Optimal control of shear-thickening flows

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Abstract

We study optimal control problems of systems describing the flow of incompressible shear-thickening fluids. We prove existence of solutions and derive necessary optimality conditions under precise restrictions on the optimal control.

Key words. Optimal control, Navier-Stokes, shear-dependent viscosity, necessary optimality conditions.

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1 Introduction

This paper deals with the mathematical analysis of an optimal control problem associated with a viscous, incompressible fluid. Control is effected through a distributed mechanical force and the objective is to match the velocity field to a given target field. The controls and states are constrained to satisfy a system of partial differential equations, consisting of a generalized Navier-Stokes system with shear dependent viscosity given by

$$\begin{cases} -\nabla \cdot (\tau(Dy)) + (y \cdot \nabla)\, y + \nabla \pi = u & \text{in } \Omega, \\ \nabla \cdot y = 0 & \text{in } \Omega, \\ y = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where y is the velocity field, π is the pressure, τ is the Cauchy stress tensor, $Dy = \frac{1}{2}\left(\nabla y + (\nabla y)^T\right)$ is the symmetric part of the velocity gradient ∇y, u is the given body force and $\Omega \subset I\!R^n$ ($n = 2$ or $n = 3$) is a bounded domain. We assume that $\tau : I\!R^{n\times n}_{sym} \longrightarrow I\!R^{n\times n}_{sym}$ has a potential, i.e. there exists a function $\Phi \in C^2(I\!R^+_n, I\!R^+_n)$ with $\Phi(0) = 0$ such that

$$\tau_{ij}(\eta) = \frac{\partial \Phi(|\eta|^2)}{\partial \eta_{ij}} = 2\Phi'(|\eta|^2)\, \eta_{ij}, \qquad \tau(0) = 0$$

for all $\eta \in I\!R^{n\times n}_{sym}$. $\big($Here $I\!R^{n\times n}_{sym}$ consists of all symetric $(n \times n)$-matrices.$\big)$ Moreover, we assume that for some $\alpha \geq 2$ the following assumptions hold

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$\mathbf{A}_1$ - There exists a positive constant γ such that for all $i, j, k, \ell = 1, \cdots, n$

$$\left|\frac{\partial \tau_{k\ell}(\eta)}{\partial \eta_{ij}}\right| \leq \gamma \left(1 + |\eta|^2\right)^{\frac{\alpha-2}{2}} \qquad \text{for all } \eta \in I\!R^{n\times n}_{sym}.$$

$\mathbf{A}_2$ - There exists a positive constant μ such that

$$\tau'(\eta) : \zeta : \zeta = \sum_{i,j=1}^{n} \sum_{k,\ell=1}^{n} \frac{\partial \tau_{k\ell}(\eta)}{\partial \eta_{ij}} \zeta_{k\ell}\zeta_{ij} \geq \mu \left(1 + |\eta|^2\right)^{\frac{\alpha-2}{2}} |\zeta|^2$$

for all $\eta, \zeta \in I\!R^{n\times n}_{sym}$.

These assumptions are usually used in the literature and cover a wide range of applications in non-Newtonian fluids. Typical prototypes of extra tensors used in applications are

$$\tau(\eta) = \left(1 + |\eta|^2\right)^{\frac{\alpha-2}{2}} \eta \qquad \text{or} \qquad \tau(\eta) = (1 + |\eta|)^{\alpha-2} \eta.$$

We recall that a fluid is called shear thickening if $\alpha > 2$ and shear thinning if $\alpha < 2$. For the special case $\tau(\eta) = \mu\eta$ ($\alpha = 2$), we recover the Navier-Stokes equation with viscosity coeficient $\mu > 0$.

The paper is concerned with the following optimal control problem

$$(P_\alpha) \begin{cases} \text{Minimize} & J(y,u) = \frac{1}{2} \displaystyle\int_\Omega |y - y_d|^2 \, dx + \frac{\nu}{2} \int_\Omega |u|^2 \, dx \\ \text{Subject to} & (y,u) \in W_0^{1,\alpha}(\Omega) \times U_{ad} \text{ satisfies (1.1) for some } \pi \in L^\alpha(\Omega), \end{cases}$$

where $\alpha \geq 2$, $\nu \geq 0$, y_d is some desired velocity field and U_{ad}, the set of admissible controls, is a nonempty closed convex subset of $L^2(\Omega)$.

A first and fundamental step when deriving the optimality conditions is devoted to the mathematical analysis of the state equation. The main problems of such analysis are related with existence, uniqueness and regularity of solutions in an appropriate functional setting. If the solution is not unique or not "sufficiently" regular, deriving first and second order optimality conditions can be a non trivial issue.

The considered class of fluids is described by partial differential equations of the quasi-linear type. It was first proposed by Ladyzhenskaya in [10], [11] and [12] as a modification of the Navier-Stokes system (the viscosity depending on the shear-rate), and was similarly suggested by Lions in [13]. Existence of weak solutions was proved by both authors using compactness arguments and the theory of monotone operators. Much work has been done since these pioneering results and, without ambition for completness, we emphasize the works by Nečas *et al.* who proved existence of weak and measure-valued solutions under less restrictive assumptions (see for example [16] and [6]).

Existence of an optimal control can be established using this basic regularity.

However, it may prove insufficient for deriving the necessary optimality conditions and the standard arguments to show the Gâteaux differentiability of the control-to-state mapping may fail (this is particularly the case when considering systems describing the shear-thinning flows). The difficulty is a consequence of the nonlinearity of the stress tensor and can be overcome if the gradient of the velocity is bounded. The corresponding viscosity, although non constant, is also bounded and the system can be studied as in the case of Navier-Stokes equations. These higher regularity results are few, difficult to obtain in general and do not seem to be available for the three-dimensional case. For the two-dimensional steady case, the boundedness of the gradient was proved by Kaplický *et al.* in [15] enabling Slawig to derive the corresponding optimality conditions in [18]. Similarly, Wachsmuth and Roubíček used the regularity results established in [14] to derive the optimality conditions for a two-dimensional unsteady system describing the flow of shear-tickening fluids (see [19]).

An other difficulty in deriving the optimality conditions is related with the convective term and the uniqueness of the state variable, guaranteed under some constraints on the data. It is similarly encountered when studying problems governed by the Navier-Stokes equations and the necessary optimality conditions can be established by taking into account these restrictions on the set of admissible controls (see for example [4], [5] and [17]).

In this paper, we consider the shear-thickening case. Observing that the underlying difficulties appear identically in a class of optimal control problems governed by quasilinear elliptic equations, we follow Casas and Fernández (see [2] and [3]) to deal with the nonlinear stress tensor and use an adequate functional setting involving weighted Sobolev spaces to analyse the properties of the control-to-state mapping as well as the adjoint system. We establish explicite estimates, carefully analyse the related equations and derive the optimality conditions in both two-dimensional and three-dimensional cases, without supposing that the gradient of the velocity is bounded and without restraining the set of admissible controls. The only constraint we need to impose concerns the optimal control and is related, in the particular case of problems governed by Navier-Stokes equations, to the property (C) at the optimal pair introduced in [8]. As a consequence, we recover a qualified version of the optimality conditions established in [1].

The plan of the paper is as follows. Assumptions, notation and some preliminary results are given in Section 2. Section 3 is devoted to existence and uniqueness results for the state and the adjoint equations. In Section 4, we prove the existence of an optimal solution and we state the necessary optimality conditions in Section 5. The properties of the control-to-state mapping are analysed in Section 6 and the proof of the optimality condition is achieved in Section 7.

2 Notation and preliminary results

Throughout the paper Ω is a bounded domain in $I\!R^n$ ($n = 2$ or $n = 3$). The boundary of Ω is denoted by $\partial\Omega$ and is of class C^2. Since many of the quantities occuring in the paper are vector-valued functions, the notation will be abreged for the sake of brevity and we will use the same notation of norms for scalar, vector and matrix-valued functions.

2.1 Auxiliary algebraic lemmas

For $\eta, \zeta \in I\!R^{n\times n}$, we define the scalar product and the corresponding norm by

$$\eta : \zeta = \sum_{i,j=1}^{n} \eta_{ij}\zeta_{ij} \quad \text{and} \quad |\eta| = (\eta : \eta)^{\frac{1}{2}} .$$

For $\eta \in I\!R^{n\times n\times n\times n}$ and $\zeta \in I\!R^{n\times n}$, the scalar product is defined by

$$\eta : \zeta = \left(\sum_{k,\ell=1}^{n} \eta_{ijk\ell}\zeta_{k\ell} \right)_{i,j=1,\cdots,n} \in I\!R^{n\times n},$$

and we can verify that

$$(\eta : \zeta_1) : \zeta_2 = (\zeta_2 : \eta) : \zeta_1, \qquad \eta \in I\!R^{n\times n\times n\times n}, \zeta_1, \zeta_2 \in I\!R^{n\times n}.$$

Assumptions $\mathbf{A}_1$-$\mathbf{A}_2$ imply the following standard properties for τ.

