z^r}, \qquad (z\to\infty) \tag{A.1}$$

for any $z, h \in \mathbb{C}$ and where $B_r(\cdot)$ is the Bernoulli polynomial of degree r.

Assuming $z, \alpha, \beta \in \mathbb{C}$ and $m \in \mathbb{R}$, the application of (A.1) leads to

$$\log\left[\frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)}\right]^m \approx m\,(\alpha-\beta)\log\,z + \sum_{r=1}^{\infty}\delta_{r,m}(\alpha,\beta)\,z^{-r}, \qquad (z\to\infty)$$

where

$$\delta_{r,m}(\alpha,\beta) = (-1)^r m\,\frac{B_{r+1}(\beta)-B_{r+1}(\alpha)}{r(r+1)}\,. \tag{A.2}$$

Therefore, we may write

$$\left[\frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)}\right]^m \approx z^{m\,(\alpha-\beta)}\,e^{\sum_{r=1}^{\infty}\delta_{r,m}(\alpha,\beta)\,z^{-r}}, \qquad (z\to\infty)\ ,$$

from which, expanding the exponential function according to expressions (2.7) and (2.8) in Moschopoulos (1985) we obtain the asymptotic expansion

$$\left[\frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)}\right]^m \approx \sum_{k=0}^{\infty}\nu_{k,m}(\alpha,\beta)\,z^{m\,(\alpha-\beta)-k}, \qquad (z\to\infty)\ , \tag{A.3}$$

for the power of a ratio of two gamma functions, with $\nu_{k,m}(\alpha,\beta)$ $(k = 0, 1, \ldots)$ given by (12).

However, it is indeed possible to improve the series expansion in (A.3), achieving a faster convergence series after a convenient parameter manipulation, in case we are willing to use a power basis which is also function of α and β.

From the property of the Bernoulli polynomials in expression 23.1.8 in Abramowitz & Stegun (1972), we note that, when n is even,

$$B_n(1-x) = B_n(x)\,. \tag{A.4}$$

Consider now $c \in \mathbb{C}$ and let $z^* = z + c$, $\alpha^* = \alpha - c$ and $\beta^* = \beta - c$. Then, from (A.3) we may write

$$\left[\frac{\Gamma(z^*+\alpha^*)}{\Gamma(z^*+\beta^*)}\right]^m \approx \sum_{k=0}^{\infty}\nu_{k,m}(\alpha^*,\beta^*)\,(z^*)^{m\,(\alpha-\beta)-k}, \qquad (z\to\infty)\ . \tag{A.5}$$

Now, the proper choice of c will allow us to reduce the number of terms in the above series. For this purpose we will force $\nu_{1,m}(\alpha^*,\beta^*) = 0$. Since $\nu_{1,m}(\alpha^*,\beta^*) = \delta_{1,m}(\alpha^*,\beta^*)$, it is enough to determine c such that

$$B_2(\beta-c) = B_2(\alpha-c)\,. \tag{A.6}$$

But then, according to (A.4), we have

$$B_2(\beta - c) = B_2(1 - (\beta - c)),$$

which, together with (A.6), entails

$$c = (\alpha + \beta - 1)/2. \tag{A.7}$$

Moreover, this choice of c implies that, for every odd j

$$\delta_{j,m}(\alpha - c, \beta - c) = 0. \tag{A.8}$$

We are now in conditions to prove that, for every odd k

$$\nu_{k,m}(\alpha - c, \beta - c) = 0,$$

and that therefore all the odd terms of the series represented in (A.5) vanish.

The proof is done by induction. As induction basis, the statement $\nu_{1,m}(\alpha - c, \beta - c) = 0$ holds, according to the choice of c in (A.7). Assuming as induction hypothesis that, for any given odd k, the statement

$$\nu_{1,m}(\alpha - c, \beta - c) = \nu_{3,m}(\alpha - c, \beta - c) = \ldots = \nu_{k-2,m}(\alpha - c, \beta - c) = \nu_{k,m}(\alpha - c, \beta - c) = 0$$

is true, we have, by (A.2),

$$\nu_{k+2,m}(\alpha - c, \beta - c) = \frac{1}{k+2} \sum_{j=1}^{k+2} j\, \delta_{j,m}(\alpha - c, \beta - c)\, \nu_{k+2-j,m}(\alpha - c, \beta - c).$$

Now all the terms in the summation are zero since, when j is odd the factor $\delta_{j,m}(\alpha - c, \beta - c)$ is zero, according to (A.8). Otherwise, when j is even, the index $k + 2 - j$ is odd and, by the induction hypothesis, the correspondent coefficient $\nu_{k+2-j,m}(\alpha - c, \beta - c)$ is also zero.

Finally, expression (A.5) may be written as

$$\left[\frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)}\right]^m \approx \sum_{k=0}^{\infty} \nu_{2k,m}(\alpha - c, \beta - c)\, (z + \tfrac{\alpha+\beta-1}{2})^{m\,(\alpha-\beta)-2k}, \quad (z \to \infty), \tag{A.9}$$

where each coefficient $\nu_{2k,m}(\alpha - c, \beta - c)$ is given by (12), with $c = (\alpha + \beta - 1)/2$.

Setting $m = 1$ in (A.3) we obtain a power expansion which is equivalent to the asymptotic series expansion for the ratio of two gamma functions proposed by Tricomi Tricomi & Erdélyi (1951), here in the form given by Fields Fields (1966), while by setting $m = 1$ in (A.9) we obtain the equivalent to the asymptotic series expansion for the ratio of two gamma functions proposed by Fields Fields (1966).

Appendix B. Notation and expressions for the probability density and cumulative distribution functions of the gamma, GIG and GNIG distributions

We use this appendix to settle some notation which is used throughout the paper and also to introduce the expressions for the probability density and cumulative distribution functions of the generalized integer gamma (GIG) and generalized near-integer gamma (GNIG) distributions.

We say that the random variable X has a gamma distribution with shape parameter $r(>0)$ and rate parameter $\lambda(>0)$, and we will denote this fact by $X \sim Gamma(r, \lambda)$, if the probability density function of X is

$$f_X(x) = \frac{\lambda^r}{\Gamma(r)}\, e^{-\lambda x}\, x^{r-1} \quad (x > 0)\,.$$

Let $X_j \sim Gamma(r_j, \lambda_j)$ $(j = 1, \ldots, p)$ be a set of p independent random variables and consider the random variable

$$Z = \sum_{j=1}^{p} X_j\,.$$

Then, given the independence of the p random variables X_j, the c.f. of Z is

$$\Phi_Z(t) = \prod_{j=1}^{p} \Phi_{X_j}(t) = \prod_{j=1}^{p} \lambda_j^{r_j} \left(\lambda_j - \mathrm{i}t\right)^{-r_j}\,, \tag{B.1}$$

so that if all the λ_j are equal, say equal to λ, the distribution of Z is a gamma distribution with shape parameter $r = \sum_{j=1}^{p} r_j$ and rate parameter λ. But in case all or some of the λ_j are different, the distribution of Z is a bit more complicated. In case two or more of the r_j are non-integer, the distribution of Z is an infinite mixture of gamma distributions (see Moschopoulos (1985)), but in case all the $r_j \in \mathbb{N}$, then the distribution of Z is very manageable and it is what we call a GIG distribution Coelho (1998, 1999). If all the λ_j are different, Z has a GIG distribution of depth p, with shape parameters r_j and rate parameters λ_j. The random variable Z has probability density function

$$f_Z(z) = f^{GIG}(z\,|\,\{r_j\}_{j=1:p}; \{\lambda_j\}_{j=1:p}; p) = K \sum_{j=1}^{p} P_j(z)\, e^{-\lambda_j z}\,, \quad (z > 0)$$

and cumulative distribution function

$$F_Z(z) = F^{GIG}(z\,|\,\{r_j\}_{j=1:p}; \{\lambda_j\}_{j=1:p}; p) = 1 - K \sum_{j=1}^{p} P_j^*(z)\, e^{-\lambda_j z}\,, \quad (z > 0)$$

where

$$K = \prod_{j=1}^{p} \lambda_j^{r_j}\,, \qquad P_j(z) = \sum_{k=1}^{r_j} c_{j,k}\, z^{k-1}$$

and

$$P_j^*(z) = \sum_{k=1}^{r_j} c_{j,k}(k-1)! \sum_{i=0}^{k-1} \frac{z^i}{i!\,\lambda_j^{k-i}}$$

with

$$c_{j,r_j} = \frac{1}{(r_j-1)!}\prod_{\substack{i=1\\i\neq j}}^{p}(\lambda_i-\lambda_j)^{-r_i}\,, \quad j=1,\dots,p\,, \tag{B.2}$$

and

$$c_{j,r_j-k} = \frac{1}{k}\sum_{i=1}^{k}\frac{(r_j-k+i-1)!}{(r_j-k-1)!}\,R(i,j,p)c_{j,r_j-(k-i)}\,, \quad (k=1,\dots,r_j-1;\ j=1,\dots,p) \tag{B.3}$$

where

$$R(i,j,p) = \sum_{\substack{k=1\\k\neq j}}^{p} r_k(\lambda_j-\lambda_k)^{-i} \quad (i=1,\dots,r_j-1)\,. \tag{B.4}$$

As we may see, in the above expressions all the λ_j are assumed to be different, however, in case some of the λ_j assume the same value as other λ_j's, the distribution of Z still is a GIG distribution, as long as all the r_j remain integer, but in this case with a reduced depth. In this more general case, let $\{\lambda_\ell;\ell=1,\dots,g(\leq p)\}$ be the set of different λ_j's and let $\{r_\ell;\ell=1,\dots,g(\leq p)\}$ be the set of the corresponding shape parameters, with r_ℓ being the sum of all r_j ($j\in\{1,\dots,p\}$) which correspond to the λ_j assuming the value λ_ℓ. In this case Z will have a GIG distribution of depth g, with shape parameters r_ℓ and rate parameters λ_ℓ ($\ell=1,\dots,g$).

