

Bonds historical simulation value at risk

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Abstract

Bonds historical returns can not be used directly to compute VaR by historical simulation because the maturities of the interest rates implied by the historical prices are not the relevant maturities at time VaR is computed.

In this paper we adjust bonds historical returns so that the adjusted returns can be used directly to compute VaR by historical simulation. The adjustment is based on the prices, implied by the historical prices, at the times to maturity relevant for the VaR computation.

Besides other features, we show that the obtained VaR values agree with the usual market trend of smaller times to maturity being traded with smaller interest rates, hence, carrying smaller risk and consequently having a smaller VaR.

1 Introduction

Despite all criticisms [5], historical simulation is by far the most popular VaR method [4].

It is well known that VaR computation, by historical simulation, of bond portfolios differs in important ways from VaR computation of stock portfolios [2]. Essentially, this is because the market historical prices of bonds imply a historical term structure of interest rates with maturities that are not the relevant maturities, at time VaR is computed. They are greater than the relevant maturities at time VaR is computed because they correspond to past times when the bond maturity was further away then it is when VaR is

computed. This moves away the possibility of using market bonds historical returns, directly in VaR computation.

The popular method to overcome this issue of cash flow mapping in risk factors, besides ignoring the portfolio specific VaR, being subjective, complex [1] and using lots of information sources, ruins the objectivity and the simplicity of the historical simulation method.

In this paper we develop a method of adjusting bonds historical returns so that they can be used directly in VaR computations. The method is based on computing the returns of the prices implied, by the historical prices, at the times to maturity relevant for the VaR computation.

We show that the developed method provides results consistent with the usual market observed trend, in which smaller times to maturity imply smaller yields, carrying smaller risk and consequently having smaller VaR. We also show that the developed method strongly preserves the market implicit correlations between the instruments in the portfolio.

2 Time to maturity adjusted bond returns

Consider the VaR computation at day n_{VaR} , with time horizon N days, and confidence level α percent, of a portfolio with a zero coupon bond with maturity $T > n_{VaR} + N$ and principal P . See the time line in Figure 1 for a graphical representation of these instants. Clearly, the relevant maturities for this VaR computation are $T - n_{VaR}$ and $T - (n_{VaR} + N)$.

Following the general historical simulation¹ method, the bond's N days market observed historical returns should be used to compute VaR. Denoting by $p(n)$, the historical price of the zero coupon bond, at day $N < n \leq n_{VaR}$, and denoting by $HR(n, N)$ the N days historical return at day n , defined as in [3], the N days possibly overlapping historical returns are given by:

$$HR(n, N) = \frac{p(n)}{p(n - N)}, \quad n = N + 1, \dots, n_{VaR}. \quad (1)$$

These market observed historical returns should be applied to the bond market value at day n_{VaR} as it follows:

$$p(n_{VaR})HR(n, N) = p(n_{VaR})\frac{p(n)}{p(n - N)}, \quad n = N + 1, \dots, n_{VaR}. \quad (2)$$

¹VaR historical simulation method is referred by some authors as non-parametric VaR.

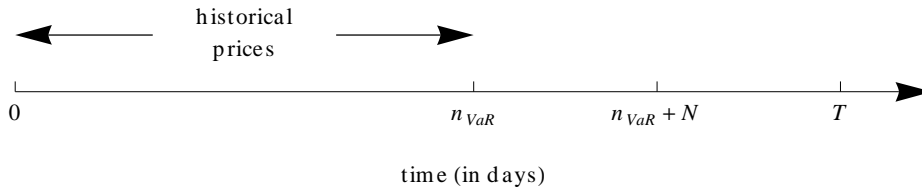


Figure 1: VaR computation time line.

The resulting returns define an empirical distribution of possible N days bond returns at time n_{VaR} . The VaR should be the potential loss of the $1 - \alpha/100$ quantile of this empirical distribution.

The problem with this general approach is that the historical price sequence $p(n)$ for $1 < n < n_{VaR}$, used in Equation (2), implies a sequence of term structure daily compounded interest rates² $r(n, T - n)$, given by:

$$r(n, T - n) = \left(\frac{P}{p(n)} \right)^{\frac{1}{T-n}} - 1. \quad (3)$$

But, the relevant maturities for this VaR computation do not occur in the implied historical interest rates. This is why the bond's historical returns can not be used directly in VaR computation.