Lemma 2.1 *Let $\alpha \geq 2$ and τ satisfying $\mathbf{A}_1$-$\mathbf{A}_2$. Then the following properties hold*

Continuity.

$$|\tau(\eta)| \leq \tfrac{n^2\gamma}{\alpha-1} \left(1 + |\eta|^2\right)^{\frac{\alpha-2}{2}} |\eta| \tag{2.1}$$

Coercivity.

$$\tau(\eta) : \eta \geq \mu|\eta|^2 \qquad \textit{and} \qquad \tau(\eta) : \eta \geq \tfrac{\mu}{\alpha-1}|\eta|^\alpha \tag{2.2}$$

Monotonicity.

$$\begin{aligned} (\tau(\eta) - \tau(\zeta)) : (\eta - \zeta) &\geq \mu \left|\eta - \zeta\right|^2 \\ (\tau(\eta) - \tau(\zeta)) : (\eta - \zeta) &\geq \tfrac{\mu}{2^{2\alpha+1}} \left|\eta - \zeta\right|^\alpha \end{aligned} \tag{2.3}$$

where γ and μ are the constants appearing in the assumptions $\mathbf{A}_1$-$\mathbf{A}_2$.

The proof of estimates (2.1)-(2.3) can be found in [16], Chapter 5 and is based on arguments similar to those applied in the proof of the following useful auxiliary result.

Lemma 2.2 *Assume that $\mathbf{A}_2$ is fulfilled with $\alpha \geq 2$. Then, for all $\eta, \zeta \in I\!R^{n\times n}_{sym}$, we have*

$$2^{-\frac{3(\alpha-2)}{2}} (1 + |\eta|^2 + |\zeta|^2)^{\frac{\alpha-2}{2}} \left|\zeta - \eta\right|^2 \leq \tfrac{1}{\mu} \left(\tau(\eta) - \tau(\zeta)\right) : (\eta - \zeta) + |\eta - \zeta|^\alpha$$

Proof. Notice first that for every $\eta, \zeta \in I\!R^{n\times n}_{sym}$, we have

$$(\tau(\eta) - \tau(\zeta)) : (\eta - \zeta)$$

$$= \sum_{i,j=1}^{n} (\tau_{ij}(\eta) - \tau_{ij}(\zeta))\, (\eta - \zeta)_{ij} = \sum_{i,j=1}^{n} \int_0^1 \tfrac{d}{ds}\tau_{ij}(\zeta + s(\eta - \zeta))\, ds\, (\eta - \zeta)_{ij}$$

$$= \sum_{i,j=1}^{n} \sum_{k,\ell=1}^{n} \int_0^1 \tfrac{\partial \tau_{ij}}{\partial \eta_{k\ell}}(\zeta + s(\eta - \zeta))\, (\eta - \zeta)_{k\ell}\; ds\, (\eta - \zeta)_{ij}$$

$$= \int_0^1 \tau'(\zeta + s(\eta - \zeta)) : (\eta - \zeta) : (\eta - \zeta)\; ds$$

and by taking into account assumption $\mathbf{A}_2$, we deduce that

$$(\tau(\eta) - \tau(\zeta)) : (\eta - \zeta) \geq \mu \int_0^1 \left(1 + |\zeta + s(\eta - \zeta)|^2\right)^{\frac{\alpha-2}{2}} |\eta - \zeta|^2\, ds. \tag{2.4}$$

On the other hand, for every $s \in [0, 1]$, we have

$$|\eta| = |\zeta + s(\eta - \zeta) + (1 - s)(\eta - \zeta)| \leq |\zeta + s(\eta - \zeta)| + (1 - s)\, |\eta - \zeta|$$

$$|\zeta| = |\zeta + s(\eta - \zeta) - s(\eta - \zeta)| \leq |\zeta + s(\eta - \zeta)| + s\, |\eta - \zeta|$$

and thus

$$\begin{aligned} 1 + |\eta|^2 + |\zeta|^2 \quad &\leq 1 + 4\left(|\zeta + s(\eta - \zeta)|^2 + |\eta - \zeta|^2\right) \\ &\leq 4\left(1 + |\zeta + s(\eta - \zeta)|^2 + |\eta - \zeta|^2\right). \end{aligned}$$

This estimate, together with standard calculations show that

$$\begin{aligned} \left(1 + |\eta|^2 + |\zeta|^2\right)^{\frac{\alpha-2}{2}} \quad &\leq 2^{\alpha-2}\left(1 + |\zeta + s(\eta - \zeta)|^2 + |\eta - \zeta|^2\right)^{\frac{\alpha-2}{2}} \\ &\leq 2^{\frac{3(\alpha-2)}{2}} \left(\left(1 + |\zeta + s(\eta - \zeta)|^2\right)^{\frac{\alpha-2}{2}} + |\eta - \zeta|^{\alpha-2}\right) \end{aligned}$$

and by integrating, we get

$$2^{-\frac{3(\alpha-2)}{2}} \left(1 + |\eta|^2 + |\zeta|^2\right)^{\frac{\alpha-2}{2}}$$

$$\leq \int_0^1 \left(1 + |\zeta + s(\eta - \zeta)|^2\right)^{\frac{\alpha-2}{2}}\; ds + |\eta - \zeta|^{\alpha-2}. \tag{2.5}$$

Combining (2.4) and (2.5), we obtain

$$2^{-\frac{3(\alpha-2)}{2}} (1 + |\eta|^2 + |\zeta|^2)^{\frac{\alpha-2}{2}}\, |\eta - \zeta|^2$$

$$\leq \int_0^1 \left(1 + |\zeta + s(\eta - \zeta)|^2\right)^{\frac{\alpha-2}{2}} |\eta - \zeta|^2\, ds + |\eta - \zeta|^{\alpha}$$

$$\leq \tfrac{1}{\mu}\, (\tau(\eta) - \tau(\zeta)) : (\eta - \zeta) + |\eta - \zeta|^{\alpha}$$

and the claimed result is proven. ∎

2.2 Functional setting

In this section, we give the definitions and basic properties of function spaces appropriate for the mathematical analysis of our problem, and we collect some results useful for the sequel.

The space of infinitely differentiable functions with compact support in Ω will be denoted by $\mathcal{D}(\Omega)$. The standard Sobolev spaces are denoted by $W^{k,\alpha}(\Omega)$ ($k \in I\!N$ and $1 < \alpha < \infty$), and their norms by $\|\cdot\|_{k,\alpha}$. We set $W^{0,\alpha}(\Omega) \equiv L^{\alpha}(\Omega)$ and $\|\cdot\|_{L^{\alpha}} \equiv \|\cdot\|_{\alpha}$. The dual space of $W_0^{1,\alpha}(\Omega)$ is denoted by $W^{-1,\alpha'}(\Omega)$ and its norm by $\|\cdot\|_{-1,\alpha'}$. In order to eliminate the pressure in the weak formulation of the state equation, we will work in divergence-free spaces. Consider

$$\mathcal{V} = \big\{\varphi \in \mathcal{D}(\Omega) \mid \nabla \cdot \varphi = 0\big\},$$

and denote by V_α the closure of $\mathcal{V}$ with respect to the norm $\|\nabla \cdot\|_{\alpha}$, i.e.

$$V_\alpha = \Big\{\varphi \in W_0^{1,\alpha}(\Omega) \mid \nabla \cdot \varphi = 0\Big\}.$$

Let y be in $W_0^{1,\alpha}(\Omega)$. Following [2] and [3], we can associate two weighted Sobolev spaces V_α^y and H_α^y, where V_α^y is the set of functions $z \in V_2$ such that the norm

$$\|z\| = \left(\int_\Omega \big(1 + |Dy|^2\big)^{\frac{\alpha-2}{2}} |Dz|^2 \, dx\right)^{\frac{1}{2}}$$

is finite, and H_α^y is the completion of $\mathcal{V}$ with respect to this norm. It may be verified that V_α^y and H_α^y are Hilbert spaces and that $H_\alpha^y \subset V_\alpha^y$. Moreover, since $\alpha \geq 2$, we have

$$V_\alpha \subset H_\alpha^y \subset V_2$$

with continuous injections.

In the remaining part of this section, we first recall two classical inequalities and next, we point out some notable facts related with the the trilinear form b defined by

$$b(y_1, y_2, y_3) = \left(\left(y_1 \cdot \nabla\right) y_2, y_3\right).$$

Lemma 2.3 (*Poincaré's inequality.*) *Let y be in $H_0^1(\Omega)$. Then the following estimate holds*

$$\|y\|_2 \leq C_P \, \|\nabla y\|_2 \qquad \textit{with} \quad C_P = \tfrac{n-1}{\sqrt{n}} |\Omega|^{\frac{1}{n}}.$$

Proof. See for example [7], Chapter 2. ∎

Lemma 2.4 (*Korn's inequality.*) *Let y be in $H_0^1(\Omega)$. Then we have*

$$\|\nabla y\|_2 \leq \sqrt{2} \, \|Dy\|_2 .$$

with the equality if we suppose that $y \in V_2$.

Proof. See for example [9]. ∎

Lemma 2.5 *Let y_1, y_2 and y_3 be in $H_0^1(\Omega)$. Then the following estimate holds*

$$|b(y_1, y_2, y_3)| \leq \kappa_1 \|Dy_1\|_2 \|Dy_2\|_2 \|Dy_3\|_2$$

with $\kappa_1 = \frac{2^{\frac{3}{2}}(n-1)}{n}|\Omega|^{\frac{1}{n(n-1)}}$.