In case all the r_j but one are integer, then Z has a GNIG distribution of depth p. More precisely, let

$$Y = Z_1 + Z_2$$

where Z_1 has a GIG distribution of depth p, with shape parameters $r_1,\dots,r_p$ and rate parameters $\lambda_1,\dots,\lambda_p$ and Z_2 is an independent random variable with a gamma distribution with a non-integer shape parameter r and rate parameter $\lambda\neq\lambda_j$ for all $j\in\{1,\dots,p\}$. Then, the random variable Y has a GNIG distribution of depth $p+1$, with probability density function

$$f_Y(y) = f^{GNIG}(y\,|\,\{r_j\}_{j=1:p},r;\{\lambda_j\}_{j=1:p},\lambda;p+1) = \\ K\lambda^r\sum_{j=1}^{p}e^{-\lambda_j y}\sum_{k=1}^{r_j}c_{j,k}\,\frac{\Gamma(k)}{\Gamma(k+r)}\,y^{k+r-1}\,{}_1F_1(r,k+r;-(\lambda-\lambda_j)y)\,, \quad (y>0)$$

and cumulative distribution function

$$F_Y(y) = F^{GNIG}(y\,|\,\{r_j\}_{j=1:p},r;\{\lambda_j\}_{j=1:p},\lambda;p+1) = \frac{\lambda^r\,y^r}{\Gamma(r+1)}\,{}_1F_1(r,r+1;-\lambda y) \\ -K\lambda^r\sum_{j=1}^{p}e^{-\lambda_j y}\sum_{k=1}^{r_j}c^*_{j,k}\sum_{i=0}^{k-1}\frac{y^{r+i}\,\lambda_j^i}{\Gamma(r+1+i)}\,{}_1F_1(r,r+1+i;-(\lambda-\lambda_j)y)\,, \quad (y>0)$$

where

$$c^*_{j,k} = \frac{c_{j,k}}{\lambda_j^k}\,\Gamma(k)$$

with $c_{j,k}$ given by (B.2)-(B.4) above. In the above expressions ${}_1F_1(a,b;y)$ is the Kummer confluent hypergeometric function. This function has usually very good convergence properties and is nowadays easily handled by a number of software packages.

Appendix C. Tables of values of Δ for the several approaches and scenarios considered in Section 7

Table 1. – Scenario I – values of Δ for BT, BC and BWFM

m	BT	BC	BWFM
10	8.01E+04	9.01E+01	5.48E−05
20	2.59E+04	2.28E+01	4.10E−09
30	5.04E+03	3.86E+00	3.17E−13
40	9.32E+02	6.42E−01	3.05E−17
50	1.72E+02	1.09E−01	3.67E−21
60	3.17E+01	1.86E−02	5.31E−25
70	4.71E+00	2.01E−03	8.83E−29
80	1.92E+04	1.87E+01	1.28E−19
90	1.11E+09	9.99E+05	1.54E−05
100	1.90E+14	1.61E+11	2.80E−03

Table 2. – Scenario I - values of Δ for BWFM2

	number of exact moments matched ($m^{**} - m^{*}$)				
m^{*}	2	5	10	15	20
-1	6.87E−02	5.27E−03	5.48E−05	4.83E−07	4.10E−09
0	2.54E−01	8.68E−03	5.00E−05	3.28E−07	2.32E−09
5	1.23E+00	1.10E−02	1.49E−05	3.90E−08	1.47E−10
10	8.81E−01	4.48E−03	2.94E−06	4.53E−09	1.13E−11
15	3.82E−01	1.37E−03	5.59E−07	5.90E−10	1.07E−12
20	1.42E−01	3.95E−04	1.13E−07	8.76E−11	1.22E−13

Table 3. – Scenario I - values of Δ for near-exact distributions

	number of exact moments matched (m)				
	2	5	10	15	20
NE-1	1.61E−06	8.51E−11	1.84E−17	1.85E−21	1.46E−25
NE-2	4.74E−06	1.62E−10	1.13E−17	4.53E−22	5.71E−26
NE-3	1.47E−06	2.10E−11	9.44E−18	2.91E−22	2.30E−25
NE-4	7.61E−06	4.38E−10	1.12E−16	2.88E−22	9.73E−26
NE-5	2.71E−06	5.33E−11	6.85E−18	4.84E−22	8.29E−26
NE-6	1.52E−06	7.73E−11	1.40E−17	1.84E−21	1.32E−25
NE-7	9.31E−07	3.93E−11	2.09E−18	1.47E−21	3.51E−25

//

Table 4. – Scenario II – values of Δ for BT, BC and BWFM

m	BT	BC	BWFM
10	3.01E+01	2.45E−02	3.70E−07
20	4.88E−02	3.11E−05	6.98E−13
30	4.63E−05	2.54E−08	1.89E−18
40	3.69E−08	1.81E−11	6.81E−24
50	2.68E−11	1.19E−14	3.02E−29
60	9.75E−12	1.33E−14	1.52E−34
70	4.09E−09	4.99E−12	3.88E−39
80	6.82E−06	7.65E−09	2.96E−34
90	3.87E−02	4.07E−05	2.20E−23
100	6.58E+02	6.56E−01	2.47E−09

Table 5. – Scenario II - values of Δ for BWFM2

	number of exact moments matched ($m^{**} - m^{*}$)				
m^{*}	2	5	10	15	20
-1	1.50E−02	2.89E−04	3.70E−07	4.87E−10	6.98E−13
0	1.82E−02	1.99E−04	1.68E−07	1.78E−10	2.23E−13
5	2.94E−03	9.69E−06	2.46E−09	1.24E−12	9.12E−16
10	1.27E−04	2.46E−07	3.22E−11	9.88E−15	4.87E−18
15	3.83E−06	5.27E−09	4.31E−13	8.97E−17	3.18E−20
20	1.02E−07	1.08E−10	6.04E−15	9.09E−19	2.42E−22

Table 6. – Scenario II - values of Δ for near-exact distributions

	number of exact moments matched (m)				
	2	5	10	15	20
NE-1	2.20E−05	7.40E−09	4.45E−14	5.37E−19	7.28E−24
NE-2	4.28E−06	4.28E−10	3.26E−16	2.99E−22	1.14E−25
NE-3	9.08E−06	1.34E−09	2.49E−15	1.13E−20	1.53E−25
NE-4	3.19E−06	3.08E−10	1.76E−16	1.56E−22	9.53E−26
NE-5	8.92E−06	1.30E−09	2.37E−15	1.05E−20	1.48E−25
NE-6	2.30E−06	2.45E−10	8.80E−17	1.25E−22	3.56E−25
NE-7	3.83E−07	1.57E−10	1.77E−17	4.68E−22	8.17E−26

Table 7. – Scenario III – values of Δ for BT, BC and BWFM

m	BT	BC	BWFM
10	1.40E+10	1.59E+07	2.57E−03
20	2.23E+11	1.93E+08	1.62E−06
30	4.35E+11	3.20E+08	4.89E−10
40	3.30E+11	2.17E+08	1.07E−13
50	1.52E+11	9.13E+07	2.05E−17
60	5.27E+10	2.94E+07	3.66E−21
70	1.53E+10	8.02E+06	3.77E−10
80	3.98E+09	1.97E+06	3.84E−10
90	5.60E+10	4.62E+07	5.07E+03
100	8.72E+15	6.21E+12	1.76E+01

Table 8. – Scenario III - values of Δ for BWFM2

	number of exact moments matched ($m^{**} - m^{*}$)				
m^{*}	2	5	10	15	20
-1	2.36E−01	5.49E−02	2.57E−03	7.43E−05	1.62E−06
0	2.34E+01	8.45E−01	9.94E−03	1.52E−04	2.29E−06
5	1.18E+04	1.18E+02	2.17E−01	8.18E−04	4.32E−06
10	1.59E+05	8.67E+02	6.64E−01	1.20E−03	3.44E−06
15	6.04E+05	2.19E+03	9.23E−01	9.97E−04	1.83E−06
20	1.16E+06	3.11E+03	8.29E−01	6.00E−04	7.72E−07

Table 9. – Scenario III - values of Δ for near-exact distributions

	number of exact moments matched (m)				
	2	5	10	15	20
NE-1	8.89E−06	4.22E−10	8.90E−17	5.16E−23	2.07E−27
NE-2	1.81E−06	4.53E−11	5.35E−18	2.62E−24	7.26E−29
NE-3	1.02E−06	1.88E−11	1.21E−18	1.51E−24	3.62E−29
NE-4	2.85E−06	9.29E−11	1.72E−17	9.61E−24	5.85E−29
NE-5	2.45E−06	7.29E−11	1.16E−17	5.84E−24	6.85E−29
NE-6	4.80E−08	2.59E−12	5.44E−20	3.38E−25	2.51E−28
NE-7	2.31E−07	3.77E−12	2.07E−20	8.23E−25	1.75E−28

Table 10. – Scenario IV – values of Δ for BT, BC and BWFM

m	BT	BC	BWFM
10	2.93E+04	2.26E+01	2.94E−06
20	2.82E+02	1.64E−01	7.61E−12
30	5.57E−01	2.77E−04	1.47E−17
40	6.14E−04	2.75E−07	2.93E−23
50	5.29E−07	2.18E−10	6.58E−29
60	2.07E−10	8.98E−14	1.72E−34
70	4.84E−08	6.14E−12	5.20E−25
80	3.46E−05	4.32E−08	1.86E−25
90	1.04E−02	3.80E−04	2.89E−16
100	2.95E+03	8.03E−00	1.93E−09

Table 11. – Scenario IV - values of Δ for BWFM2

	number of exact moments matched ($m^{**} - m^{*}$)				
m^{*}	2	5	10	15	20
-1	3.47E−02	1.23E−03	2.94E−06	5.09E−09	7.61E−12
0	1.49E−01	2.09E−03	2.52E−06	3.09E−09	3.72E−12
5	4.71E−01	1.47E−03	3.31E−07	1.36E−10	7.64E−14
10	1.04E−01	1.69E−04	1.61E−08	3.84E−12	1.18E−15
15	8.71E−03	9.38E−06	5.05E−10	6.91E−14	1.61E−17
20	4.44E−04	3.56E−07	1.26E−11	1.21E−15	2.10E−19