Nevertheless, the implied historical interest rates, for times $1 < n \leq n_{VaR}$, provide future bond valuation at times n_{VaR} and $n_{VaR} + N$.

The future value at time $n_{VaR} < m < T$ of the bond, bought at time $1 < n < n_{VaR}$, at historical price $p(n)$, is given by the valuation of the future cash flow at maturity time, at daily compounded interest rate $r(n, T - n)$,

²Daily compounded interest rates are used because typical VaR time horizons are specified in days.

fixed by price $p(n)$. The possibility of a default event is assumed to be implicitly incorporated in the price $p(n)$ itself. Denoting by $f(m, n)$ the future value of the bond at time $n_{VaR} < m < T$ fixed by the price $p(n)$ at time $1 < n < n_{VaR}$, $f(m, n)$ is given by:

$$f(m, n) = \frac{P}{(1 + r(n, T - n))^{T-m}} = \frac{P}{\left(\frac{P}{p(n)}\right)^{\frac{T-m}{T-n}}}. \quad (4)$$

In this paper we define the n_{VaR} time to maturity adjusted N days historical return at day n , denoted by $AHR(n, N, n_{VaR})$, as the quotient between $f(n_{VaR} + N, n)$ and $f(n_{VaR}, n - N)$:

$$AHR(n, N, n_{VaR}) = \frac{f(n_{VaR} + N, n)}{f(n_{VaR}, n - N)}, \quad n = N + 1, \dots, n_{VaR}. \quad (5)$$

Substituting Equation (4) in Equation (5), the defined adjusted historical return is given by:

$$AHR(n, N, n_{VaR}) = \frac{\left(\frac{P}{p(n-N)}\right)^{\frac{T-n_{VaR}}{T-(n-N)}}}{\left(\frac{P}{p(n)}\right)^{\frac{T-(n_{VaR}+N)}{T-n}}}, \quad n = N + 1, \dots, n_{VaR}. \quad (6)$$

Note that the $AHR(n, N, n_{VaR})$ value is fixed by historical market prices $p(n)$ and $p(n - N)$, thus capturing the market changes between $n - N$ and n , while being adjusted to the VaR computation relevant maturities, namely, $T - n_{VaR}$ and $T - (n_{VaR} + N)$.

Our proposal is to replace the original historical returns of Equation (1) by those of Equation (6).

Using this adjusted historical returns directly in the VaR computation, the VaR is the potential loss of the $1 - \alpha/100$ quantile of the following time to maturity adjusted empirical distribution:

$$p(n_{VaR}) \frac{\left(\frac{P}{p(n-N)}\right)^{\frac{T-n_{VaR}}{T-(n-N)}}}{\left(\frac{P}{p(n)}\right)^{\frac{T-(n_{VaR}+N)}{T-n}}}, \quad n = N + 1, \dots, n_{VaR}. \quad (7)$$

3 Extensions

In this section we discuss the extension of the bonds historical simulation VaR method developed in the previous section to other scenarios beside computing the VaR for zero coupon bonds at the time the historical sequence of prices end.

3.1 Coupon bonds

The extension to coupon bonds is straight forward. In order to compute the future value of a coupon bond at time m , based on the market price of the bond at time $n < m$, two differences from the zero coupon bond case arise:

1. the yield to maturity at time n is computed using the bond's dirty price and accounting for all future cash flows after time n ;
2. the value of the bond at time m accounts for all future cash flows after time m .

Then, the adjusted returns are defined by Equation (6) as in the case of a zero coupon bond and the VaR is computed as the loss corresponding to the quantile of the empirical distribution of Equation (7).

3.2 Adjusting for past values

Suppose a bond B has already expired and the issuer of the bond issues a new bond, B_1 , equal to bond B , i.e., with the same type, principal, maturity, number of coupons, coupon rate (if applicable), etc.

Consider a portfolio that contains the bond B_1 . The portfolio VaR with time horizon N days is to be computed by historical simulation at day 1. The only historical prices available from bond's B_1 issuer are those of bond B .