Proof. Due to Lemma 1.1, Chapter VIII in [7], we have

$$|b(y_1, y_2, y_3)| \leq C \|\nabla y_1\|_2 \|\nabla y_2\|_2 \|\nabla y_3\|_2$$

with $C = \frac{n-1}{n}|\Omega|^{\frac{1}{n(n-1)}}$. The conclusion follows by using the Korn inequality. ∎

Lemma 2.6 *Let y_1 be in V_2 and let y_2 and y_3 be in $H_0^1(\Omega)$. Then*

$$b(y_1, y_2, y_3) = -b(y_1, y_3, y_2) \qquad \text{and} \qquad b(y_1, y_2, y_2) = 0.$$

3 State equation and adjoint equation

3.1 State equation

For the subsequent analysis, we state existence and uniqueness results for the state equation and derive useful explicite estimates.

Definition 3.1 *Let $u \in W^{-1,\alpha'}(\Omega)$. A function $y \in V_\alpha$ is a weak solution of* (1.1) *if*

$$(\tau(Dy), D\varphi) + b(y, y, \varphi) = \langle u, \varphi \rangle \qquad \text{for all } \varphi \in V_\alpha.$$

Remark 3.2 *Let us recall that, having a solution satisfying the formulation given in Definition* 3.1*, it is standard to construct the corresponding pressure $\pi \in L_0^\alpha(\Omega)$ such that*

$$(\tau(Dy), D\varphi) + b(y, y, \varphi) - (\pi, \nabla \cdot \varphi) = \langle u, \varphi \rangle \qquad \text{for all } \varphi \in W_0^{1,\alpha}(\Omega).$$

We will involve the pressure only in the formulations of the theorems and lemmas but not in the proofs, since it can always be reconstructed uniquely.

First mathematical investigations of system (1.1) under conditions (2.1)-(2.3), were performed by J. L. Lions who proved existence of a weak solution for $\alpha \geq \frac{3n}{n+2}$ (see [13] for more details). The restriction on the exponent α ensures that the convective term belongs to L^1 when considering test functions in V_α. Due to Lemma 2.5, we can see that this condition is obviously satisfied when dealing with shear-thickening flows.

Proposition 3.3 *Assume that $\mathbf{A}_1$-$\mathbf{A}_2$ are fulfilled with $\alpha \geq 2$ and that $u \in L^2(\Omega)$. Then problem* (1.1) *admits a weak solution $y_u \in V_\alpha$ and the following estimates hold*

$$\|Dy_u\|_2 \leq \kappa_2 \frac{\|u\|_2}{\mu} \tag{3.1}$$

$$\|Dy_u\|_\alpha^\alpha \leq (\alpha - 1)\left(\kappa_2 \frac{\|u\|_2}{\mu}\right)^2 \tag{3.2}$$

with $\kappa_2 = \frac{\sqrt{2}(n-1)}{\sqrt{n}}|\Omega|^{\frac{1}{n}}$.

Proof. Since $\alpha \geq 2 > \frac{2n}{n+2}$, we have $L^2(\Omega) \hookrightarrow W^{-1,\alpha'}(\Omega)$ and our problem admits at least a solution (see [13]). To prove the estimate, we set $\varphi = y_u$ in the weak formulation of (1.1) and use Lemma 2.6, Lemma 2.3 and Lemma 2.4 to get

$$\begin{aligned}&(\tau(Dy_u), Dy_u) = -b(y_u, y_u, y_u) + (u, y_u) = (u, y_u)\\ &\leq \|u\|_2 \|y_u\|_2 \leq C_P \|u\|_2 \|\nabla y_u\|_2 = \kappa_2 \|u\|_2 \|Dy_u\|_2 \end{aligned} \tag{3.3}$$

with $\kappa_2 = \sqrt{2}\, C_P$. On the other hand, by taking into account the coercivity condition $(2.2)_1$, we have

$$\mu \|Dy_u\|_2^2 \leq (\tau(Dy_u), Dy_u). \tag{3.4}$$

Combining (3.3) and (3.4), we obtain estimate (3.1). Similarly, by using $(2.2)_2$ and (3.1) we obtain

$$\tfrac{\mu}{\alpha-1} \|Dy_u\|_\alpha^\alpha \leq (\tau(Dy_u), Dy_u) = (u, y_u) \leq \kappa_2 \|u\|_2 \|Dy_u\|_2 \leq \tfrac{(\kappa_2 \|u\|_2)^2}{\mu}$$

and the proof is complete. ∎

Proposition 3.4 *Assume that* $\mathbf{A}_1$-$\mathbf{A}_2$ *are fulfilled with* $\alpha \geq 2$ *and that* $u \in L^2(\Omega)$ *satisfies*

$$\tfrac{\|u\|_2}{\mu^2} < \tfrac{\sqrt{n^3}}{4(n-1)^2 |\Omega|^{\frac{1}{n-1}}}. \tag{3.5}$$

Then, equation (1.1) *admits a unique weak solution* $y_u \in V_\alpha$.

Proof. Assume that y_u and χ_u are two weak solutions of (1.1) corresponding to u. Substituing in the weak formulation of (1.1), setting $\varphi = y_u - \chi_u$ and taking into account Lemma 2.5, Lemma 2.6 and estimate (3.1), we obtain

$$\begin{aligned}&(\tau(Dy_u) - \tau(D\chi_u), D(y_u - \chi_u))\\ &= -b(y_u, y_u, y_u - \chi_u) + b(\chi_u, \chi_u, y_u - \chi_u)\\ &= -b(y_u, y_u - \chi_u, y_u - \chi_u) - b(y_u - \chi_u, \chi_u, y_u - \chi_u)\\ &= -b(y_u - \chi_u, \chi_u, y_u - \chi_u)\\ &\leq \kappa_1 \|D(y_u - \chi_u)\|_2^2 \|D\chi_u\|_2 \leq \kappa_2\kappa_1 \|D(y_u - \chi_u)\|_2^2 \tfrac{\|u\|_2}{\mu}.\end{aligned} \tag{3.6}$$

On the other hand, due to the monotonicity condition $(2.3)_1$, we have

$$\mu \|D(y_u - \chi_u)\|_2^2 \leq (\tau(Dy_u) - \tau(D\chi_u), D(y_u - \chi_u)). \tag{3.7}$$

Combining (3.6) and (3.7), we deduce that

$$\left(\mu - \kappa_2\kappa_1 \tfrac{\|u\|_2}{\mu}\right) \|D(y_u - \chi_u)\|_2^2 \leq 0$$

and thus $y_u \equiv \chi_u$ if $\mu^2 > \kappa_2\kappa_1 \|u\|_2$. ∎

3.2 Linearized equation

To derive the optimality conditions, we need to investigate the following linearized equation

$$\begin{cases} -\nabla\cdot\left(\tau'(Dy_u):Dz\right)+\left(z\cdot\nabla\right)y_u+\left(y_u\cdot\nabla\right)z+\nabla\pi=w & \text{in } \Omega,\\ \nabla\cdot z=0 & \text{in } \Omega,\\ z=0 & \text{on } \partial\Omega,\end{cases}\tag{3.8}$$

where $u \in L^2(\Omega)$ satisfies (3.5), $y_u \in V_\alpha$ being the corresponding (unique) solution of (1.1) and $w \in L^2(\Omega)$.

Definition 3.5 *Let $w \in L^2(\Omega)$. A function z is a weak solution of* (3.8) *if*

$$(\tau'(Dy_u):Dz, D\varphi)+b\left(z,y_u,\varphi\right)+b\left(y_u,z,\varphi\right)=(w,\varphi) \quad \textit{for all } \varphi\in H_\alpha^{y_u}.$$

Proposition 3.6 *Let $u \in L^2(\Omega)$ satisfying* (3.5) *and let $y_u \in V_\alpha$ be the corresponding solution of* (1.1)*. For $w \in L^2(\Omega)$, problem* (3.8) *admits a unique solution z_{uw} in $H_\alpha^{y_u}$. Moreover, the following estimate holds*

$$\|z_{uw}\|_{H_\alpha^{y_u}} \le \mathcal{L}\left(\|u\|_2\right)\|w\|_2$$

where $\mathcal{L}$ is defined by

$$\mathcal{L}(t)=\tfrac{\kappa_2\mu}{\mu^2-\kappa_2\kappa_1 t}=\tfrac{\sqrt{2}n(n-1)|\Omega|^{\frac{1}{n}}\mu}{\sqrt{n^3}\mu^2-4(n-1)^2|\Omega|^{\frac{1}{n-1}}t} \qquad 0\le t<\tfrac{\mu^2}{\kappa_1\kappa_2}.$$

Proof. Consider the bilinear form defined by

$$B(z_1,z_2)=(\tau'(Dy_u):Dz_1,Dz_2)+b\left(z_1,y_u,z_2\right)+b\left(y_u,z_1,z_2\right).$$

Taking into account Lemma 2.6, we have

$$\begin{aligned} B(z,z)&=(\tau'(Dy_u):Dz,Dz)+b\left(z,y_u,z\right)+b\left(y_u,z,z\right)\\ &=(\tau'(Dy_u):Dz,Dz)+b\left(z,y_u,z\right)\end{aligned}$$

for every $z \in H_\alpha^{y_u}$. On the other hand, taking into account Lemma 2.5 and assumption $\mathbf{A}_2$, we deduce that

$$(\tau'(Dy_u):Dz,Dz)\ge\mu\int_\Omega\left(1+|Dy_u|^2\right)^{\frac{\alpha}{2}-1}|Dz|^2\,dx=\mu\,\|z\|^2_{H_\alpha^{y_u}},$$

and

$$|b\left(z,y_u,z\right)| \quad \le\kappa_1\|Dy_u\|_2\|Dz\|_2^2\le\kappa_1\|Dy_u\|_2\|z\|^2_{H_\alpha^{y_u}}.$$