Table 12. – Scenario IV - values of Δ for near-exact distributions

	number of exact moments matched (m)				
	2	5	10	15	20
NE-1	2.13E−05	3.90E−09	1.00E−14	6.20E−20	6.47E−25
NE-2	1.14E−05	1.07E−09	7.73E−16	1.33E−21	4.79E−27
NE-3	2.02E−05	2.88E−09	4.20E−15	1.47E−20	8.87E−26
NE-4	3.54E−06	1.75E−10	3.37E−17	6.73E−24	1.85E−27
NE-5	7.13E−06	4.89E−10	2.03E−16	1.89E−22	6.79E−28
NE-6	1.54E−06	1.16E−10	3.23E−17	6.81E−23	3.70E−27
NE-7	2.10E−07	7.07E−11	5.18E−18	2.31E−23	5.06E−27

Table 13. – Scenario V – values of Δ for BT, BC and BWFM

m	BT	BC	BWFM
10	1.03E+09	9.36E+05	2.40E−04
20	1.55E+09	1.07E+06	1.57E−08
30	2.28E+08	1.34E+05	4.76E−13
40	1.17E+07	6.15E+03	1.06E−17
50	3.52E+05	1.70E+02	2.12E−22
60	7.94E+03	3.59E+00	4.15E−27
70	1.53E+02	6.51E−02	2.91E−14
80	2.66E+00	1.08E−03	1.14E−12
90	4.03E+01	3.92E−02	1.29E−03
100	7.60E+05	6.39E+02	3.38E−01

Table 14. – Scenario V - values of Δ for BWFM2

	number of exact moments matched ($m^{**} - m^{*}$)				
m^{*}	2	5	10	15	20
-1	1.27E−01	1.55E−02	2.40E−04	2.25E−06	1.57E−08
0	6.92E+00	1.50E−01	6.61E−04	3.45E−06	1.71E−08
5	1.31E+03	7.04E+00	4.52E−03	5.77E−06	1.01E−08
10	6.09E+03	1.74E+01	4.46E−03	2.69E−06	2.54E−09
15	7.30E+03	1.36E+01	1.89E−03	6.74E−07	4.07E−10
20	4.13E+03	5.65E+00	4.93E−04	1.18E−07	5.02E−11

Table 15. – Scenario V - values of Δ for near-exact distributions

	number of exact moments matched (m)				
	2	5	10	15	20
NE-1	7.57E−06	5.27E−10	3.03E−16	4.77E−22	1.30E−27
NE-2	3.89E−06	1.31E−10	1.89E−17	7.21E−24	8.13E−30
NE-3	6.93E−06	3.54E−10	1.03E−16	7.98E−23	1.16E−28
NE-4	1.19E−06	2.14E−11	8.22E−19	4.36E−26	1.11E−30
NE-5	2.42E−06	5.96E−11	4.95E−18	9.98E−25	3.70E−30
NE-6	5.05E−07	1.45E−11	8.20E−19	4.06E−25	1.83E−29
NE-7	5.11E−08	9.25E−12	1.49E−19	7.72E−26	1.32E−29

Table 16. – Scenario VI – values of Δ for BT, BC and BWFM

m	BT	BC	BWFM
10	6.18E−02	7.47E−05	4.03E−07
20	5.33E−04	4.77E−07	1.52E−11
30	4.63E−06	3.43E−09	7.34E−16
40	4.02E−08	2.59E−11	3.91E−20
50	2.90E−09	1.18E−11	2.15E−24
60	1.47E−06	6.95E−09	4.94E−28
70	4.50E−03	2.23E−05	1.38E−30
80	5.74E+01	3.03E−01	9.41E−27
90	2.54E+06	1.46E+04	3.65E−15
100	3.35E+11	2.15E+09	2.33E−10

Table 17. – Scenario VI - values of Δ for BWFM2

	number of exact moments matched ($m^{**} - m^{*}$)				
m^{*}	2	5	10	15	20
-1	3.54E−03	9.02E−05	4.03E−07	2.35E−09	1.52E−11
0	1.07E−03	2.29E−05	8.24E−08	4.11E−10	2.36E−12
5	1.88E−05	1.86E−07	2.63E−10	6.63E−13	2.23E−15
10	8.23E−07	4.68E−09	3.26E−12	4.75E−15	1.02E−17
15	4.67E−08	1.77E−10	7.06E−14	6.53E−17	9.55E−20
20	3.01E−09	8.25E−12	2.09E−15	1.31E−18	1.38E−21

Table 18. – Scenario VII – values of Δ for BT, BC and BWFM

m	BT	BC	BWFM
10	4.94E−11	5.54E−13	1.25E−13
20	1.98E−14	9.62E−17	6.18E−19
30	1.11E−15	1.81E−18	8.67E−23
40	1.16E−15	6.63E−19	2.35E−26
50	1.30E−14	1.42E−18	3.81E−29
60	1.01E−12	3.22E−16	1.29E−31
70	3.99E−10	1.94E−13	2.21E−34
80	6.50E−07	3.83E−10	9.99E−33
90	3.65E−03	2.38E−06	5.26E−22
100	6.21E+01	4.27E−02	1.63E−12

Table 19. – Scenario VII - values of Δ for BWFM2

	number of exact moments matched ($m^{**} - m^*$)				
m^*	2	5	10	15	20
-1	1.13E−06	1.50E−09	1.25E−13	2.07E−16	6.18E−19
0	1.42E−07	8.56E−11	2.39E−14	1.60E−17	1.29E−19
5	1.82E−12	1.51E−15	4.28E−18	5.57E−21	6.27E−23
10	2.20E−15	1.18E−17	9.66E−21	3.00E−23	3.46E−25
15	3.62E−17	2.11E−19	6.77E−23	7.96E−25	8.36E−27
20	4.54E−19	5.95E−21	7.26E−24	6.25E−26	5.96E−28

Table 20. – Scenario VIII – values of Δ for BT, BC and BWFM

m	BT	BC	BWFM
10	3.80E+00	4.48E−03	2.55E−06
20	5.75E−02	5.06E−05	9.62E−11
30	7.14E−04	5.24E−07	4.64E−15
40	8.07E−06	5.19E−09	2.49E−19
50	7.57E−08	3.92E−11	1.41E−23
60	7.98E−06	7.92E−09	8.40E−28
70	3.13E−02	3.05E−05	1.94E−31
80	5.14E+02	4.88E−01	4.96E−26
90	2.95E+07	2.73E+04	2.93E−12
100	5.16E+12	4.66E+09	6.51E−08

Table 21. – Scenario VIII - values of Δ for BWFM2

	number of exact moments matched ($m^{**} - m^*$)				
m^*	2	5	10	15	20
-1	1.63E−02	5.32E−04	2.55E−06	1.49E−08	9.62E−11
0	9.62E−03	2.26E−04	8.22E−07	4.06E−09	2.31E−11
5	6.35E−04	5.91E−06	7.86E−09	1.91E−11	6.26E−14
10	4.71E−05	2.54E−07	1.67E−10	2.34E−13	4.87E−16
15	3.75E−05	1.36E−08	5.16E−12	4.60E−15	6.55E−18
20	3.11E−07	8.24E−10	1.99E−13	1.21E−16	1.24E−19

Table 22. – Scenario IX – values of Δ for BT, BC and BWFM

m	BT	BC	BWFM
10	8.01E−10	1.44E−12	1.64E−14
20	2.03E−13	1.98E−16	7.98E−20
30	7.01E−15	2.04E−18	2.32E−23
40	6.18E−15	1.44E−18	6.50E−27
50	6.30E−14	3.09E−17	7.26E−30
60	4.61E−12	2.99E−15	1.52E−32
70	1.76E−09	1.31E−12	1.06E−35
80	2.79E−06	2.21E−09	1.40E−34
90	1.54E−02	1.25E−05	2.87E−23
100	2.57E+02	2.12E−01	6.21E−12

Table 23. – Scenario IX - values of Δ for BWFM2

	number of exact moments matched ($m^{**} - m^*$)				
m^*	2	5	10	15	20
-1	1.02E−06	5.55E−10	1.64E−14	1.16E−16	7.98E−20
0	3.05E−07	1.44E−10	1.89E−14	3.11E−18	6.08E−20
5	8.05E−12	1.00E−14	7.45E−18	2.99E−21	5.87E−23
10	2.03E−15	1.43E−17	2.60E−20	2.55E−23	4.31E−25
15	1.04E−16	5.16E−19	2.93E−22	1.01E−24	1.28E−26
20	2.69E−18	2.27E−20	8.70E−24	1.09E−25	1.09E−27

Table 24. – Scenario X - values of Δ for near-exact distributions

	number of exact moments matched (m)				
	2	5	10	15	20
NE-1	9.39E−04	1.11E−05	2.82E−08	1.56E−10	1.29E−12
NE-2	4.17E−04	5.14E−06	2.47E−08	3.66E−10	9.50E−12
NE-3	3.68E−04	4.10E−06	1.61E−08	1.92E−10	3.95E−12
NE-4	3.73E−04	4.20E−06	1.69E−08	2.05E−10	4.33E−12
NE-5	3.01E−04	2.87E−06	8.27E−09	6.99E−11	1.01E−12
NE-6	1.86E−04	8.78E−07	5.04E−10	7.23E−13	1.62E−15
NE-7	4.73E−05	4.07E−07	1.35E−10	1.72E−13	3.03E−16

Table 25. – Scenario XI - values of Δ for near-exact distributions

	number of exact moments matched (m)				
	2	5	10	15	20
NE-1	2.98E−04	1.62E−06	1.34E−09	2.52E−12	7.05E−15
NE-2	1.27E−04	3.68E−07	1.12E−10	8.91E−14	1.14E−16
NE-3	1.55E−04	5.10E−07	2.06E−10	2.19E−13	3.80E−16
NE-4	7.55E−05	1.73E−07	2.65E−11	1.07E−14	6.74E−18
NE-5	9.30E−05	2.29E−07	4.61E−11	2.39E−14	1.98E−17
NE-6	6.69E−05	1.84E−07	4.21E−11	2.28E−14	1.89E−17
NE-7	8.11E−06	6.70E−08	1.96E−12	7.47E−16	1.84E−19