The adjustment method proposed in this paper can still be applied to adjust the historical prices of bond B to past times, times before the historical prices were observed, namely, for day 1. The process is the same as for future values: for each historical price compute the daily yield to maturity implied by the historical price; then compute the bond's value at a previous time valuing the bond's cash flows following the time considered, with the implied daily yield to maturity; finally use the previous times values to get the adjusted returns of Equation (5), and compute the VaR.

4 Application

In this section we illustrate the usage of the bond adjusted historical return of Equation (6), by computing the VaR of the simplest possible portfolio, namely, a portfolio with a unique real zero coupon bond. We use a sequence of real historical prices of an alive zero coupon bond and compute the VaR at time the historical prices sequence ends.

We first present the used portfolio, than we detail the adjustment of single historical return, and then we adjust all the available historical returns and compute the VaR.

Additionally we illustrate the adjustment for past values by computing the VaR at the time the historical prices sequence begins.

4.1 Portfolio

Consider the zero coupon bond, B , with principal $P = 1000$, maturing at day $T = 731$ whose real market historical prices are in Figure 2. The prices were obtained from a quote service that delivers market prices aggregated from different dealers responsible for trading (market makers) this particular bond.

The prices in Figure 2 imply the market observed term structure interest rates represented in Figure 3. Recall from Figure 3 that each day corresponds to a different time to maturity.

Figure 3 clearly shows the usual trend observed in the market, in which smaller time to maturities are traded with smaller implied interest rates.

4.2 Adjustment of a single return

Consider the $N = 10$ days historical return at day $n = 190$ of bond B , detailed in Table 1(a). The prices that determine this historical return are highlighted in Figure 4 with the black circles. The maturities underlying this historical return are $731 - (190 - 10) = 551$ and $731 - 190 = 541$ days to maturity.

Now, consider the VaR, computed by historical simulation, at day $n_{VaR} = 372$, with a time horizon of $N = 10$ days, of the portfolio containing the bond B . The empirical distribution of the possible 10 days returns at $731 - 372 = 359$ days to maturity must be used. In order to obtain this distribution we adjust each of the historical returns with Equation (6).

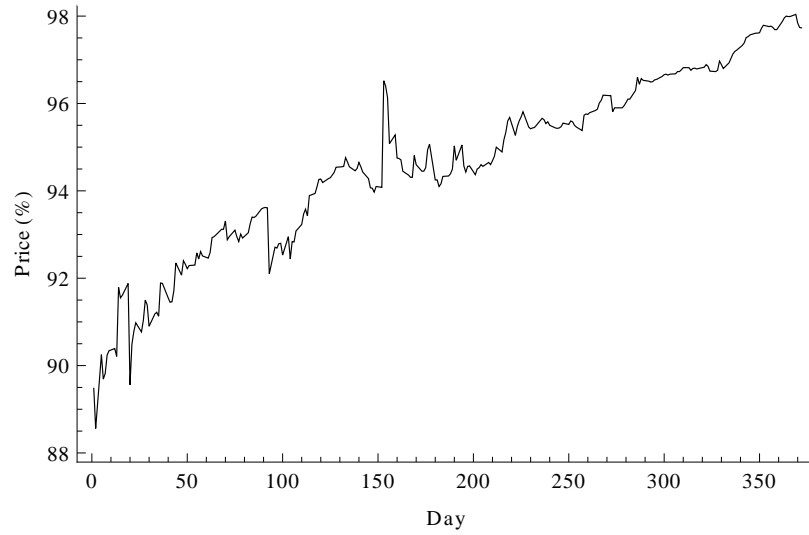


Figure 2: Real historical prices of a zero coupon bond with principal $P = 1000$ and maturing at day $T = 731$, as a percentage of the principal.