Hence, due to (3.1) we obtain

$$B(z,z)\ge\left(\mu-\kappa_1\|Dy_u\|_2\right)\|z\|^2_{H_\alpha^{y_u}}\ge\left(\mu-\kappa_2\kappa_1\tfrac{\|u\|_2}{\mu}\right)\|z\|^2_{H_\alpha^{y_u}}\tag{3.9}$$

and B is coercive on $H_\alpha^{y_u}$ since u satisfies (3.5). Let us now prove that B is continuous. Lemma 2.5 and assumption $\mathbf{A}_1$ yield

$$|(\tau'(Dy_u) : Dz_1, Dz_2)| \leq \gamma \int_\Omega \left(1 + |Dy_u|^2\right)^{\frac{\alpha-2}{2}} |Dz_1||Dz_2|\, dx$$

$$\leq \gamma \left\| (1+|Dy_u|^2)^{\frac{\alpha-2}{4}} Dz_1 \right\|_2 \left\| (1+|Dy_u|^2)^{\frac{\alpha-2}{4}} Dz_2 \right\|_2 = \gamma \left\| z_1 \right\|_{H_\alpha^{y_u}} \left\| z_2 \right\|_{H_\alpha^{y_u}}$$

and

$$|b(z_1, y_u, z_2) + b(y_u, z_1, z_2)|$$

$$\leq 2\kappa_1 \left\| Dy_u \right\|_2 \left\| Dz_1 \right\|_2 \left\| Dz_2 \right\|_2 \leq 2\kappa_2\kappa_1 \frac{\|u\|_2}{\mu} \left\| Dz_1 \right\|_2 \left\| Dz_2 \right\|_2$$

$$\leq 2\mu \left\| Dz_1 \right\|_2 \left\| Dz_2 \right\|_2 \leq 2\mu \left\| z_1 \right\|_{H_\alpha^{y_u}} \left\| z_2 \right\|_{H_\alpha^{y_u}}$$

for every $z_1, z_2 \in H_\alpha^{y_u}$. Therefore,

$$B(z_1, z_2) \leq (\gamma + 2\mu) \left\| z_1 \right\|_{H_\alpha^{y_u}} \left\| z_2 \right\|_{H_\alpha^{y_u}}.$$

The bilinear form B is then continuous and coercive on $H_\alpha^{y_u}$. Applying the Lax-Milgram theorem, we deduce that problem (3.8) admits a unique solution z_{uw} in $H_\alpha^{y_u}$. Taking into account (3.9), we obtain

$$\left(\mu - \kappa_2\kappa_1 \frac{\|u\|_2}{\mu}\right) \left\| z_{uw} \right\|^2_{H_\alpha^{y_u}} \leq B(z_{uw}, z_{uw}) = (w, z_{uw})$$

$$\leq \kappa_2 \left\| w \right\|_2 \left\| Dz_{uw} \right\|_2 \leq \kappa_2 \left\| w \right\|_2 \left\| z_{uw} \right\|_{H_\alpha^{y_u}}$$

which gives the estimate. ∎

Remark 3.7 *Notice that the existence and uniqueness result remains valid if in the definition of equation* (3.8)*, the state y_u is replaced by a function $y \in V_\alpha$ satisfying $\|Dy\|_2 < \frac{\mu}{\kappa_1}$.*

3.3 Adjoint equation

Let $u \in L^2(\Omega)$ satisfying (3.5) and let $y_u \in V_\alpha$ be the corresponding solution of (1.1). Consider the adjoint system

$$\begin{cases} -\nabla \cdot \left(\tau'(Dy_u)^T : Dp\right) + (\nabla y_u)^T p - (y_u \cdot \nabla) p + \nabla \pi = f & \text{in } \Omega, \\ \nabla \cdot p = 0 & \text{in } \Omega, \\ p = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.10}$$

where $f \in L^2(\Omega)$.

Definition 3.8 *A function p is a weak solution of* (3.10) *if*

$$(\tau'(Dy_u) : D\varphi, Dp) + \left((\nabla y_u)^T p - (y_u \cdot \nabla) p, \varphi\right) = (f, \varphi) \quad \textit{for all } \varphi \in H_\alpha^{y_u}.$$

Proposition 3.9 *Let $u \in L^2(\Omega)$ satisfying (3.5) and let $y_u \in V_\alpha$ be the corresponding solution of (1.1). For $f \in L^2(\Omega)$, problem (3.10) admits a unique solution p in $H_\alpha^{y_u}$. Moreover, the following estimate holds*

$$\|p\|_{H_\alpha^{y_u}} \leq \mathcal{L}\left(\|u\|_2\right)\|f\|_2$$

where $\mathcal{L}$ is defined in Proposition 3.6, and we have

$$(\tau'(Dy_u) : D\varphi, Dp) + b\left(\varphi, y_u, p\right) + b\left(y_u, \varphi, p\right) = (f, \varphi) \quad \text{for all } \varphi \in H_\alpha^{y_u}. \tag{3.11}$$

Proof. Existence and uniqueness of a solution as well as the estimate can be obtained with arguments similar to those used in the proof of Proposition 3.6. Moreover, observing that

$$\left((\nabla y_u)^T p, \varphi\right) = \left((\varphi\cdot\nabla)\, y_u, p\right) \quad \text{and} \quad -\left((y_u\cdot\nabla)\, p, \varphi\right) = \left((y_u\cdot\nabla)\,\varphi, p\right) \tag{3.12}$$

we obtain (3.11). ∎

4 Existence of an optimal control

Theorem 4.1 *Assume that $\mathbf{A}_1$-$\mathbf{A}_2$ are fulfilled with $\alpha \geq 2$ and that either U_{ad} is bounded in $L^2(\Omega)$ or $\nu > 0$. Then problem (P_α) admits at least a solution.*

Proof. Let $(y_k, u_k)_k \subset V_\alpha \times U_{ad}$ be a minimizing sequence. Since $(u_k)_k$ is uniformly bounded in the closed convex set U_{ad}, we may extract a subsequence, still indexed by k, weakly convergent to some $u \in U_{ad}$ in $L^2(\Omega)$. On the other hand, Due to estimate (3.2), we have

$$\|Dy_k\|_\alpha^\alpha \leq (\alpha-1)\left(\kappa_2 \tfrac{\|u_k\|_2}{\mu}\right)^2$$

and the sequence $(y_k)_k$ is then bounded in V_α. The previous estimate together with (2.1) imply

$$\|\tau(Dy_k)\|_{\alpha'}^{\alpha'} \leq \left(\tfrac{n^2\gamma}{\alpha-1}\right)^{\alpha'} \int_\Omega \left(1+|Dy_k|^2\right)^{\frac{\alpha-2}{2}\alpha'} |Dy_k|^{\alpha'}\, dx$$

$$\leq \left(\tfrac{n^2\gamma}{\alpha-1}\right)^{\alpha'} \int_\Omega \left(1+|Dy_k|^2\right)^{\frac{\alpha}{2}} dx \leq \left(\tfrac{n^2\gamma}{\alpha-1}\right)^{\alpha'} 2^{\frac{\alpha-2}{2}}\left(|\Omega| + \|Dy_k\|_\alpha^\alpha\right)$$

$$\leq \left(\tfrac{n^2\gamma}{\alpha-1}\right)^{\alpha'} 2^{\frac{\alpha-2}{2}}\left(|\Omega| + (\alpha-1)\left(\kappa_2\tfrac{\|u_k\|_2}{\mu}\right)^2\right)$$

and the sequence $(\tau(Dy_k))_k$ is uniformly bounded in $L^{\alpha'}(\Omega)$. There then exist a subsequence, still indexed by k, $y \in V_\alpha$ and $\widetilde{\tau} \in L^{\alpha'}(\Omega)$ such that $\left(y_k\right)_k$ weakly converges to y in V_α and $\left(\tau(Dy_k)\right)_k$ weakly converges to $\widetilde{\tau}$ in $L^{\alpha'}(\Omega)$. Moreover, Since $\alpha \geq 2 > \frac{3n}{n+2}$, by using compactness results on Sobolev spaces, we deduce that $(y_k)_k$ strongly converges to y in $L^{\frac{2\alpha}{\alpha-1}}(\Omega)$ and for all $\varphi \in V_\alpha$, we have

$$\left|b\left(y_k, y_k, \varphi\right) - b\left(y, y, \varphi\right)\right|$$

$$\leq |b(y_k - y, y_k, \varphi)| + |b(y, y_k - y, \varphi)| = |b(y_k - y, y_k, \varphi)| + |b(y, \varphi, y_k - y)|$$
$$\leq \left(\|\nabla y_k\|_\alpha \|\varphi\|_{\frac{2\alpha}{\alpha-1}} + \|y\|_{\frac{2\alpha}{\alpha-1}} \|\nabla\varphi\|_\alpha\right) \|y_k - y\|_{\frac{2\alpha}{\alpha-1}}$$
$$\longrightarrow 0 \qquad \text{when } k \to +\infty. \tag{4.1}$$

Taking into account these convergence results and passing to the limit in the weak formulation corresponding to y_k, we obtain

$$(\widetilde{\tau}, D\varphi) + b(y, y, \varphi) = (u, \varphi) \qquad \text{for all } \varphi \in V_\alpha. \tag{4.2}$$

In particular, by taking into account Lemma 2.6 we have

$$(\widetilde{\tau}, Dy) = (\widetilde{\tau}, Dy) + b(y, y, y) = (u, y). \tag{4.3}$$