Table 26. – Scenario XII - values of Δ for near-exact distributions

	number of exact moments matched (m)				
	2	5	10	15	20
NE-1	9.03E−04	5.31E−06	3.85E−09	6.63E−12	1.89E−14
NE-2	1.18E−04	2.63E−07	4.11E−11	1.84E−14	1.52E−17
NE-3	1.08E−04	2.34E−07	3.34E−11	1.37E−14	1.03E−17
NE-4	2.07E−04	6.02E−07	1.76E−10	1.50E−13	2.43E−16
NE-5	1.87E−04	5.11E−07	1.32E−10	9.97E−14	1.42E−16
NE-6	1.21E−04	2.43E−07	3.27E−11	1.22E−14	8.11E−18
NE-7	1.88E−05	8.53E−08	1.76E−12	5.88E−16	7.02E−20

Table 27. – Scenario XIII - values of Δ for near-exact distributions

	number of exact moments matched (m)				
	2	5	10	15	20
NE-1	4.81E−03	9.79E−05	4.06E−07	3.24E−09	3.82E−11
NE-2	4.75E−04	4.56E−06	1.22E−08	1.03E−10	1.66E−12
NE-3	5.81E−04	6.16E−06	2.03E−08	2.15E−10	4.41E−12
NE-4	5.03E−04	4.96E−06	1.41E−08	1.27E−10	2.18E−12
NE-5	5.50E−04	5.67E−06	1.77E−08	1.76E−10	3.37E−12
NE-6	3.51E−04	1.64E−06	7.35E−10	8.48E−13	1.60E−15
NE-7	4.71E−05	7.59E−07	3.74E−10	7.67E−13	2.55E−15

Appendix D. Tables of values of Δ for the examples of application in Section 8

Appendix D.1. Values of Δ for the statistic Λ_1 in subsection 8.1

Table 28 - Values of Δ for the near-exact distributions for Λ_1

	m (number of exact moments matched)				m (number of exact moments matched)			
	4	6	10	15	4	6	10	15
$p_k = \{3, 5, 6\}$	$n = 16$				$n = 100$			
NE-1	1.63E-09	2.50E-12	1.70E-17	1.81E-23	7.07E-13	3.93E-17	2.66E-25	3.31E-35
NE-2	2.37E-11	8.56E-15	1.34E-21	3.40E-26	1.43E-14	2.39E-19	7.73E-28	2.37E-36
NE-6	4.73E-13	1.20E-15	1.53E-21	1.40E-25	6.27E-17	7.86E-21	3.55E-28	2.60E-36
NE-7	1.00E-12	1.01E-15	1.93E-21	1.06E-25	9.92E-17	6.42E-21	3.32E-28	2.69E-36
NE-II	1.64E-15	9.19E-19	1.06E-23	3.42E-28	1.20E-22	9.40E-29	2.99E-39	2.66E-50
$p_k = \{7, 11, 15\}$	$n = 35$				$n = 100$			
NE-1	3.21E-09	3.05E-12	5.90E-18	1.27E-24	1.69E-10	4.58E-14	6.40E-21	4.25E-29
NE-2	1.39E-10	5.94E-14	2.93E-20	1.42E-27	7.08E-12	8.10E-16	2.42E-23	2.72E-32
NE-6	5.48E-12	1.03E-15	9.53E-23	1.27E-30	1.01E-13	5.60E-18	3.98E-26	5.42E-35
NE-7	6.98E-12	1.84E-15	3.14E-22	3.83E-30	1.91E-13	1.40E-17	1.57E-25	7.00E-35
NE-II	1.32E-21	3.44E-27	1.54E-36	4.80E-46	1.31E-24	9.40E-29	3.03E-43	8.76E-56
$p_k = \{7, 7, 9, 10\}$	$n = 35$				$n = 100$			
NE-1	7.17E-11	1.88E-14	3.26E-21	3.99E-29	5.29E-12	4.52E-16	7.31E-24	3.74E-33
NE-2	8.72E-13	6.57E-17	1.06E-24	7.24E-34	6.76E-14	1.66E-18	2.44E-27	6.64E-38
NE-6	4.22E-14	1.85E-18	9.79E-27	3.75E-36	1.22E-15	1.87E-20	1.04E-29	5.72E-40
NE-7	7.26E-14	3.83E-18	2.61E-26	6.24E-36	2.39E-15	4.29E-20	2.91E-29	6.20E-40
NE-II	1.19E-21	2.94E-27	1.18E-36	3.18E-46	9.46E-25	1.27E-31	1.51E-43	3.26E-56
$p_k = \{5, 5, 7, 7, 9\}$	$n = 35$				$n = 100$			
NE-1	1.42E-10	3.53E-14	4.30E-21	2.65E-29	1.13E-11	9.28E-16	1.10E-23	3.00E-33
NE-2	3.94E-12	4.01E-16	1.15E-23	1.58E-32	3.16E-13	1.05E-17	2.73E-26	1.50E-36
NE-6	4.37E-15	2.60E-18	4.20E-26	1.57E-35	4.90E-15	1.46E-19	1.66E-28	2.46E-39
NE-7	2.08E-13	1.40E-17	1.49E-25	5.43E-35	1.11E-14	2.61E-19	2.64E-28	3.80E-39
NE-II	2.32E-21	5.61E-27	2.17E-36	5.54E-46	1.70E-24	2.16E-31	2.34E-43	4.51E-56
$p_k = \{11, 11, 15, 15, 19, 21\}$	$n = 100$				$n = 200$			
NE-1	7.16E-10	1.65E-13	8.02E-21	5.82E-30	1.61E-10	2.03E-14	2.94E-22	4.63E-32
NE-2	3.18E-13	7.00E-18	5.85E-27	5.99E-38	7.02E-14	8.52E-19	2.12E-28	4.57E-40
NE-6	9.39E-15	6.14E-20	1.56E-29	2.24E-40	3.18E-16	4.92E-21	1.87E-30	3.13E-42
NE-7	2.02E-14	3.33E-19	1.58E-28	8.39E-40	3.89E-15	3.73E-20	5.57E-30	6.53E-42
NE-II	6.57E-28	3.14E-36	5.12E-51	3.23E-67	8.41E-30	7.02E-39	3.49E-55	2.86E-73