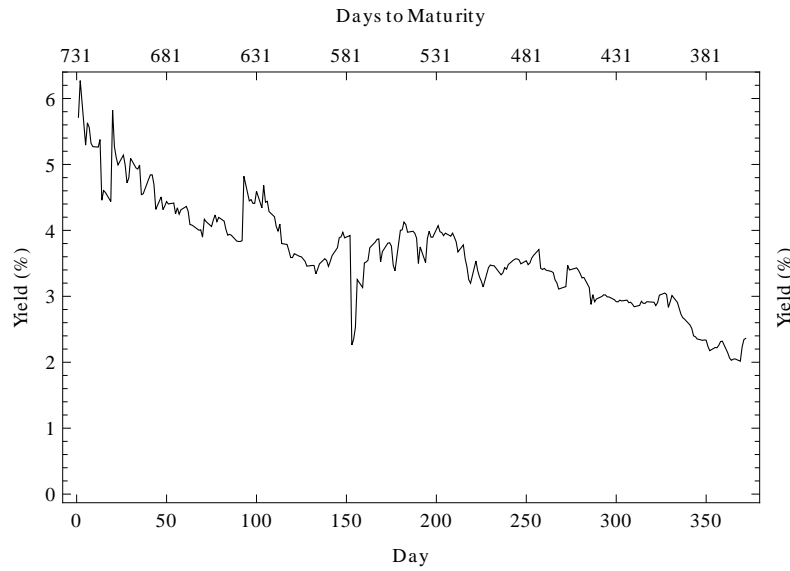


Figure 3: Term structure annualized daily compounded interest rates, implied by the historical prices of Figure 2, as a function of both time and time to maturity.

$N = 10$ days historical return at day $n = 190$		
$p(190 - 10)$	$p(190)$	$\frac{p(190)}{p(180)}$
94.25	95.03	1.00828

(a)

Annualized daily YTM (%)	
$r(180, 551)$	$r(190, 541)$
4.001	3.499

(b)

Adjusted return for $n_{VaR} = 372$		
$f(372, 190 - 10)$	$f(372 + 10, 190)$	$\frac{f(382, 190)}{f(372, 180)}$
96.215	96.765	1.00571

(c)

Table 1: (a) $N = 10$ days market observed historical return at day $n = 190$. (b) Days $n = 180$ and $n = 190$ annualized daily yields to maturity. (c) Adjusted $N = 10$ days return at day $n = 190$, for VaR computed at day $n_{VaR} = 372$.

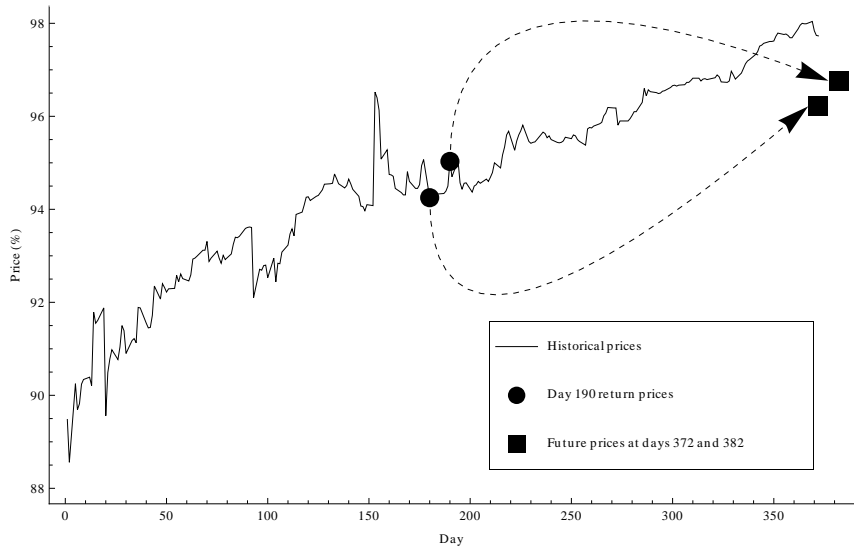


Figure 4: The prices that determine the $N = 10$ days historical return at time $n = 190$, the corresponding future prices at times $n_{VaR} = 372$ and $n_{VaR} + N = 372 + 10 = 382$, along with the historical prices sequence. The arrows represent future values.

In the particular case of the prices highlighted in Figure 4 with the black circles, the corresponding adjusted prices, with Equation (4), are highlighted with the black squares.

Table 1 shows the market observed historical return at day $n = 190$, as well as the corresponding adjusted return for VaR computed at day $n_{VaR} = 372$. The annualized daily yields to maturity and future prices used to compute the adjusted return are also detailed.

As it can be observed from table 1 the adjusted return is closer to one than the historical return. This is in accordance with the trend observed in Figure 3. Once the interest rates of smaller maturities tend to be smaller, the returns at time to maturity 369 should be closer than one than those at time to maturity 541.