On the other hand, the monotonicity assumption (2.3) implies

$$(\tau(Dy_k) - \tau(D\varphi), Dy_k - D\varphi) \geq 0 \qquad \text{for all } \varphi \in V_\alpha. \tag{4.4}$$

Since $(\tau(Dy_k), Dy_k) = (u_k, y_k)$, by substituing in (4.4), we obtain

$$(u_k, y_k) - (\tau(Dy_k), D\varphi) - (\tau(D\varphi), Dy_k - D\varphi) \geq 0$$

and by passing to the limit, we get

$$(u, y) - (\widetilde{\tau}, D\varphi) - (\tau(D\varphi), Dy - D\varphi) \geq 0 \qquad \text{for all } \varphi \in V_\alpha.$$

This inequality together with (4.3) then yields

$$(\widetilde{\tau} - \tau(D\varphi), Dy - D\varphi) \geq 0 \qquad \text{for all } \varphi \in V_\alpha$$

and by setting $\varphi = y - t\psi$ with $t > 0$, we obtain

$$(\widetilde{\tau} - \tau(Dy - tD\psi), D\psi) \geq 0 \qquad \text{for all } \psi \in V_\alpha.$$

Letting t tend to zero and using the continuity of τ, we deduce that

$$(\widetilde{\tau} - \tau(Dy), D\psi) \geq 0 \qquad \text{for all } \psi \in V_\alpha$$

and thus

$$(\widetilde{\tau}, D\psi) = (\tau(Dy), D\psi) \qquad \text{for all } \psi \in V_\alpha. \tag{4.5}$$

Combining (4.2) and (4.5), we deduce that

$$(\tau(Dy), D\varphi) + b(y, y, \varphi) = (u, \varphi) \qquad \text{for all } \varphi \in V_\alpha$$

showing that

$$y_k \longrightarrow y \qquad \text{weakly in } W_0^{1,\alpha}(\Omega)$$

and that (y, u) satisfies (1.1). From the convexity and continuity of J, it follows the lower semicontinuity of J in the weak topology and

$$J(y, u) \leq \liminf_k J(y_k, u_k) = \inf(P_\alpha),$$

showing that (y, u) is a solution for (P_α). ∎

5 Statement of the optimality conditions

In order to obtain the first order optimality conditions for (P_α) stated in Theorem 5.1 below, the analysis of the control-to-state mapping is carried out in Section 6 leading to the proof of the main result in Section 7.

Theorem 5.1 *Assume that* $\mathbf{A}_1$-$\mathbf{A}_2$ *are fulfilled with* $\alpha \geq 2$. *Let* $(\bar{u}, \bar{y})$ *be a solution of* (P_α) *with* $\bar{u}$ *satisfying the following condition*

$$\|\bar{u}\|_2 < \tfrac{\mu^2\sqrt{n^3}}{4(n-1)^2}. \tag{5.1}$$

There then exists a unique $\bar{p} \in H_\alpha^{\bar{y}}$ *such that*

$$\begin{cases} -\nabla \cdot (\tau(D\bar{y})) + (\bar{y}\cdot\nabla)\,\bar{y} + \nabla\bar{\pi} = \bar{u} & \text{in } \Omega, \\ \nabla\cdot\bar{y} = 0 & \text{in } \Omega, \\ \bar{y} = 0 & \text{on } \partial\Omega, \end{cases}$$

$$\begin{cases} -\nabla \cdot (\tau'(D\bar{y}) : D\bar{p}) - (\bar{y}\cdot\nabla)\,\bar{p} + (\nabla\bar{y})^T\bar{p} + \nabla\widetilde{\pi} = \bar{y} - y_d & \text{in } \Omega, \\ \nabla\cdot\bar{p} = 0 & \text{in } \Omega, \\ \bar{p} = 0 & \text{on } \partial\Omega, \end{cases}$$

$$(\bar{p} + \nu\bar{u}, v - \bar{u}) \geq 0 \qquad \text{for all } v \in U_{ad}.$$

It is obvious that these optimality conditions can be written using the weak formulations corresponding to the state and adjoint state systems which read as

$$(\tau\,(D\bar{y})\,, D\varphi) + ((\bar{y}\cdot\nabla)\,\bar{y}, \varphi) = (\bar{u}, \varphi) \qquad \text{for all } \varphi \in V_\alpha,$$

$$(\tau'\,(D\bar{y}) : D\bar{p}, D\varphi) + \Big((\nabla\bar{y})^T\,\bar{p} - (\bar{y}\cdot\nabla)\,\bar{p}, \varphi\Big) = (\bar{y} - y_d, \varphi) \quad \text{for all } \varphi \in H_\alpha^{\bar{y}}.$$

As a consequence, we have the following regularity result for the optimal control.

Corollary 5.2 *Assume that the assumption of Theorem* 5.1 *are fulfilled with* $\nu > 0$ *and that* $U_{ad} = \big\{v \in L^2(\Omega) \mid \|v\|_2 \leq 1\big\}$. *Then* $\bar{u} \in H_\alpha^{\bar{y}}$.

Proof. Due to the necessary condition for the optimal control, we have

$$\begin{aligned} \bar{u}(x) &= \mathrm{Proj}_{U_{ad}}\Big(-\tfrac{\bar{p}(x)}{\nu}\Big) \qquad \text{a.e. } x \in \Omega \\ &= \begin{cases} -\tfrac{\bar{p}(x)}{\nu} & \text{for a.e. } x \in \Omega \quad \text{if } \tfrac{\|\bar{p}\|_2}{\nu} \leq 1 \\ -\tfrac{\bar{p}(x)}{\|\bar{p}\|_2} & \text{for a.e. } x \in \Omega \quad \text{if } \tfrac{\|\bar{p}\|_2}{\nu} > 1 \end{cases} \end{aligned}$$

where $\mathrm{Proj}_{U_{ad}}$ denotes the projection on U_{ad}. The result follows from the regularity of $\bar{p}$. ∎

Let us finish this section by considering the case of the Navier-Stokes equations. For $\alpha = 2$, $V_\alpha^{\bar{y}} \equiv H_\alpha^{\bar{y}} \equiv V_2$. The first order optimality conditions we obtain in this case are less restrictive than the ones obtained in [4], [5], [17] where all the admissible controls are subject to a condition that ensures the uniqueness of the corresponding states. Condition (5.1) guarantees uniqueness of the optimal state and of the optimal adjoint state and implies that the set U_{ad} of admissible controls satisfies the property (C), introduced by Gunzburger *et al.* [8], at $(\bar{y}, \bar{u})$. Our result can then be seen as a qualified version of the optimality conditions already established by Abergel and Casas in [1] for a slightly different functional.

Corollary 5.3 *Assume that the extra-stress tensor has the form $\tau(\eta) = 2\mu\eta$.Let $(\bar{u}, \bar{y})$ be a solution of (P_α) with $\bar{u}$ satisfying* (5.1)*. There then exists a unique $\bar{p} \in V_2$ such that the following conditions hold*

$$\begin{cases} -\mu\,\Delta\bar{y} + (\bar{y}\cdot\nabla)\,\bar{y} + \nabla\bar{\pi} = \bar{u} & \text{in } \Omega, \\ \nabla\cdot\bar{y} = 0 & \text{in } \Omega, \\ \bar{y} = 0 & \text{on } \partial\Omega, \end{cases}$$

$$\begin{cases} -\mu\,\Delta\bar{p} - (\bar{y}\cdot\nabla)\,\bar{p} + (\nabla\bar{y})^T\bar{p} + \nabla\widetilde{\pi} = \bar{y} - y_d & \text{in } \Omega, \\ \nabla\cdot\bar{p} = 0 & \text{in } \Omega, \\ \bar{p} = 0 & \text{on } \partial\Omega, \end{cases}$$

$$(\bar{p} + \nu\bar{u}, v - \bar{u}) \geq 0 \qquad \text{for all } v \in U_{ad}.$$

6 Analysis of the control-to-state mapping

We first establish some useful estimates related with local Lipschitz continuity of the state with respect to the control.

Lemma 6.1 *Let u_1 and u_2 be in $L^2(\Omega)$ with u_2 satisfying* (3.5)*, and let y_{u_1} and y_{u_2} be corresponding solutions of* (1.1)*. Then the following estimates hold*

$$\|D\,(y_{u_1} - y_{u_2})\|_2 \leq \mathcal{L}\,(\|u_2\|_2)\,\|u_1 - u_2\|_2$$

$$\tfrac{1}{2^{2\alpha+1}}\,\|D\,(y_{u_1} - y_{u_2})\|_\alpha^\alpha \leq \left(\mathcal{L}\,(\|u_2\|_2)\,\|u_1 - u_2\|_2\right)^2,$$

where $\mathcal{L}$ is defined as in Proposition 3.6

Proof. Testing the weak formulation of (1.1) for u_1 and u_2 by $\varphi = y_{u_1} - y_{u_2}$, and using Lemma 2.6 we get

$$\begin{aligned} &(\tau(Dy_{u_1}) - \tau(Dy_{u_2}), D\,(y_{u_1} - y_{u_2})) \\ &= (u_1 - u_2, y_{u_1} - y_{u_2}) - b\,(y_{u_1}, y_{u_1}, y_{u_1} - y_{u_2}) + b\,(y_{u_2}, y_{u_2}, y_{u_1} - y_{u_2}) \\ &= (u_1 - u_2, y_{u_1} - y_{u_2}) - b\,(y_{u_1} - y_{u_2}, y_{u_2}, y_{u_1} - y_{u_2})\,. \end{aligned} \tag{6.1}$$