Appendix D.2. Values of Δ for the statistic Λ_2 in subsection 8.2

Table 29 - Values of Δ for the near-exact distributions for Λ_2

		First approach m (number of exact moments matched)					Second approach m (number of exact moments matched)			
$p = 3$		4	6	10	15		4	6	10	15
	NE-1	8.64E-07	1.02E-07	1.82E-09	3.55E-11	NE2-1	3.68E-08	1.46E-08	1.90E-10	2.75E-12
	NE-2	2.35E-06	5.27E-08	1.55E-10	3.37E-12	NE2-2	5.84E-07	2.14E-09	3.80E-11	5.99E-13
$n = 5$	NE-4	1.01E-06	1.41E-08	3.10E-10	7.19E-12	NE2-4	2.13E-07	1.42E-09	1.61E-11	3.72E-13
	NE-6	1.92E-07	8.50E-08	1.72E-09	3.85E-11	NE2-6	4.47E-07	1.73E-08	2.25E-10	4.20E-12
	NE-7	1.66E-07	7.68E-08	1.57E-09	3.54E-11	NE2-7	6.47E-08	1.20E-08	1.54E-10	2.07E-12
	NE-1	1.91E-09	5.87E-12	1.72E-14	9.75E-18	NE2-1	1.48E-09	7.90E-12	5.69E-15	3.58E-18
	NE-2	2.15E-09	1.78E-11	6.53E-15	2.20E-18	NE2-2	2.23E-09	5.72E-12	3.23E-15	2.30E-19
$n = 15$	NE-4	3.97E-10	1.91E-11	4.91E-15	3.62E-18	NE2-4	1.21E-09	1.08E-11	1.26E-15	1.53E-18
	NE-6	2.21E-09	2.01E-11	7.14E-15	9.43E-18	NE2-6	4.25E-10	1.22E-11	8.23E-15	6.03E-19
	NE-7	2.21E-09	2.03E-11	6.85E-15	9.35E-18	NE2-7	5.95E-10	1.35E-11	7.91E-15	6.39E-19
$p = 6$		4	6	10	15		4	6	10	15
	NE-1	1.12E-08	2.87E-11	2.17E-16	1.21E-20	NE2-1	4.94E-11	9.71E-14	1.30E-17	2.01E-21
	NE-2	6.65E-08	4.14E-10	3.69E-14	7.35E-19	NE2-2	2.29E-09	3.25E-12	1.07E-17	4.33E-22
$n = 8$	NE-4	2.49E-08	1.10E-10	4.88E-15	3.88E-20	NE2-4	2.29E-09	3.25E-12	1.07E-17	4.33E-22
	NE-6	1.05E-10	2.40E-13	5.75E-17	4.16E-21	NE2-6	2.79E-11	1.97E-13	1.87E-17	1.07E-21
	NE-7	2.84E-10	1.18E-13	6.30E-17	2.32E-21	NE2-7	3.74E-11	2.20E-13	1.70E-17	1.53E-21
	NE-1	2.95E-10	1.50E-13	6.36E-20	5.66E-26	NE2-1	2.54E-12	1.88E-15	6.01E-21	3.60E-26
	NE-2	1.90E-09	2.19E-12	5.51E-18	1.09E-24	NE2-2	1.18E-10	3.99E-14	1.14E-20	2.07E-26
$n = 18$	NE-4	6.64E-10	5.34E-13	6.58E-19	5.32E-26	NE2-4	1.18E-10	3.99E-14	1.14E-20	2.07E-26
	NE-6	2.94E-12	1.32E-15	9.24E-21	1.57E-26	NE2-6	1.95E-13	6.70E-16	1.19E-20	3.69E-26
	NE-7	4.80E-12	2.06E-15	6.93E-21	7.68E-27	NE2-7	3.40E-13	7.74E-16	1.24E-20	3.93E-26
	NE-1	2.26E-11	4.03E-15	2.49E-22	3.76E-29	NE2-1	2.23E-13	6.39E-17	4.33E-23	1.72E-29
	NE-2	1.47E-10	5.85E-14	1.69E-20	2.21E-28	NE2-2	1.08E-11	1.34E-15	6.19E-23	1.01E-29
$n = 30$	NE-4	5.05E-11	1.40E-14	1.95E-21	2.54E-29	NE2-4	1.08E-11	1.34E-15	6.19E-23	1.01E-29
	NE-6	2.28E-13	5.65E-17	6.65E-24	2.25E-29	NE2-6	8.99E-15	7.00E-18	2.79E-23	7.11E-30
	NE-7	3.11E-13	6.75E-17	1.11E-23	2.34E-29	NE2-7	3.43E-15	8.86E-18	2.92E-23	6.66E-30
$p = 10$		4	6	10	15		4	6	10	15
	NE-1	3.26E-10	1.66E-13	9.82E-20	9.01E-27	NE2-1	3.66E-13	1.51E-16	2.42E-23	2.54E-28
	NE-2	1.50E-09	1.40E-12	2.97E-18	7.05E-25	NE2-2	7.11E-11	1.83E-14	2.60E-21	4.23E-29
$n = 12$	NE-4	3.80E-10	2.30E-13	2.06E-19	1.56E-26	NE2-4	7.11E-11	1.83E-14	2.60E-21	4.23E-29
	NE-6	4.30E-13	2.56E-17	1.18E-21	4.57E-27	NE2-6	2.92E-14	9.35E-17	6.27E-22	1.60E-27
	NE-7	2.78E-12	5.20E-16	9.61E-22	4.04E-27	NE2-7	7.06E-14	1.09E-16	6.41E-22	1.62E-27
	NE-1	5.79E-11	1.23E-14	1.07E-21	7.58E-30	NE2-1	8.66E-14	1.95E-17	7.46E-24	2.57E-30
	NE-2	2.85E-10	1.13E-13	3.41E-20	5.31E-28	NE2-2	1.84E-11	2.28E-15	6.27E-23	1.14E-30
$n = 22$	NE-4	6.84E-11	1.72E-14	2.14E-21	1.23E-29	NE2-4	1.84E-11	2.28E-15	6.27E-23	1.14E-30
	NE-6	1.59E-13	2.20E-17	3.18E-24	2.03E-30	NE2-6	1.51E-14	1.63E-18	7.37E-24	3.25E-31
	NE-7	3.76E-13	4.08E-17	1.20E-24	2.54E-30	NE2-7	1.04E-14	2.44E-18	7.70E-24	4.11E-31
	NE-1	7.03E-12	6.26E-16	9.26E-24	1.06E-32	NE2-1	1.15E-14	1.17E-18	1.05E-25	3.20E-33
	NE-2	3.52E-11	5.84E-15	2.95E-22	4.62E-31	NE2-2	2.58E-12	1.40E-16	7.27E-25	1.29E-33
$n = 35$	NE-4	8.28E-12	8.66E-16	1.78E-23	1.26E-32	NE2-4	2.58E-12	1.40E-16	7.27E-25	1.29E-33
	NE-6	2.35E-14	1.51E-18	3.54E-26	5.29E-33	NE2-6	3.14E-15	1.17E-19	3.50E-26	2.70E-33
	NE-7	3.96E-14	2.09E-18	4.60E-26	5.37E-33	NE2-7	2.76E-15	8.89E-20	3.73E-26	2.64E-33
	NE-1	1.27E-12	5.67E-17	2.07E-25	3.98E-35	NE2-1	2.16E-15	1.13E-19	2.79E-27	1.07E-35
	NE-2	6.43E-12	5.32E-16	6.58E-24	1.73E-33	NE2-2	5.01E-13	1.39E-17	1.86E-26	3.11E-36
$n = 50$	NE-4	1.50E-12	7.79E-17	3.89E-25	4.95E-35	NE2-4	5.01E-13	1.39E-17	1.86E-26	3.11E-36
	NE-6	4.61E-15	1.53E-19	1.46E-27	2.24E-35	NE2-6	6.96E-16	2.07E-20	2.91E-28	1.64E-35
	NE-7	6.63E-15	1.89E-19	1.61E-27	2.23E-35	NE2-7	6.47E-16	1.88E-20	3.30E-28	1.63E-35

Table 30 - Values of Δ for the near-exact distributions for Λ_2

		First approach					Second approach			
		m (number of exact moments matched)					m (number of exact moments matched)			
$p = 15$		4	6	10	15		4	6	10	15
$n = 17$	NE-1	2.06E-11	2.58E-15	1.22E-22	3.61E-29	NE2-1	3.29E-14	6.95E-18	4.92E-24	6.34E-30
	NE-2	1.06E-10	2.29E-14	2.02E-21	4.15E-30	NE2-2	6.56E-12	4.66E-16	7.70E-25	1.29E-30
	NE-4	1.30E-11	1.45E-15	4.56E-23	7.52E-30	NE2-4	5.07E-12	3.20E-16	1.32E-24	1.13E-30
	NE-6	5.87E-14	1.29E-17	2.28E-24	1.37E-29	NE2-6	1.50E-14	1.67E-18	1.26E-23	7.96E-30
	NE-7	1.56E-14	1.54E-17	4.09E-24	1.16E-29	NE2-7	1.29E-14	2.06E-18	1.27E-23	7.90E-30
$n = 27$	NE-1	1.15E-11	9.93E-16	1.81E-23	2.15E-30	NE2-1	2.25E-14	4.64E-18	7.06E-25	1.85E-31
	NE-2	6.15E-11	9.24E-15	3.32E-22	7.46E-31	NE2-2	4.53E-12	2.43E-16	1.72E-26	3.68E-31
	NE-4	7.23E-12	5.51E-16	6.37E-24	5.27E-31	NE2-4	3.49E-12	1.67E-16	6.54E-25	4.17E-31
	NE-6	2.31E-14	5.45E-18	3.99E-24	1.68E-30	NE2-6	1.53E-14	1.45E-18	2.04E-24	7.19E-31
	NE-7	2.62E-15	5.89E-18	4.06E-24	1.70E-30	NE2-7	1.45E-14	1.34E-18	2.08E-24	7.36E-31
$n = 50$	NE-1	8.84E-13	2.65E-17	4.23E-26	5.51E-34	NE2-1	1.90E-15	1.61E-19	1.06E-26	3.91E-34
	NE-2	4.81E-12	2.52E-16	1.06E-24	3.53E-34	NE2-2	4.03E-13	8.03E-18	3.62E-27	9.80E-35
	NE-4	5.48E-13	1.44E-17	1.17E-26	3.33E-34	NE2-4	3.09E-13	5.49E-18	3.78E-28	6.82E-35
	NE-6	1.12E-15	1.25E-19	1.59E-26	3.30E-34	NE2-6	1.61E-15	9.56E-20	1.13E-27	3.17E-34
	NE-7	7.22E-17	1.29E-19	1.57E-26	3.16E-34	NE2-7	1.58E-15	9.41E-20	1.02E-27	3.15E-34
$n = 100$	NE-1	3.36E-14	2.69E-19	2.45E-29	1.84E-38	NE2-1	7.48E-17	1.79E-21	1.10E-29	8.91E-39
	NE-2	1.84E-13	2.59E-18	7.60E-28	4.43E-39	NE2-2	1.66E-14	9.15E-20	6.55E-30	1.04E-38
	NE-4	2.06E-14	1.45E-19	3.92E-30	1.37E-38	NE2-4	1.27E-14	6.26E-20	3.69E-30	9.95E-39
	NE-6	2.72E-17	1.04E-21	9.25E-30	4.47E-39	NE2-6	7.08E-17	1.32E-21	5.66E-30	1.49E-38
	NE-7	7.57E-18	1.06E-21	9.12E-30	4.67E-39	NE2-7	7.02E-17	1.31E-21	5.62E-30	1.49E-38
$p = 20$		4	6	10	15		4	6	10	15
$n = 22$	NE-1	2.55E-12	1.17E-16	4.47E-25	3.22E-35	NE2-1	1.82E-16	8.04E-21	6.06E-29	1.31E-37
	NE-2	1.64E-11	1.41E-15	1.76E-23	5.13E-33	NE2-2	6.66E-13	1.69E-17	1.75E-26	1.80E-37
	NE-4	1.84E-12	7.90E-17	2.67E-25	1.64E-35	NE2-4	6.66E-13	1.69E-17	1.75E-26	1.80E-37
	NE-6	1.18E-15	1.44E-20	4.89E-29	5.92E-38	NE2-6	1.60E-16	7.37E-22	4.74E-29	4.59E-39
	NE-7	4.16E-15	5.38E-20	1.06E-29	1.02E-37	NE2-7	1.30E-16	2.44E-22	5.01E-29	6.62E-39
$n = 35$	NE-1	2.01E-12	7.49E-17	1.61E-25	4.61E-36	NE2-1	1.41E-16	5.78E-21	3.43E-29	2.53E-38
	NE-2	1.33E-11	9.42E-16	6.76E-24	7.84E-34	NE2-2	6.14E-13	1.34E-17	8.95E-27	4.82E-38
	NE-4	1.44E-12	5.02E-17	9.56E-26	2.29E-36	NE2-4	6.14E-13	1.34E-17	8.95E-27	4.82E-38
	NE-6	1.26E-16	5.43E-22	2.41E-30	4.52E-38	NE2-6	1.76E-16	1.95E-21	7.24E-30	2.76E-38
	NE-7	2.47E-15	2.72E-20	1.10E-29	4.68E-38	NE2-7	1.60E-16	1.73E-21	8.03E-30	2.72E-38
$n = 50$	NE-1	5.18E-13	1.11E-17	7.72E-27	5.24E-38	NE2-1	3.51E-17	8.88E-22	1.91E-30	2.30E-40
	NE-2	3.48E-12	1.42E-16	3.31E-25	9.09E-36	NE2-2	1.70E-13	2.21E-18	5.03E-28	7.77E-40
	NE-4	3.71E-13	7.40E-18	4.53E-27	2.55E-38	NE2-4	1.70E-13	2.21E-18	5.03E-28	7.77E-40
	NE-6	7.04E-17	7.56E-22	6.05E-31	6.14E-40	NE2-6	5.22E-17	4.14E-22	3.42E-32	4.90E-40
	NE-7	5.32E-16	3.49E-21	8.84E-31	6.13E-40	NE2-7	4.93E-17	3.90E-22	2.06E-32	4.88E-40
$n = 100$	NE-1	2.29E-14	1.40E-19	7.73E-30	2.17E-42	NE2-1	1.46E-18	1.14E-23	2.16E-33	2.07E-45
	NE-2	1.56E-13	1.81E-18	3.39E-28	3.84E-40	NE2-2	8.18E-15	3.11E-20	6.02E-31	4.82E-44
	NE-4	1.63E-14	9.23E-20	4.47E-30	1.03E-42	NE2-4	8.18E-15	3.11E-20	6.02E-31	4.82E-44
	NE-6	8.15E-18	1.91E-23	1.13E-33	2.42E-44	NE2-6	2.67E-18	7.17E-24	5.53E-34	2.64E-44
	NE-7	1.82E-17	3.60E-23	1.26E-33	2.39E-44	NE2-7	2.61E-18	7.02E-24	5.38E-34	2.64E-44