4.3 Portfolio VaR

Consider the VaR, with a time horizon of $N = 10$ days and confidence level $\alpha = 99\%$, computed by historical simulation, at day $n_{VaR} = 372$, of the portfolio containing the bond B .

The VaR is computed using the empirical distribution of the adjusted returns of Equation (6). In order to obtain this distribution the adjustment of the single return detailed in the previous section is repeated for all available historical returns.

Figure 5 shows the real, market observed historical prices, and also, the corresponding future prices, $f(m, n)$ of Equation (4), at days $m = n_{VaR} = 372$ and $m = n_{VaR} + N = 372 + 10 = 382$. The future prices are plotted as a function of the day n of the historical price $p(n)$ which fixes the future value $f(m, n)$. The prices highlighted in Figure 4 with the black circles and squares are highlighted again in Figure 5, but now plotted as a function of n .

Figure 6 shows the sequence of the bond's historical returns along with the sequence of the corresponding adjusted returns computed from the future prices of Figure 5. Figure 7 shows the respective histograms.

Finally, Table 2 shows the VaR value computed from the empirical distribution of the overlapping adjusted returns of Equation (7), along with the possible loss corresponding to $1 - \alpha/100$ quantile of the overlapping historical returns empirical distribution of Equation (2), for comparison purposes. It also shows the correlation coefficient between the original and the adjusted returns.

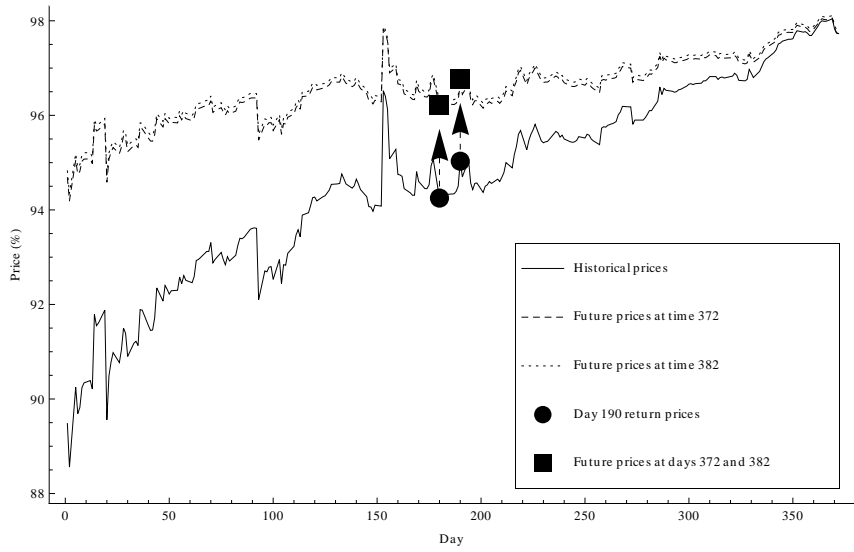


Figure 5: Historical prices, and the corresponding future prices at times $m = n_{VaR} = 372$ and $m = n_{VaR} + N = 372 + 10 = 382$. The future prices are plotted as a function of the time n , of the historical price $p(n)$, that fixed the future price.