Once we have

$$|(u_1 - u_2, y_{u_1} - y_{u_2})|$$

$$\leq \|u_1 - u_2\|_2 \|y_{u_1} - y_{u_2}\|_2 \leq \kappa_2 \|u_1 - u_2\|_2 \|D(y_{u_1} - y_{u_2})\|_2$$

by combining (3.6), (3.7) and (6.1), we obtain

$$\mu \|D(y_{u_1} - y_{u_2})\|_2^2 \leq (\tau(Dy_{u_1}) - \tau(Dy_{u_2}), D(y_{u_1} - y_{u_2}))$$

$$\leq \kappa_2 \left(\kappa_1 \tfrac{\|u_2\|_2}{\mu} \|D(y_{u_1} - y_{u_2})\|_2 + \|u_1 - u_2\|_2\right) \|D(y_{u_1} - y_{u_2})\|_2 .$$

Therefore,

$$\left(\mu - \kappa_2 \kappa_1 \tfrac{\|u_2\|_2}{\mu}\right) \|D(y_{u_1} - y_{u_2})\|_2 \leq \kappa_2 \|u_1 - u_2\|_2$$

and this gives the first estimate. Consequently, by using $(2.3)_2$ we have

$$\tfrac{\mu}{2^{2\alpha+1}} \|D(y_{u_1} - y_{u_2})\|_\alpha^\alpha \leq (\tau(Dy_{u_1}) - \tau(Dy_{u_2}), D(y_{u_1} - y_{u_2}))$$

$$\leq \kappa_2 \left(\kappa_1 \tfrac{\|u_2\|_2}{\mu} \|D(y_{u_1} - y_{u_2})\|_2 + \|u_1 - u_2\|_2\right) \|D(y_{u_1} - y_{u_2})\|_2$$

$$\leq \kappa_2 \left(\kappa_1 \tfrac{\|u_2\|_2}{\mu} \mathcal{L}(\|u_2\|_2) + 1\right) \mathcal{L}(\|u_2\|_2) \|u_1 - u_2\|_2^2$$

$$= \mu (\mathcal{L}(\|u_2\|_2))^2 \|u_1 - u_2\|_2^2 \tag{6.2}$$

which gives the second estimate and completes the proof. ∎

Lemma 6.2 *Let u_1 and u_2 be in $L^2(\Omega)$ with u_2 satisfying* (3.5), *and let y_{u_1} and y_{u_2} be corresponding solutions of* (1.1). *Then the following estimate holds*

$$\left\| \left(1 + |Dy_{u_1}|^2 + |Dy_{u_2}|^2\right)^{\frac{\alpha-2}{4}} D(y_{u_1} - y_{u_2}) \right\|_2 \leq \kappa_3 \mathcal{L}(\|u_2\|_2) \|u_1 - u_2\|_2$$

with $\mathcal{L}$ defined in Proposition 3.6 *and* $\kappa_3 = 2^{\frac{3(\alpha-2)}{4}} \left(1 + 2^{2\alpha+1}\right)^{\frac{1}{2}}$.

Proof. Taking into account Lemma 2.2, we obtain

$$2^{-\frac{3(\alpha-2)}{2}} \left\| \left(1 + |Dy_{u_1}|^2 + |Dy_{u_2}|^2\right)^{\frac{\alpha-2}{4}} D(y_{u_1} - y_{u_2}) \right\|_2^2$$

$$\leq \tfrac{1}{\mu} (\tau(Dy_{u_1}) - \tau(Dy_{u_2}), D(y_{u_1} - y_{u_2})) + \|D(y_{u_1} - y_{u_2})\|_\alpha^\alpha .$$

The conclusion follows then from (6.2). ∎

Lemma 6.3 *Let u_1 and u_2 be in $L^2(\Omega)$ with u_2 satisfying* (3.5), *and let y_{u_1} and y_{u_2} be corresponding solutions of* (1.1). *Then the following estimates hold*

$$\|D(y_{u_1} - y_{u_2})\|_2 \leq \|y_{u_1} - y_{u_2}\|_{H_\alpha^{y_{u_i}}} \leq \kappa_3 \mathcal{L}(\|u_2\|_2) \|u_1 - u_2\|_2 \quad i = 1, 2.$$

Proof. It is a direct consequence of Lemma 6.2, the definition of $H_\alpha^{y_{u_i}}$ and the fact that $\alpha \geq 2$. ∎

As already mentioned in the introduction, the ideas dealing with the Gâteaux differentiability of the control-to-state mapping are mainly due to Casas and Fernández and were developed in [2] and [3] to study optimal control problems governed by quasi-linear elliptic equations. The uniform ellipticity of the quasi-linear term induces uniqueness of the weak solution and this subjacent property is important in the proof of the differentiability. By taking into account the corresponding estimates, we can use very similar arguments to derive optimality conditions for problems governed by generalized Stokes systems. The case of problems governed by Navier-Stokes equations and generalized Navier-Stokes equations is more delicate since a direct adaptation of these arguments, managing the convective term, may restrain all the admissible controls to satisfy condition (3.5) (see for example [4], [5], [17] and [18]). To overcome this difficulty, we observe that in the estimates stated in Lemma 6.2, Lemma 6.1 and Lemma 6.3 and dealing with the difference $y_{u_1} - y_{u_2}$, the restriction on the control guarenteeing the uniqueness of the corresponding state solution is only imposed on u_2. This fact is particularly important and enables us, when deriving the necessary optimality conditions, to restraint only the optimal control.

For u in $L^2(\Omega)$ satisfying (3.5), w in $L^2(\Omega)$ and ρ in $]0,1[$, set $u_\rho = u + \rho w$, let y_u be the unique solution of (1.1) corresponding to u, and y_{u_ρ} be a solution of (1.1) corresponding to u_ρ. In the remaining part of this section, and in order to simplify the notation, we set y_ρ instead of y_{u_ρ}, y instead of y_u and $z_\rho = \frac{y_\rho - y}{\rho}$. Substituing in the weak formulation of (1.1), we obtain

$$(\tau(Dy_\rho) - \tau(Dy), D\varphi) + b\left(y_\rho, y_\rho, \varphi\right) - b\left(y, y, \varphi\right) = \rho(w, \varphi) \tag{6.3}$$

for all $\varphi \in V_\alpha$.

Lemma 6.4 *Let u be in $L^2(\Omega)$ satisfying* (3.5) *and w be in $L^2(\Omega)$. Then, the following estimates hold*

$$\left\|Dz_\rho\right\|_2 \leq \mathcal{L}\left(\left\|u\right\|_2\right)\left\|w\right\|_2, \qquad \rho^{\alpha-2}\left\|Dz_\rho\right\|_\alpha^\alpha \leq 2^{2\alpha+1}\left(\mathcal{L}\left(\left\|u\right\|_2\right)\left\|w\right\|_2\right)^2,$$

$$\left\|z_\rho\right\|_{H_\alpha^{y_u}} \leq \kappa_3\,\mathcal{L}\left(\left\|u\right\|_2\right)\left\|w\right\|_2,$$

where $\mathcal{L}$ is defined in Proposition 3.6.

Proof. It is a direct consequence of Lemma 6.1 and Lemma 6.3. ∎

Lemma 6.5 *If $\left(z_{\rho_k}\right)_k$ weakly converges to z in V_2 for some sequence $(\rho_k)_k$ converging to zero, then*

$$\lim_{k\to+\infty} \frac{1}{\rho_k}\left(b\left(y_{\rho_k}, y_{\rho_k}, \varphi\right) - b\left(y, y, \varphi\right)\right) = b\left(z, y, \varphi\right) + b\left(y, z, \varphi\right)$$

for all $\varphi \in V_\alpha$.

Proof. Notice that

$$\begin{aligned}
&\left|\tfrac{1}{\rho_k}\left(b\left(y_{\rho_k}, y_{\rho_k}, \varphi\right)-b\left(y, y, \varphi\right)\right)-\left(b\left(z, y, \varphi\right)+b\left(y, z, \varphi\right)\right)\right| \\
&=\left|b\left(z_{\rho_k}, y_{\rho_k}, \varphi\right)+b\left(y, z_{\rho_k}, \varphi\right)-\left(b\left(z, y, \varphi\right)+b\left(y, z, \varphi\right)\right)\right| \\
&\leq\left|b\left(z_{\rho_k}, y_{\rho_k}, \varphi\right)-b\left(z, y, \varphi\right)\right|+\left|b\left(y, z_{\rho_k}, \varphi\right)-b\left(y, z, \varphi\right)\right| \\
&\leq\left|b\left(z_{\rho_k}-z, y_{\rho_k}, \varphi\right)\right|+\left|b\left(z, y_{\rho_k}-y, \varphi\right)\right|+\left|b\left(y, z_{\rho_k}-z, \varphi\right)\right| \\
&\leq\left\|z_{\rho_k}-z\right\|_4\left\|\nabla y_{\rho_k}\right\|_2\|\varphi\|_4+\|z\|_4\left\|\nabla\left(y_{\rho_k}-y\right)\right\|_2\|\varphi\|_4+\left|b\left(y, z_{\rho_k}-z, \varphi\right)\right|
\end{aligned}$$

The result is then a consequence of the strong convergence of $\left(y_{\rho_k}\right)_k$ to y_u in $W_0^{1,\alpha}(\Omega)$, the weak convergence of $\left(z_{\rho_k}\right)_k$ to z in $H_0^1(\Omega)$ and its the strong convergence in $L^4(\Omega)$. ∎

Lemma 6.6 *If $\left(z_{\rho_k}\right)_k$ weakly converges to z in H_α^y for some sequence $(\rho_k)_k$ converging to zero, then*

$$\lim_{k\to+\infty}\tfrac{1}{\rho_k}\left(\tau\left(Dy_{\rho_k}\right)-\tau\left(Dy\right), D\varphi\right)=\left(\tau'(Dy):Dz, D\varphi\right)$$

for all $\varphi\in\mathcal{V}$.