Table 31 - Values of Δ for the near-exact distributions for Λ_2

		First approach					Second approach			
		m (number of exact moments matched)					m (number of exact moments matched)			
$p = 25$		4	6	10	15		4	6	10	15
	NE-1	6.16E-13	1.38E-17	1.10E-26	1.15E-36	NE2-1	1.16E-16	4.16E-21	2.23E-29	6.55E-38
	NE-2	3.17E-12	1.25E-16	2.94E-25	9.50E-36	NE2-2	1.99E-13	2.68E-18	6.73E-28	6.24E-38
$n = 27$	NE-4	3.17E-13	6.02E-18	3.63E-27	2.28E-38	NE2-4	1.71E-13	2.15E-18	4.66E-28	7.13E-38
	NE-6	3.50E-17	5.21E-21	9.04E-29	4.88E-37	NE2-6	1.20E-16	1.69E-21	4.31E-29	1.91E-37
	NE-7	5.93E-16	8.26E-21	9.18E-29	4.90E-37	NE2-7	1.14E-16	1.58E-21	4.42E-29	1.95E-37
	NE-1	3.69E-13	6.04E-18	2.24E-27	2.08E-38	NE2-1	7.26E-17	2.27E-21	1.69E-29	3.57E-38
	NE-2	1.97E-12	5.76E-17	6.43E-26	6.56E-37	NE2-2	1.39E-13	1.46E-18	2.00E-28	1.38E-38
$n = 50$	NE-4	1.89E-13	2.63E-18	7.27E-28	3.53E-38	NE2-4	1.19E-13	1.17E-18	1.40E-28	1.27E-38
	NE-6	7.61E-17	2.22E-21	2.26E-29	2.65E-38	NE2-6	9.61E-17	1.64E-21	2.78E-30	3.09E-38
	NE-7	2.75E-16	2.90E-21	2.21E-29	2.48E-38	NE2-7	9.39E-17	1.62E-21	2.61E-30	3.08E-38
	NE-1	1.96E-14	9.81E-20	3.34E-30	4.53E-42	NE2-1	3.82E-18	3.91E-23	3.52E-32	2.22E-42
	NE-2	1.06E-13	9.56E-19	9.91E-29	4.83E-41	NE2-2	8.03E-15	2.66E-20	3.70E-31	2.81E-42
$n = 100$	NE-4	1.00E-14	4.25E-20	1.07E-30	2.81E-42	NE2-4	6.88E-15	2.13E-20	2.62E-31	2.84E-42
	NE-6	7.00E-18	3.29E-23	2.61E-32	1.65E-42	NE2-6	5.84E-18	3.55E-23	2.00E-32	4.14E-42
	NE-7	1.22E-17	3.82E-23	2.55E-32	1.72E-42	NE2-7	5.78E-18	3.53E-23	1.99E-32	4.14E-42

Appendix D.3. Values of Δ for the statistic Λ_3 in subsection 8.3

Table 32 - Values of Δ for the near-exact distributions for Λ_3

		First approach					Second approach			
		m (number of exact moments matched)					m (number of exact moments matched)			
$p = 3, q = 5$		4	6	10	15		4	6	10	15
	NE-1	1.84E-06	2.52E-08	9.65E-12	1.43E-15	NE2-1	5.20E-07	5.80E-09	2.72E-12	2.10E-15
	NE-2	3.18E-06	5.38E-08	2.83E-11	4.42E-15	NE2-2	3.36E-07	3.13E-09	6.03E-13	6.16E-17
$n = 5$	NE-4	2.17E-06	3.22E-08	1.30E-11	1.46E-15	NE2-4	2.36E-07	1.92E-09	3.12E-13	6.25E-17
	NE-6	6.83E-08	3.83E-10	1.83E-13	1.57E-16	NE2-6	3.81E-08	1.88E-10	1.75E-13	1.80E-16
	NE-7	1.49E-08	4.17E-11	2.30E-14	1.09E-16	NE2-7	5.23E-09	1.98E-11	2.39E-14	1.15E-16
	NE-1	5.94E-11	1.18E-14	8.20E-22	6.72E-29	NE2-1	1.71E-11	3.14E-15	2.41E-22	7.56E-29
	NE-2	6.57E-11	1.21E-14	6.68E-22	4.08E-29	NE2-2	5.91E-12	6.76E-16	5.41E-23	4.69E-29
$n = 50$	NE-4	4.19E-11	6.74E-15	3.20E-22	4.64E-29	NE2-4	3.87E-12	4.12E-16	3.08E-23	4.24E-29
	NE-6	6.18E-13	1.07E-16	8.76E-23	1.73E-29	NE2-6	4.88E-13	9.38E-17	9.11E-23	1.90E-29
	NE-7	3.70E-13	8.07E-17	8.25E-23	1.35E-29	NE2-7	3.64E-13	7.62E-17	8.60E-23	1.50E-29
	NE-1	1.72E-12	8.33E-17	3.59E-25	9.07E-34	NE2-1	4.96E-13	2.22E-17	1.17E-25	8.45E-34
	NE-2	1.85E-12	8.11E-17	2.57E-25	2.12E-34	NE2-2	1.65E-13	4.57E-18	3.61E-26	6.67E-34
$n = 100$	NE-4	1.18E-12	4.50E-17	1.29E-25	3.89E-34	NE2-4	1.07E-13	2.83E-18	2.85E-26	7.25E-34
	NE-6	1.67E-14	5.84E-19	2.05E-26	4.83E-34	NE2-6	1.40E-14	5.02E-19	2.18E-26	4.74E-34
	NE-7	1.33E-14	4.94E-19	1.92E-26	5.01E-34	NE2-7	1.23E-14	4.41E-19	2.05E-26	4.93E-34
$p = 5, q = 7$		4	6	10	15		4	6	10	15
	NE-1	9.81E-05	2.91E-06	2.69E-09	4.96E-13	NE2-1	1.75E-05	3.28E-07	1.50E-10	1.52E-14
	NE-2	1.00E-05	1.64E-07	7-09E-11	9.27E-15	NE2-2	8.84E-07	8.70E-09	1.82E-12	1.26E-16
$n = 7$	NE-4	2.21E-06	2.41E-08	5.10E-12	2.96E-16	NE2-4	2.36E-07	1.70E-09	1.99E-13	6.71E-18
	NE-6	1.47E-08	2.20E-11	7.54E-16	7.17E-20	NE2-6	3.82E-09	3.94E-12	8.25E-16	5.41E-20
	NE-7	3.12E-08	1.56E-10	7.63E-15	9.81E-20	NE2-7	2.13E-08	9.79E-11	4.71E-15	6.07E-20
	NE-1	3.60E-09	1.70E-12	3.72E-19	1.83E-27	NE2-1	6.87E-10	2.10E-13	2.38E-20	6.75E-29
	NE-2	5.18E-10	1.36E-13	1.20E-20	2.70E-29	NE2-2	3.72E-11	5.55E-15	2.19E-22	2.97E-31
$n = 50$	NE-4	7.82E-11	1.22E-14	4.38E-22	4.00E-31	NE2-4	6.80E-12	6.86E-16	1.39E-23	1.44E-32
	NE-6	4.74E-13	4.26E-17	8.00E-25	8.93E-32	NE2-6	4.20E-13	3.59E-17	9.79E-25	9.28E-32
	NE-7	6.99E-13	5.70E-17	9.87E-25	9.04E-32	NE2-7	5.51E-13	4.43E-17	1.11E-24	9.17E-32
	NE-1	1.05E-10	1.20E-14	1.55E-22	2.22E-32	NE2-1	2.00E-11	1.49E-15	9.98E-24	8.29E-34
	NE-2	1.52E-11	9.71E-16	4.94E-24	3.14E-34	NE2-2	1.08E-12	3.90E-17	8.91E-26	3.90E-36
$n = 100$	NE-4	2.25E-12	8.43E-17	1.74E-25	5.14E-36	NE2-4	1.94E-13	4.69E-18	5.36E-27	7.27E-37
	NE-6	1.51E-14	3.23E-19	5.19E-28	1.16E-36	NE2-6	1.30E-14	2.68E-19	5.59E-28	8.97E-37
	NE-7	1.83E-14	3.72E-19	5.51E-28	1.12E-36	NE2-7	1.49E-14	2.97E-19	5.79E-28	8.55E-37