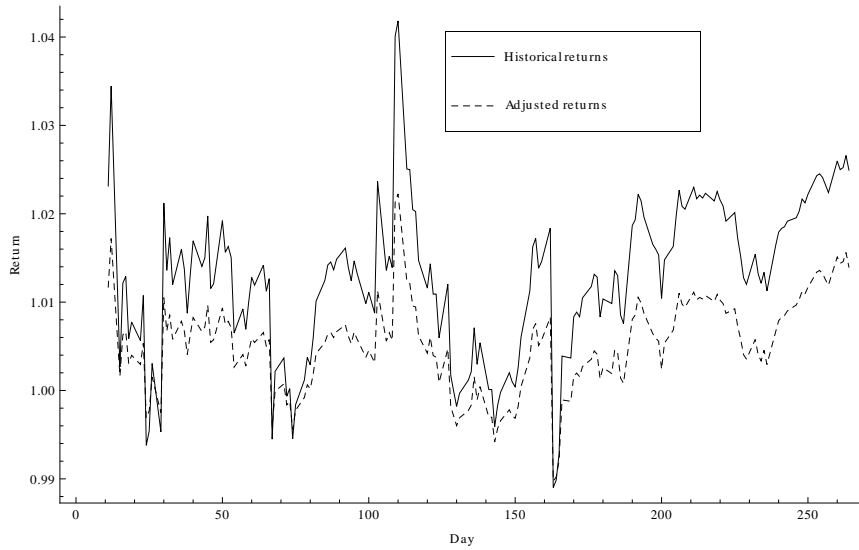


Figure 6: Historical returns and the corresponding adjusted returns for $n_{VaR} = 372$.

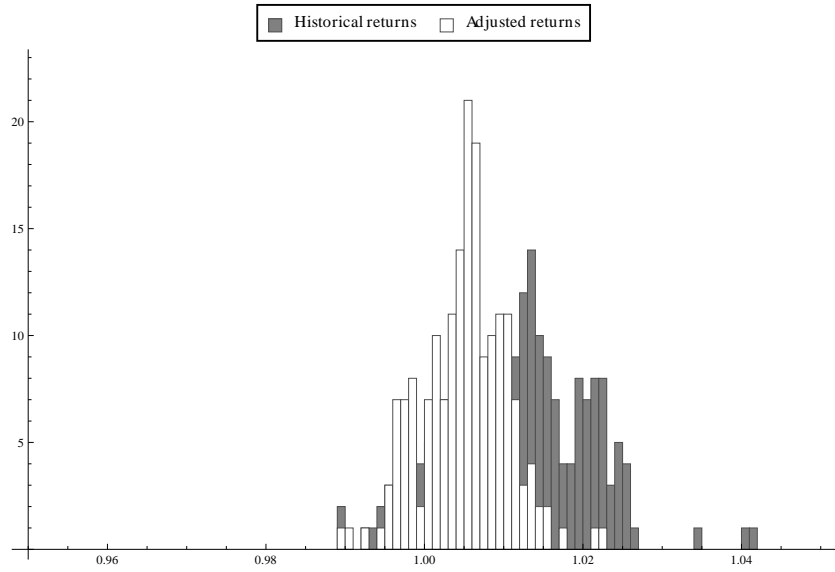


Figure 7: Historical returns and the corresponding adjusted for $n_{VaR} = 372$ returns, histograms.

Time horizon	Confidence level	VaR	Quantile	Correlation
$N = 10$	$\alpha = 99\%$	-9.576	-9.935	0.984

Table 2: Time horizon $N = 10$, confidence level $\alpha = 99\%$, bond B VaR, computed at day $n_{VaR} = 372$ by historical simulation using adjusted historical returns.

Time horizon	Confidence level	VaR	Quantile	Correlation
$N = 10$	$\alpha = 99\%$	-20.291	-9.935	0.985

Table 3: Time horizon $N = 10$, confidence level $\alpha = 99\%$, bond B VaR, computed at day $n_{VaR} = 1$ by historical simulation using adjusted historical returns.

As it can be observed from Figure 6, the adjusted returns are closer to one than the historical ones. This can be observed again in Figure 7 where the adjusted returns histogram is more concentrated towards one than the historical returns histogram. This results in a VaR value smaller than the corresponding loss of the $1 - \alpha/100$ quantile of the possible historical returns. Again, this result conforms with Figure 3 which shows a clear decreasing trend in interest rate as time to maturity decreases.

4.4 Adjusting for past values

Suppose that the bond B has already expired and its issuer issues a new bond, B_1 , equal to bond B .

Consider the portfolio containing the bond B_1 and the VaR computed by historical simulation at day $n = 1$ with the historical prices of bond B , showed in Figure 2.

Following section 3.2 the past values of Equation (4), $f(m, n)$ with $m = 1 \leq n$ are used to compute the adjusted historical returns of Equation (6) and the vaR is computed from the resulting empirical distribution. In this section we illustrate this process by repeating the figures and the table of the previous section, but now, for $n_{VaR} = 1$.