Proof. Let $\varphi\in\mathcal{V}$ be fixed. Arguing as in the proof of Lemma 2.2, we obtain

$$\begin{aligned}
&\tfrac{1}{\rho_k}\left(\tau(Dy_{\rho_k})-\tau(Dy), D\varphi\right) \\
&=\int_\Omega\int_0^1\tau'\left(Dy(x)+sD\left(y_{\rho_k}-y\right)(x)\right):Dz_{\rho_k}(x):D\varphi(x)\,ds\,dx \\
&=\int_\Omega\tau'\left(\sigma_{\rho_k}^{\varphi}(x)\right):Dz_{\rho_k}(x):D\varphi(x)\,dx=\left(\tau'\left(\sigma_{\rho_k}^{\varphi}\right):Dz_{\rho_k}, D\varphi\right),
\end{aligned}\tag{6.4}$$

where $\sigma_{\rho_k}^{\varphi}(x)=Dy(x)+s_{\rho_k}^{\varphi}(x)D\left(y_{\rho_k}-y\right)(x)$ with $0<s_{\rho_k}^{\varphi}(x)<1$ being a number (depending on $\varphi(x)$) arising when applying the mean values theorem to the integral in the interval $[0,1]$. Since $\alpha\geq 2$, we have

$$\left(1+|\sigma_{\rho_k}^{\varphi}(x)|^2\right)^{\frac{\alpha-2}{2}}\leq 2^{\alpha-2}\left(\left(1+|Dy(x)|^2\right)^{\frac{\alpha-2}{2}}+\left|D\left(y_{\rho_k}-y\right)(x)\right|^{\alpha-2}\right)$$

and by taking into account Lemma 6.4, we deduce that

$$\begin{aligned}
&2^{2-\alpha}\left\|\left(1+|\sigma_{\rho_k}^{\varphi}|^2\right)^{\frac{\alpha-2}{4}}Dz_{\rho_k}\right\|_2^2 \\
&\leq\int_\Omega\left(1+|Dy(x)|^2\right)^{\frac{\alpha-2}{2}}\left|Dz_{\rho_k}(x)\right|^2dx+\int_\Omega\left|D\left(y_{\rho_k}-y\right)(x)\right|^{\alpha-2}\left|Dz_{\rho_k}(x)\right|^2dx \\
&=\left(\left\|z_{\rho_k}\right\|_{H_\alpha^y}^2+\rho^{\alpha-2}\left\|Dz_{\rho_k}\right\|_\alpha^\alpha\right)\leq\left(2^{2\alpha+1}+\kappa_3^2\right)\left(\mathcal{L}\left(\|u\|_2\right)\|w\|_2\right)^2.
\end{aligned}$$

Moreover, due to Lemma 6.4, we deduce the convergence of $(\sigma^{\varphi}_{\rho_k})_k$ to Dy in $L^{\alpha}(\Omega)$ and thus, for every $\psi \in \mathcal{V}$ we have

$$\lim_{k\to+\infty} \left\| \psi(1+|\sigma^{\varphi}_{\rho_k}|^2)^{\frac{\alpha-2}{4}} - \psi(1+|Dy|^2)^{\frac{\alpha-2}{4}} \right\|_2 = 0.$$

This result together with the convergence of $(Dz_{\rho_k})_{\rho_k}$ to Dz in the weak topology of $L^2(\Omega)$ imply that

$$\lim_{k\to+\infty} \left((1+|\sigma^{\varphi}_{\rho_k}|^2)^{\frac{\alpha-2}{4}} Dz_{\rho_k}, \psi\right) = \left((1+|Dy|^2)^{\frac{\alpha-2}{4}} Dz, \psi\right) \tag{6.5}$$

and we conclude that the sequence $\left(\left(1+|\sigma^{\varphi}_{\rho_k}|^2\right)^{\frac{\alpha-2}{4}} Dz_{\rho_k}\right)_k$ weakly converges to $\left(1+|Dy|^2\right)^{\frac{\alpha-2}{4}} Dz$ in $L^2(\Omega)$. On the other hand, let us consider the operator

$$\begin{array}{rcl} \mathcal{A} : L^{\alpha}(\Omega) & \longrightarrow & L^2(\Omega) \\ g & \longmapsto & \mathcal{A}(g) = \frac{D\varphi : \tau'(g)}{(1+|g|^2)^{\frac{\alpha-2}{4}}}. \end{array}$$

Using $\mathbf{A}_1$, we can easily verify that

$$|\mathcal{A}(g)| = \frac{|D\varphi : \tau'(g)|}{(1+|g|^2)^{\frac{\alpha-2}{4}}} \le \gamma n^2 \frac{|D\varphi|(1+|g|^2)^{\frac{\alpha-2}{2}}}{(1+|g|^2)^{\frac{\alpha-2}{4}}} = \gamma n^2 |D\varphi|(1+|g|^2)^{\frac{\alpha-2}{4}}$$

which shows that the operator $\mathcal{A}$ is continuous. Therefore, since $\left(\sigma^{\varphi}_{\rho_k}\right)_k$ converges to Dy in $L^{\alpha}(\Omega)$, we deduce that

$$\lim_{k\to+\infty} \left\|\mathcal{A}\left(\sigma^{\varphi}_{\rho_k}\right) - \mathcal{A}\left(Dy\right)\right\|_2 = 0. \tag{6.6}$$

Finally, by combining (6.5) and (6.6), we obtain

$$\lim_{k\to+\infty} \left(\tau'\left(\sigma^{\varphi}_{\rho_k}\right) : Dz_{\rho_k}, D\varphi\right) = \lim_{k\to+\infty} \left(D\varphi : \tau'\left(\sigma^{\varphi}_{\rho_k}\right), Dz_{\rho_k}\right)$$

$$= \lim_{k\to+\infty} \left(\mathcal{A}(\sigma^{\varphi}_{\rho_k}), (1+|\sigma^{\varphi}_{\rho_k}|^2)^{\frac{\alpha-2}{4}} Dz_{\rho_k}\right) = \left(\mathcal{A}(Dy), (1+|Dy|^2)^{\frac{\alpha-2}{4}} Dz\right)$$

$$= \left(D\varphi : \tau'\left(Dy\right), Dz\right) = \left(\tau'\left(Dy\right) : Dz, D\varphi\right)$$

which completes the proof. ∎

Proposition 6.7 *If $(z_{\rho_k})_k$ weakly converges to z in H^y_{α} for some sequence $(\rho_k)_k$ converging to zero, then z is the unique weak solution of* (3.8) *corresponding to w. Moreover, $(z_{\rho_k})_k$ converges strongly to z in V_2.*

Proof. The first assertion is a direct consequence of Lemma 6.5, Lemma 6.6 and of the density of $\mathcal{V}$ in H^y_{α}. To prove the strong convergence, let us set

$$M = \tau'(Dy(x)), \qquad M_{\rho}(x) = \tau'(\sigma_{\rho}(x)),$$

where σ_ρ is as in (6.4) with φ substituted by $y_\rho - y$. Due to $\mathbf{A}_2$, the matrices

$$M^s(x) = \tfrac{M(x)+M^T(x)}{2}, \qquad M^s_\rho(x) = \tfrac{M_\rho(x)+M^T_\rho(x)}{2}$$

are symmetric and positive definite. Applying the Cholesky method to $M^s(x)$ and $M^s_\rho(x)$, we deduce the existence of lower triangular matrices $L(x)$ and $L_\rho(x)$ such that

$$M^s(x) = L(x)L^T(x) \qquad \text{and} \qquad M^s_\rho(x) = L_\rho(x)L^T_\rho(x).$$