Table 33 - Values of Δ for the near-exact distributions for Λ_3

		First approach					Second approach			
		m (number of exact moments matched)					m (number of exact moments matched)			
$p=7, q=5$		4	6	10	15		4	6	10	15
	NE-1	5.70E-07	2.71E-09	8.43E-14	3.34E-19	NE2-1	1.18E-07	3.84E-10	7.11E-15	1.80E-20
	NE-2	4.38E-07	2.01E-09	6.19E-14	2.63E-19	NE2-2	1.79E-08	3.63E-11	3.02E-16	3.48E-22
$n=9$	NE-4	1.25E-07	3.91E-10	5.96E-15	1.10E-20	NE2-4	6.19E-09	9.44E-12	4.54E-17	2.02E-23
	NE-6	1.24E-09	1.39E-12	1.48E-18	7.28E-23	NE2-6	2.90E-10	2.85E-13	1.52E-18	3.39E-23
	NE-7	7.16E-10	6.93E-13	2.78E-18	2.67E-23	NE2-7	3.33E-10	2.51E-13	1.61E-18	5.80E-24
	NE-1	2.22E-10	4.18E-14	1.82E-21	1.64E-30	NE2-1	4.78E-11	6.32E-15	1.72E-22	1.18E-31
	NE-2	1.68E-10	2.93E-14	1.08E-21	7.66E-31	NE2-2	6.26E-12	4.73E-16	4.71E-24	4.95E-33
$n=50$	NE-4	4.00E-11	4.49E-15	7.52E-23	1.69E-32	NE2-4	1.85E-12	1.03E-16	7.06E-24	1.02E-32
	NE-6	8.96E-14	5.29E-18	1.49E-25	1.56E-33	NE2-6	1.18E-14	1.38E-19	6.28E-26	5.56E-33
	NE-7	4.12E-14	4.23E-19	1.12E-25	2.59E-33	NE2-7	3.22E-14	1.55E-18	4.00E-26	6.34E-33
	NE-1	6.69E-12	3.11E-16	8.18E-25	2.24E-35	NE2-1	1.45E-12	4.73E-17	7.84E-26	1.79E-36
	NE-2	5.04E-12	2.14E-16	4.68E-25	9.66E-36	NE2-2	1.86E-13	3.43E-18	2.01E-27	3.71E-38
$n=100$	NE-4	1.18E-12	3.22E-17	3.17E-26	3.03E-37	NE2-4	5.43E-14	7.33E-19	3.21E-28	9.26E-38
	NE-6	1.83E-15	2.78E-20	1.81E-29	1.74E-37	NE2-6	1.13E-16	3.90E-21	2.44E-29	1.98E-37
	NE-7	1.18E-16	7.09E-21	9.62E-30	1.78E-37	NE2-7	5.43E-16	1.00E-20	2.94E-29	1.99E-37
$p=7, q=10$		4	6	10	15		4	6	10	15
	NE-1	2.37E-05	3.28E-07	5.26E-11	8.24E-16	NE2-1	2.80E-06	1.92E-08	9.34E-13	4.36E-18
	NE-2	2.22E-06	1.34E-08	5.03E-13	1.78E-18	NE2-2	4.39E-08	8.56E-11	5.33E-16	3.62E-22
$n=9$	NE-4	2.19E-07	6.36E-10	6.65E-15	6.12E-21	NE2-4	7.26E-09	9.35E-12	2.92E-17	9.34E-24
	NE-6	6.32E-10	3.20E-13	1.69E-20	4.84E-26	NE2-6	5.70E-11	2.54E-14	9.21E-20	1.32E-26
	NE-7	7.83E-10	7.99E-13	1.34E-18	1.50E-25	NE2-7	3.82E-10	3.18E-13	4.23E-19	2.84E-26
	NE-1	9.52E-09	5.50E-12	1.44E-18	6.55E-27	NE2-1	1.21E-09	3.57E-13	2.98E-20	4.33E-29
	NE-2	1.38E-09	4.02E-13	3.01E-20	3.61E-29	NE2-2	2.37E-11	2.08E-15	2.30E-23	4.67E-33
$n=50$	NE-4	9.54E-11	1.17E-14	1.93E-22	4.65E-32	NE2-4	2.81E-12	1.49E-16	7.22E-25	6.50E-35
	NE-6	9.53E-14	5.45E-18	1.77E-26	4.47E-35	NE2-6	8.48E-14	3.51E-18	1.11E-26	4.83E-35
	NE-7	2.13E-13	9.63E-18	2.73E-26	4.55E-35	NE2-7	1.24E-13	4.70E-18	1.39E-26	4.68E-35
	NE-1	2.87E-10	4.10E-14	6.51E-22	8.90E-32	NE2-1	3.66E-11	2.67E-15	1.36E-23	5.97E-34
	NE-2	4.30E-11	3.09E-15	1.42E-23	5.10E-34	NE2-2	7.28E-13	1.58E-17	1.06E-26	6.40E-38
$n=100$	NE-4	2.87E-12	8.61E-17	8.57E-26	6.07E-37	NE2-4	8.37E-14	1.09E-18	3.16E-28	1.37E-39
	NE-6	3.72E-15	4.76E-20	9.78E-30	6.97E-40	NE2-6	2.85E-15	2.84E-20	6.01E-30	4.36E-40
	NE-7	5.49E-15	6.30E-20	1.19E-29	6.67E-40	NE2-7	3.44E-15	3.28E-20	6.58E-30	4.00E-40

Table 34 - Values of Δ for the near-exact distributions for Λ_3

		First approach					Second approach			
		m (number of exact moments matched)					m (number of exact moments matched)			
$p = 10, q = 5$		4	6	10	15		4	6	10	15
$n = 12$	NE-1	2.36E-06	1.52E-08	6.62E-13	2.83E-18	NE2-1	1.87E-09	1.15E-12	7.67E-19	3.60E-26
	NE-2	6.35E-08	1.28E-10	7.69E-16	4.51E-22	NE2-2	2.38E-10	8.02E-14	1.66E-20	1.68E-28
	NE-4	1.81E-08	2.60E-11	8.88E-17	2.87E-23	NE2-4	9.69E-11	2.57E-14	3.35E-21	1.94E-29
	NE-6	2.07E-10	9.26E-14	2.81E-20	3.01E-27	NE2-6	7.83E-12	1.64E-15	1.46E-22	7.71E-31
	NE-7	2.32E-10	1.69E-13	1.62E-19	1.22E-26	NE2-7	3.88E-12	5.14E-16	8.45E-24	2.41E-31
$n = 50$	NE-1	5.74E-09	3.14E-12	8.93E-19	6.12E-27	NE2-1	8.72E-12	5.85E-16	4.26E-24	5.88E-34
	NE-2	1.89E-10	3.37E-14	1.35E-21	1.17E-30	NE2-2	8.89E-13	3.01E-17	5.57E-26	1.26E-36
	NE-4	4.47E-11	5.39E-15	1.12E-22	4.79E-32	NE2-4	3.35E-13	8.78E-18	9.96E-27	1.28E-37
	NE-6	9.09E-15	3.48E-18	4.58E-26	1.21E-35	NE2-6	1.63E-14	4.52E-19	5.02E-28	4.94E-39
	NE-7	3.28E-13	2.14E-17	1-35E-25	1.74E-35	NE2-7	3.68E-15	1.96E-19	3.01E-28	3.68E-39
$n = 100$	NE-1	1.74E-10	2.35E-14	4.09E-22	8.49E-32	NE2-1	2.84E-13	4.85E-18	2.28E-27	1.01E-38
	NE-2	5.79E-12	2.56E-16	6.19E-25	1.58E-35	NE2-2	2.81E-14	2.38E-19	2.75E-29	1.92E-41
	NE-4	1.34E-12	3.97E-17	4.90E-26	6.13E-37	NE2-4	1.05E-14	6.86E-20	4.84E-30	1.91E-42
	NE-6	2.02E-15	5.25E-20	2.76E-29	1.81E-40	NE2-6	4.49E-16	3.35E-21	2.50E-31	1.20E-43
	NE-7	7.15E-15	1.21E-19	4.83E-29	2.13E-40	NE2-7	2.46E-16	2.31E-21	1.97E-31	1.08E-43
$p = 10, q = 15$		4	6	10	15		4	6	10	15
$n = 12$	NE-1	4.91E-06	3.18E-08	9.36E-13	1.31E-18	NE2-1	2.44E-10	4.07E-14	1.15E-21	4.67E-31
	NE-2	3.57E-07	8.52E-10	3.68E-15	5.45E-22	NE2-2	6.70E-10	1.56E-13	8.05E-21	6.38E-30
	NE-4	2.96E-08	2.89E-11	2.41E-17	5.72E-25	NE2-4	4.83E-11	4.84E-15	5.31E-23	7.02E-33
	NE-6	1.70E-11	2.15E-15	4.02E-23	4.63E-32	NE2-6	3.10E-13	1.10E-17	1.84E-26	1.07E-36
	NE-7	1.40E-11	2.68E-15	1.36E-22	1.80E-31	NE2-7	1.84E-13	5.58E-18	5.73E-27	1.56E-36
$n = 50$	NE-1	2.11E-08	1.50E-11	5.16E-18	2.58E-26	NE2-1	1.91E-12	4.42E-17	2.28E-26	5.71E-38
	NE-2	2.21E-09	6.67E-13	4.34E-20	3.05E-29	NE2-2	4.78E-12	1.50E-16	1.36E-25	6.33E-37
	NE-4	1.47E-10	1.67E-14	1.78E-22	1.64E-32	NE2-4	2.98E-13	3.88E-18	6.95E-28	5.04E-40
	NE-6	4.05E-16	1.92E-19	2.44E-28	1.85E-39	NE2-6	5.02E-16	2.60E-21	9.52E-32	3.13E-43
	NE-7	4.11E-14	9.36E-19	6.05E-28	3.06E-39	NE2-7	3.51E-16	1.24E-21	3.48E-33	3.51E-43
$n = 100$	NE-1	6.78E-10	1.22E-13	2.68E-21	4.29E-31	NE2-1	6.61E-14	3.97E-19	1.38E-29	1.18E-42
	NE-2	7.40E-11	5.72E-15	2.43E-23	5.63E-34	NE2-2	1.63E-13	1.33E-18	7.99E-29	1.26E-41
	NE-4	4.77E-12	1.37E-16	9.38E-26	2.77E-37	NE2-4	9.95E-15	3.33E-20	3.96E-31	9.62E-45
	NE-6	3.29E-16	3.01E-21	1.69E-31	3.49E-44	NE2-6	9.21E-18	1.36E-23	3.71E-35	9.27E-48
	NE-7	9.99E-16	6.09E-21	2.64E-31	4.49E-44	NE2-7	5.23E-18	3.29E-24	8.09E-36	9.41E-48

References

ABRAMOWITZ, M. & STEGUN, I. (eds.) (1972). *Handbook of Mathematical Functions: with Formulas, Graphs, and Mathematical Tables*, 10th printing. New York: Dover.