It can be observed from Figure 11 that the adjusted returns are now less concentrated towards one than the historical returns. This results in a VaR value, showed in Table 3, which is now greater than the corresponding loss of the $1 - \alpha/100$ quantile of the possible historical returns. Again, this observation is in accordance with Figure 3 and the fact that the time to maturity at time VaR is computed, $n_{VaR} = 1$, is greater than the times to maturity at following times, namely, when the historical prices were observed.

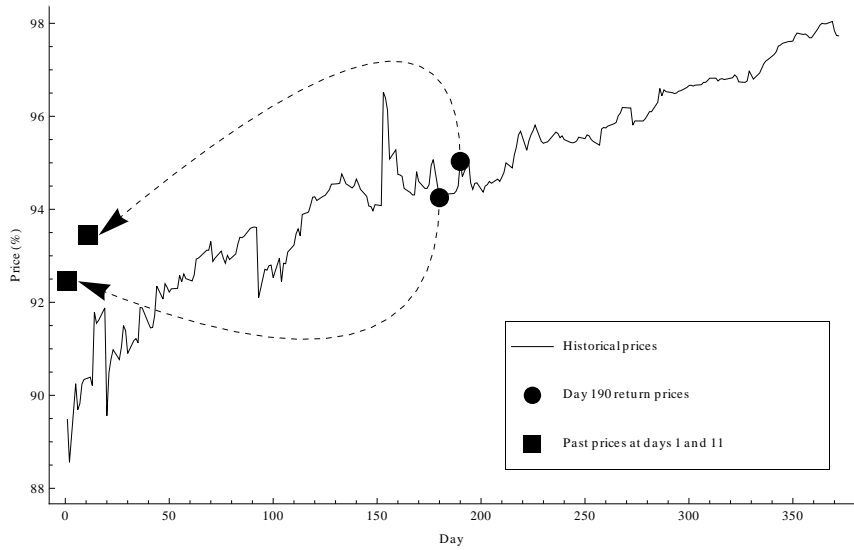


Figure 8: The prices that determine the $N = 10$ days historical return at time $n = 190$, the corresponding past prices at times $n_{VaR} = 1$ and $n_{VaR} + N = 1 + 10 = 11$, along with the historical prices sequence. The arrows represent past values.

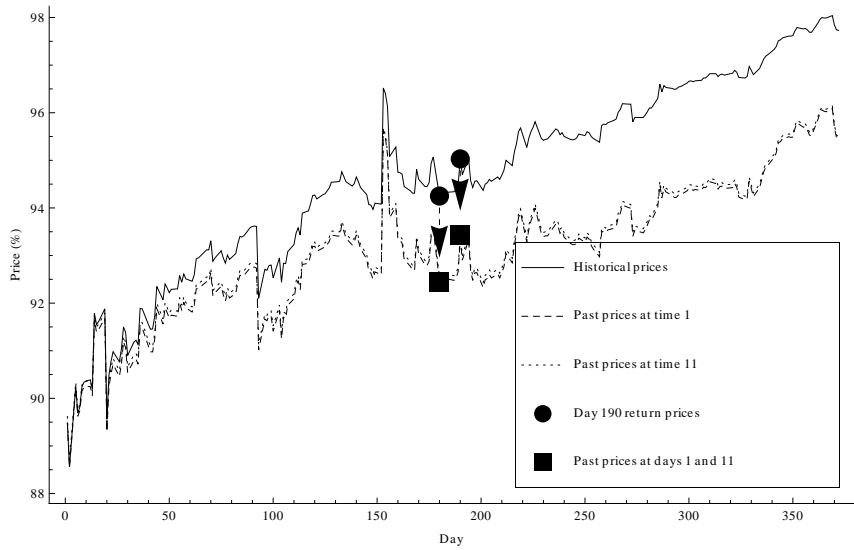


Figure 9: Historical prices, and the corresponding past prices at times $m = n_{VaR} = 1$ and $m = n_{VaR} + N = 1 + 10 = 11$. The past prices are plotted as a function of the time n , of the historical price $p(n)$, that fixed the future price.