Therefore, due to (6.3) and (6.4), we have

$$\begin{aligned}
\left\|L^T_{\rho_k} Dz_{\rho_k}\right\|_2^2 &= (M_{\rho_k} : Dz_{\rho_k}, Dz_{\rho_k}) \\
&= -\tfrac{1}{\rho_k}\left(b\left(y_{\rho_k}, y_{\rho_k}, z_{\rho_k}\right) - b\left(y, y, z_{\rho_k}\right)\right) + \left(w, z_{\rho_k}\right) \\
&= -b\left(z_{\rho_k}, y_{\rho_k}, z_{\rho_k}\right) - b\left(y, z_{\rho_k}, z_{\rho_k}\right) + \left(w, z_{\rho_k}\right) \\
&= -b\left(z_{\rho_k}, y_{\rho_k}, z_{\rho_k}\right) + \left(w, z_{\rho_k}\right) \le \left\|\nabla y_{\rho_k}\right\|_2 \left\|z_{\rho_k}\right\|_4^2 + \|w\|_2 \left\|z_{\rho_k}\right\|_2 \\
&\le \kappa_1 \left\|Dy_{\rho_k}\right\|_2 \left\|Dz_{\rho_k}\right\|_2^2 + \kappa_2 \|w\|_2 \left\|Dz_{\rho_k}\right\|_2 \\
&\le \kappa_2\kappa_1 \tfrac{\left\|u_{\rho_k}\right\|_2}{\mu} \left\|Dz_{\rho_k}\right\|_2^2 + \kappa_2 \|w\|_2 \left\|Dz_{\rho_k}\right\|_2 \\
&\le \left(\kappa_2\kappa_1 \tfrac{\|u\|_2+\|w\|_2}{\mu} \mathcal{L}\left(\|u\|_2\right) + \kappa_2\right) \mathcal{L}\left(\|u\|_2\right) \|w\|_2^2
\end{aligned} \tag{6.7}$$

and the sequence $(L^T_{\rho_k} Dz_{\rho_k})_k$ is bounded in $L^2(\Omega)$. On the other hand, due to $\mathbf{A}_1$ we have

$$\left|L_{\rho_k}(x)\right|^2 = \left|M_{\rho_k}(x)\right| \le \gamma n^2 \left(1 + |Dy(x)|^2 + \left|Dy_{\rho_k}(x)\right|^2\right)^{\frac{\alpha-2}{2}} \in L^{\frac{\alpha}{\alpha-2}}(\Omega)$$

for all $x \in \Omega$. Taking into account the convergence of $(Dy_\rho)_k$ to Dy into $L^2(\Omega)$, we deduce that there exists $h_1 \in L^{\frac{2\alpha}{\alpha-2}}(\Omega) \subset L^2(\Omega)$ such that

$$\left|L_{\rho_k}(x)\right| \le h_1(x) \qquad \text{for a.e. } x \in \Omega \text{ and } k > k_o,$$

$$L_{\rho_k}(x) \longrightarrow L(x) \qquad \text{for a.e. } x \in \Omega.$$

The dominated convergence theorem then implies

$$L_{\rho_k} \longrightarrow L \qquad \text{strongly in } L^2(\Omega) \tag{6.8}$$

which together with the weak convergence of $(z_{\rho_k})_k$ to z in V_2 gives

$$L^T_{\rho_k} Dz_{\rho_k} \longrightarrow L^T Dz \qquad \text{weakly in } L^2(\Omega).$$

Moreover, taking into account (6.7), we deduce that

$$\left\|L^T Dz\right\|_2^2 \le \liminf_k \left\|L^T_{\rho_k} Dz_{\rho_k}\right\|_2^2$$

$$\leq \limsup_k \left\| L_{\rho_k}^T Dz_{\rho_k} \right\|_2^2 = \limsup_k \left(M_{\rho_k} : Dz_{\rho_k}, Dz_{\rho_k} \right)$$
$$= \limsup_k \left(-\tfrac{1}{\rho_k} \left(b\left(y_{\rho_k}, y_{\rho_k}, z_{\rho_k}\right) - b\left(y, y, z_{\rho_k}\right)\right) + \left(w, z_{\rho_k}\right)\right)$$
$$= \limsup_k \left(-b\left(z_{\rho_k}, y_{\rho_k}, z_{\rho_k}\right) - b\left(y, z_{\rho_k}, z_{\rho_k}\right) + \left(w, z_{\rho_k}\right)\right)$$
$$= \limsup_k \left(-b\left(z_{\rho_k}, y_{\rho_k}, z_{\rho_k}\right) + \left(w, z_{\rho_k}\right)\right)$$
$$= -b\left(z, y, z\right) + \left(w, z\right) = -b\left(z, y, z\right) + b\left(y, z, z\right) + \left(w, z\right)$$
$$= \left(M : Dz, Dz\right) = \left\| L^T Dz \right\|_2^2 .$$

Weak convergence together with norm convergence implies strong convergence of $(L_{\rho_k}^T Dz_{\rho_k})_k$ to $L^T Dz$ in $L^2(\Omega)$. There then exist a subsequence, still indexed by ρ_k, and a function $h_2 \in L^2(\Omega)$ such that

$$\left| L_{\rho_k}^T Dz_{\rho_k}(x) \right| \leq h_2(x) \qquad \text{for a.e. } x \in \Omega \text{ and } k > k_1,$$
$$L_{\rho_k}^T(x) Dz_{\rho_k}(x) \longrightarrow L^T(x) Dz(x) \qquad \text{for a.e. } x \in \Omega.$$

Therefore,

$$\left| Dz_{\rho_k}(x) \right|^2 \leq \left(1 + |\sigma_{\rho_k}(x)|^2\right)^{\frac{\alpha-2}{2}} \left| Dz_{\rho_k}(x) \right|^2$$
$$\leq \tfrac{1}{\mu} M_{\rho_k}(x) : Dz_{\rho_k}(x) : Dz_{\rho_k}(x) = \tfrac{1}{\mu} Dz_{\rho_k}^T(x) : M_{\rho_k}(x) : Dz_{\rho_k}(x)$$
$$= \tfrac{1}{\mu} \left| L_{\rho_k}^T(x) Dz_{\rho_k}(x) \right|^2 \leq \tfrac{1}{\mu} (h_2(x))^2 \qquad \text{for a.e. } x \in \Omega \text{ and } k > k_1.$$

Since (6.8) implies

$$\left(L_{\rho_k}^T(x) \right)^{-1} \longrightarrow \left(L^T(x) \right)^{-1} \qquad \text{for a.e. } x \in \Omega,$$

we deduce that

$$Dz_{\rho_k}(x) = \left(L_{\rho_k}^T(x) \right)^{-1} L_{\rho_k}^T(x) Dz_{\rho_k} \longrightarrow \left(L^T(x) \right)^{-1} L^T(x) Dz(x) = Dz(x)$$

for a.e. $x \in \Omega$. The conclusion follows by applying the dominated convergence theorem. ∎

7 Proof of the main result

Let us now prove Theorem 5.1. For $\rho \in]0, 1[$ and $v \in U_{ad}$, let $u_\rho = \bar{u} + \rho(v - \bar{u})$, $y_\rho \equiv y_{u_\rho}$ and $z_\rho = \frac{y_\rho - \bar{y}}{\rho}$. Due to Lemma 6.4, we deduce that $(z_\rho)_\rho$ is bounded in $H_\alpha^{\bar{y}}$. There then exist a subsequence $(z_{\rho_k})_k$ and $z \in H_\alpha^{\bar{y}}$ such that $(z_{\rho_k})_k$ weakly converges to z in V_2. Due to Proposition 6.7, $(z_{\rho_k})_k$ strongly converges to z in V_2 and $z \equiv z_{\bar{u}v} - z_{\bar{u}\bar{u}}$, where $z_{\bar{u}w}$ is the solution of (3.8) corresponding to $(\bar{y}, w)$, and thus

$$\lim_k \frac{J(y_{\rho_k}, u_{\rho_k}) - J(\bar{y}, \bar{u})}{\rho_k}$$

$$= \lim_k \Big((z_{\rho_k}, \bar{y} - y_d) + \tfrac{\rho_k}{2} \|z_{\rho_k}\|_2^2 + \nu\,(\bar{u}, v - \bar{u}) + \tfrac{\nu\rho_k}{2} \|v - \bar{u}\|_2^2 \Big)$$

$$= (\bar{y} - y_d, z_{\bar{u}v} - z_{\bar{u}\bar{u}}) + \nu\,(\bar{u}, v - \bar{u})\,. \tag{7.1}$$

On the other hand, since $(\bar{y}, \bar{u})$ is an optimal solution and (y_{ρ_k}, u_{ρ_k}) is admissible, we have

$$\lim_k \tfrac{J(y_{\rho_k}, u_{\rho_k}) - J(\bar{y}, \bar{u})}{\rho_k} \geq 0. \tag{7.2}$$

Let $\bar{p} \in H_\alpha^{\bar{y}}$ be the unique solution of

$$\begin{cases} -\nabla \cdot \big(\tau'(D\bar{y})^T : D\bar{p}\big) + (\nabla\bar{y})^T \bar{p} - (\bar{y} \cdot \nabla)\,\bar{p} + \nabla\tilde{\pi} = \bar{y} - y_d & \text{in } \Omega, \\ \nabla \cdot \bar{p} = 0 & \text{in } \Omega, \\ \bar{p} = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\tilde{\pi}$ the adjoint pressure. Setting $\phi = z_{\bar{u}v} - z_{\bar{u}\bar{u}}$ in the weak formulation corresponding to this problem and taking into account the weak formulation of problem (3.8) and (3.11), we obtain

$$(\bar{y} - y_d, z_{\bar{u}v} - z_{\bar{u}\bar{u}})$$

$$= \big(\tau'(D\bar{y})^T : D\bar{p}, D\,(z_{\bar{u}v} - z_{\bar{u}\bar{u}})\big) + \big((\nabla\bar{y})^T\bar{p} - (\bar{y} \cdot \nabla)\,\bar{p}, z_{\bar{u}v} - z_{\bar{u}\bar{u}}\big)$$

$$= (\tau'\,(D\bar{y}) : D\,(z_{\bar{u}v} - z_{\bar{u}\bar{u}})\,, D\bar{p}) + b\,(z_{\bar{u}v} - z_{\bar{u}\bar{u}}, \bar{y}, \bar{p}) + b\,(\bar{y}, z_{\bar{u}v} - z_{\bar{u}\bar{u}}, \bar{p})$$

$$= (v - \bar{u}, \bar{p})\,. \tag{7.3}$$

The result follows by combining (7.1), (7.2) and (7.3). ∎

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