ANDERSON, T. W. (2003). *An Introduction to Multivariate Statistical Analysis*, 3rd ed.. New York: J. Wiley & Sons.

BAKSALARY, J. K., HAUKE, J. & STYAN, G. P. H. (1994). On some distributional properties of quadratic forms in normal variables and on some associated matrix partial orderings. *Multivariate Analysis and Its Applications*, IMS Lecture Notes Monogr. Ser., Vol. 24, Inst. Math. Statist., Hayward, CA, 111–121.

BARNES, E. W.(1899). The theory of the gamma function. *Messenger of Mathematics* **29**, 64–128.

BERRY, A. C. (1941). The accuracy of the Gaussian approximation to the sum of independent variates. *Trans. Amer. Math. Soc.* **49**, 122–136.

BHARGAVA, R. P. & KHATRI, C. G. (1981). The distribution of product of independent Beta random variables with applications to multivariate analysis. *Ann. Inst. Statist. Math.*, Ser. A **33**, 287–296.

BURIĆ, T. & ELEZOVIĆ, N. (2011). Bernoulli polynomials and asymptotic expansions of the quotient of gamma functions. *J. Comput. Appl. Math.* **235**, 3315–3331.

CARTER, B. D. & SPRINGER, M. D. (1977). The distribution of products, quotients and powers of independent H-function variates. *SIAM J. Appl. Math.* **33**, 542–558.

COELHO, C. A. (1998). The generalized integer gamma distribution – a basis for distributions in multivariate statistics. *J. Multiv. Analysis* **64**, 86–102.

COELHO, C. A. (1999). Addendum to the paper "The generalized integer gamma distribution – a basis for distributions in multivariate statistics". *J. Multiv. Analysis* **69**, 281–285.

COELHO, C. A. (2004). The generalized near-integer gamma distribution: a basis for 'near-exact' approximations to the distribution of statistics which are the product of an odd number of independent Beta random variables. *J .Multivariate Anal.* **89**, 191–218.

COELHO, C. A. & MARQUES, F. J. (2011a). The multi-sample block-scalar sphericity test – exact and near-exact distributions for its likelihood ratio test statistic. Preprint #19/2011, Centro de Matemática e Aplicações, Universidade Nova de Lisboa, CMA/UNL (submitted for publication).

COELHO, C. A. & MARQUES, F. J. (2011b). On the exact, asymptotic and near-exact distributions for the likelihood ratio statistics to test equality of several exponential distributions, *AIP Conference Proceedings* **1389**, 1471–1474.

COELHO, C. A., MARQUES, F. J. & ARNOLD, B. C. (2010). Near-exact distributions for certain likelihood ratio test statistics, *J. Stat. Theory Pract.* **4**, 711–725.

COELHO, C. A., MARQUES, F. J. & ARNOLD, B. C. (2011). A general approach to the exact and near-exact distributions of the main likelihood ratio test statistics used in the complex multivariate normal setting. Preprint # 15/2011, Centro de Matemática e Aplicações, Universidade Nova de Lisboa, CMA/UNL (submitted for publication).

Esseen, C.-G. (1945). Fourier Analysis of distribution functions - a mathematical study of the Laplace-Gaussian law. *Acta Mathematica* **77**, 1–125.

Fields, J.L. (1966). A note on the asymptotic expansion of a ratio of gamma functions. *Proc. Edinburgh Math. Soc.* **15**, 43–45.

Gupta, A. K. (1971). Distribution of Wilks' likelihood-ratio criterion in the complex case. *Ann. Inst. Statist. Math.* **23**, 77–87.

Heuback, S. & Mansour, T. (2009). *Combinatorics of Compositions and Words*, Boca Raton: Chapman and Hall/CRC Press.

Hsu, P. L. (1945). The approximate distributions of the mean and variance of a sample of independent variables. *Ann. Math. Statist.* **15**, 1–29.

Hwang, H.-K. (1998). On convergence rates in the central limit theorems for combinatorial structures. *European J. Combin.* **19**, 329–343.

Imhof, J. P. (1961). Computing the distribution of quadratic forms in normal variables. *Biometrika* **48**, 419–426.

Jensen, D. R. & Solomon, H. (1972). A Gaussian approximation to the distribution of a definite quadratic form. *J. Amer. Statist. Assoc.* **67**, 898–902.

Khatri, C. G. (1965a). Classical Statistical Analysis Based on a Certain Multivariate Complex Gaussian Distribution. *Ann. Math. Statist.* **36**, 98–114.

Khatri, C. G. (1965b). A test for reality of a covariance matrix in a certain complex Gaussian distribution. *Ann. Math. Statist.* **36**, 115–119.

Khatri, C. G. (1980). Quadratic forms in normal variables. *Handbook of Statistics, Volume 1: Analysis of Variance* (P. R. Krishnaiah, ed.), Amsterdam: North-Holland, 443–469.

Kotz, S., Johnson, N. L. & Boyd, D. W. (1967a). Series representations of distributions of quadratic forms in Normal variables I - central case. *Ann. Math. Statist.* **38**, 823–837.

Kotz, S., Johnson, N. L. & Boyd, D. W. (1967b). Series representations of distributions of quadratic forms in Normal variables II - non-central case. *Ann. Math. Statist.* **38**, 838–848.

Kshirsagar, A. M. (1972). *Multivariate Analysis*, New York: Marcel Dekker.

Marques, F. J. & Coelho, C. A. (2011a). The block sphericity test – exact and near-exact distributions for the likelihood ratio statistic. *Math. Methods Appl. Sci.* **35**, 373–383.

Marques, F. J. & Coelho, C. A. (2011b). The multi-sample block-matrix sphericity test. *AIP Conference Proceedings* **1389**, 1479–1482.

Marques, F. J., Coelho, C. A. & Arnold, B. C. (2011). A general near-exact distribution theory for the most common likelihood ratio test statistics used in Multivariate Analysis. *TEST* **20**, 180–203.

Mathai, A. M. (1984). Extensions of Wilks' integral equations and distributions of test statistics. *Ann. Inst. Statist. Math.* **36**, Part A, 271–288.

Moschopoulos, P. G. (1985). The distribution of the sum of independent gamma random variables. *Ann. Inst. Statist. Math.* **37**, Part A, 541–544.

Muirhead, R. J. (2005). *Aspects of Multivariate Statistical Theory*, 2nd ed., New York: J. Wiley & Sons.

Nagar, D. K., Jain, S. K. & Gupta, A. K. (1985). Distribution of LRC for testing sphericity of a complex multivariate Gaussian model. *Int. J. Math. Math. Sci.* **8**, 555–562.

Nagarsenker, B. N. & Das, M. M. (1975). Exact distribution of sphericity criterion in the complex case and its percentage points. *Comm. Statist.* **4**, 363–374.

Nagarsenker, B. N. & Suniaga, J. (1983). Distributions of a class of statistics useful in multivariate analysis. *J. Amer. Statist. Assoc.* **78**, 472–475.

Nandi, S. B. (1980). The exact distributions of certain likelihood ratio criteria. *Sankhya*, Ser. B **42**, 187–196.

Olkin, I. & Press, S. J. (1961). Testing and estimation for a circular stationary model. *Ann. Math. Statist.* **40**, 1358–1373.

Pederzoli, G. (1985). Some properties of beta functions and the distribution for the product of independent beta random variables. *Trabajos de Estadistica y de Investigacion Operativa* **36**, 122–128.

Pillai, K. C. S. & Jouris, G. M. (1971). Distribution of Wilks' likelihood-ratio criterion in the complex case. *Ann. Math. Statist.* **42**, 517–525.

Shah, B. K. (1963). Distribution of definite and indefinite quadratic forms from a non-central normal distribution. *Ann. Math. Statist.* **34**, 186–190.

Springer, M. D. (1979). *The Algebra of Random Variables*. New York: J. Wiley & Sons.

Springer, M. D. & Thompson, W. E. (1966). The distribution of products of independent random variables. *SIAM J. Appl. Math.* **14**, 511–526.

Springer, M. D. & Thompson, W. E. (1970). The distribution of products of beta, gamma and Gaussian random variables. *SIAM J. Appl. Math.* **18**, 721–737.

Tang, J. & Gupta, A. K. (1984). On the distribution of the product of independent Beta random variables. *Statist. Probab. Lett.* **2**, 165–168.

Tang, J. & Gupta, A. K. (1986). Exact distribution of certain general test statistics in multivariate analysis. *Aust. J. Stat.*, **28**, 107–114.

Tretter, M. J. & Walster, G. W. (1975). Central and noncentral distribution of Wilks' statistics in MANOVA as mixtures of incomplete beta functions. *Ann. Statist.* **3**, 467–472.

Tricomi, F. G. & Erdélyi, A. (1951). The asymptotic expansion of a ratio of gamma functions. *Pac. J. Math.* **1**, 133–142.

Tukey, J. W. & Wilks, S. S. (1946). Approximation of the distribution of the product of beta variables by a single beta variable. *Ann. Math. Statist.* **17**, 318–324.

Walster, G. W. & Tretter, M. J. (1980). Exact noncentral distributions of Wilks Lambda and Wilks-Lawley U criteria as mixtures of incomplete Beta functions: for three tests. *Ann. Statist.* **8**, 1388–1390.