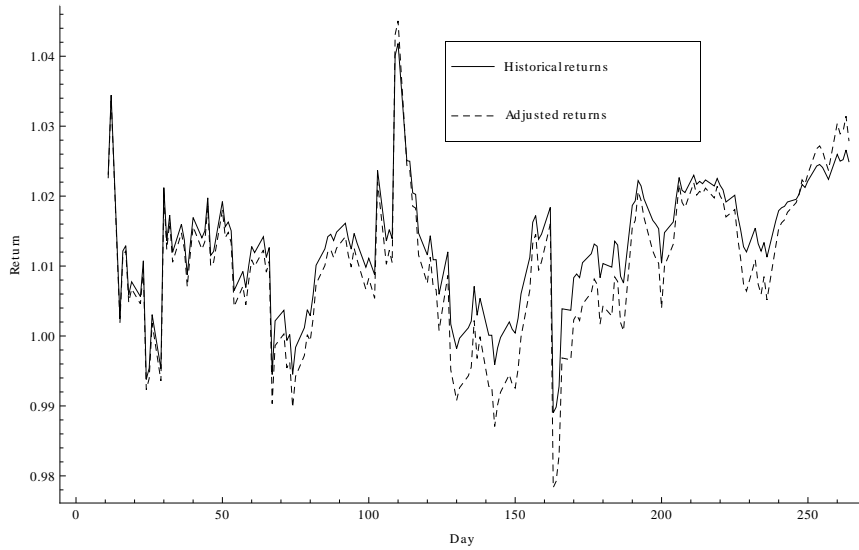


Figure 10: Historical returns and the corresponding adjusted returns for $n_{VaR} = 1$.

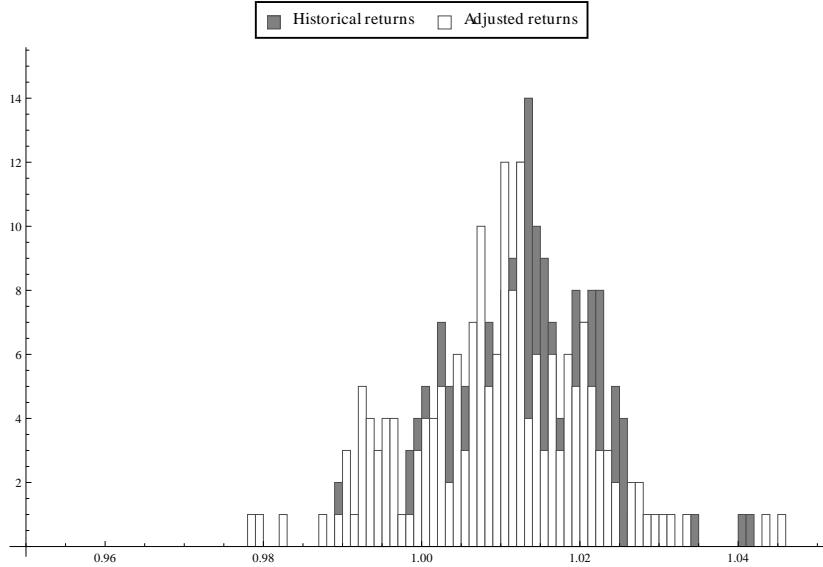


Figure 11: Historical returns and the corresponding adjusted for $n_{VaR} = 1$ returns, histograms.

5 Conclusions

Bond historical returns can not be used directly to compute VaR by historical simulation because the maturities of the interest rates implied by the historical prices are not the relevant maturities at time VaR is computed.

In this paper we adjust bonds historical returns so that the adjusted returns can be used directly to compute VaR by historical simulation. The adjustment is based on the prices, implied by the historical prices, at the times to maturity relevant for the VaR computation.

The proposed method has the following features:

- The time to maturity adjusted bond returns are used directly in the VaR historical simulation computation.
- VaR of portfolios with bonds can be computed by historical simulation keeping the simplicity of the historical simulation method.
- The portfolio specific VaR is obtained.
- The VaR values obtained are consistent with the usual market trend

of smaller times to maturity being traded with smaller interest rates, therefore carrying smaller risk and having a smaller VaR.

- The only source of information used is the market, through the bonds historical prices.
- The correlation between each bond return and the returns of the other instruments in the portfolio is strongly preserved.
- The VaR for the desired time horizon is computed directly with no VaR time scaling approximations.

We left for future work, the research of the mathematical properties of the developed method, and backtesting the method with benchmark portfolios.

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