On the reconstruction of heat and acoustic sources from boundary measurements

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Abstract

We address the identification and reconstruction of source functions in inverse boundary value problems for the Helmholtz and Laplace problems. We introduce particular affine/linear classes of sources and address the identification and reconstruction is such classes. In order to solve these inverse problems we propose a new numerical method that relies on solving several Helmholtz/Laplace boundary value problems. We establish a connection between these particular classes and full identification results using many boundary measurements. The developed methods are theoretically justified and illustrated with several numerical examples.

Keywords: Laplace and Helmholtz boundary value problems, inverse source problems, fundamental solutions, plane waves.

1. Introduction

Inverse source problems for boundary value problems arises quite often in non destructive engineering problems (eg. [18]). In this paper, we address the inverse source problem of, given a pair of Cauchy boundary data $(g, g_{\mathbf{n}})$, find the source function $f : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$, such that, for some u, we have

$$\begin{cases} (\Delta + \kappa^2)u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \\ \partial_{\mathbf{n}} u = g_{\mathbf{n}} & \text{on } \partial\Omega \end{cases}, \tag{1}$$

where the frequency κ belongs to $\mathbb{R}_0^+$ (and thus, we address both Laplace problems and Helmholtz problems). This inverse source problem is known to be ill posed, in the sense of Hadamard. In particular, it is well known (eg. [5]) that the source term f cannot be fully identified from a pair of boundary data. Usually, one has to consider some sort of extra source information. For instance, we may know *a priori* that the sources we are looking for are characteristic functions on some domain $\omega \subset\subset \Omega$ (eg. [12], [6],[16]), or point sources (eg. [9], [13]) or sources in linear/affine classes (eg. [4]). We refer the papers

[10] and [11] where several classes of heat and acoustic sources that can be retrieved from one boundary measurements are presented.

In this paper, we focus on the numerical reconstruction of sources in some linear/affine classes. We show that these classes play an important role on full source identification. In particular, any good reconstruction method for this linear/affine particular setting should provide a good numerical method for full source reconstruction.

The reconstruction of sources in the above class have been addressed by several authors (cf. [14] and [4]). The available methods rely on solving a Bilaplace (or the corresponding eigenvalue problem) fourth order boundary value (direct) problem. Once the solution is computed, say $\widetilde{u}$, the source is recovered as $\Delta\widetilde{u}$ (or $(\Delta+\kappa^2)\widetilde{u}$). Since, for some situations (eg. [14]), the arising fourth order problem is homogeneous and the differential operators are linear, elliptic (with constant coefficients), meshfree methods such as the method of fundamental solutions provide good approximation results with small computational effort. This approach was implemented and tested in, for instance, [15]. The more general case that corresponds to a fourth order non homogeneous problem was addressed in [4] and [3]. However, such passage to a higher order direct problem may be very affected by the ill posedness of the inverse problem. Moreover, the aring 4th order problem may be difficult to solve numerically.

We propose a reconstruction method that does not require a higher order direct problem. Instead, the method relies on several non homogeneous direct problems for the same operator. Since the problem is linear, we approximate the source by fitting the response (Neumann data) obtained from properly chosen basis functions to the available source response.

The paper is organized as follows: We start by describing the addressed inverse problem and the corresponding direct problem. In section three, the affine classes of sources are introduced and some related identification results (from one and several boundary measurements) are presented. Section four is devoted to the numerical methods. Here, we provide some theoretical results and discussion concerning the proposed numerical method. We finish with some numerical simulations and concluding remarks.

2. Direct and inverse problems formulation

Let $\Omega \subset \mathbb{R}^2$ be an open, bounded and simply connected C^2 domain, which we shall call a *regular domain*. Denote the boundary of Ω by Γ, ie., $\Gamma := \partial\Omega$.

Direct problem.

The direct problem consists in, given the source f, compute the normal derivative (Neumann data)

$$g_{\mathbf{n}}^{\kappa} := \partial_{\mathbf{n}} u_{\kappa}|_{\Gamma}$$

where u_κ solves

$$\begin{cases} (\Delta + \kappa^2)u_\kappa = f & \text{in } \Omega \\ u_\kappa = 0 & \text{on } \Gamma \end{cases}, \tag{2}$$

with $\kappa \geq 0$. The usual functional framework for the above boundary value problem (BVP) is $f \in L^2(\Omega)$. In this case, if κ is not an eigenfrequency, (2) is well posed, with $u \in H^2(\Omega)$ (eg. [17]). We recall that $\kappa > 0$ is not an eigenfrequency for the Helmholtz-Dirichlet problem in Ω if the problem

$$\begin{cases} (\Delta + \kappa^2)u_\kappa = 0 & \text{in } \Omega \\ u_\kappa = 0 & \text{on } \Gamma \end{cases}$$

has an unique solution. In the following we shall always consider this non resonance condition.

Inverse problem.

The inverse source problem consists in, given a pair of Cauchy boundary data $(0, g_{\mathbf{n}})$ for some frequency κ, determine $f \in L^2(\Omega)$ such that, for some u, it holds

$$\begin{cases} (\Delta + \kappa^2)u_\kappa = f & \text{in } \Omega \\ u_\kappa = 0 & \text{on } \Gamma \\ \partial_{\mathbf{n}} u_\kappa = g_{\mathbf{n}} & \text{on } \Gamma \end{cases}.$$

Of great importance, is the source to Neumann map, $\Lambda_\kappa : L^2(\Omega) \longrightarrow H^{-1/2}(\Gamma)$ defined by

$$\Lambda_\kappa(f) = \partial_{\mathbf{n}} u_\kappa|_\Gamma,$$

where $u_\kappa \in H^2(\Omega)$ solves (2). This map allows to formulate the inverse source problem as the following equation:

Given $g_{\mathbf{n}}$ determine $f \in L^2(\Omega)$ such that

$$\Lambda_\kappa(f) = g_{\mathbf{n}}.$$

The map Λ_κ is linear and continuous. Is not invertible because (eg. [5])

$$\ker \Lambda_\kappa = \left(\Delta + k^2\right)(H_0^2(\Omega)),$$

where $H_0^2(\Omega)$ is the space of functions $u \in H^2(\Omega)$ such that $u|_\Gamma = \partial_{\mathbf{n}} u|_\Gamma = 0$. The range of Λ_κ is characterized in the following result.

Lemma 1. *We have*

$$\Lambda_\kappa(L^2(\Omega)) = H^{1/2}(\Gamma)$$

hence the space of compatible Neumann data is $H^{1/2}(\Gamma)$*.*

Proof. Notice that in the considered functional framework, the normal trace $\partial_{\mathbf{n}} u_\kappa$ belongs to $H^{1/2}(\Gamma)$. Reciprocally, for $g_{\mathbf{n}} \in H^{1/2}(\Gamma)$, exists $u_\kappa \in H^2(\Omega)$ satisfying the 4th order boundary value problem (eg. [8])

$$\begin{cases} (\Delta^2 - \kappa^4)u_\kappa = 0 & \text{in } \Omega \\ u_\kappa = 0 & \text{on } \Gamma \\ \partial_{\mathbf{n}} u_\kappa = g_{\mathbf{n}} & \text{on } \Gamma \end{cases} .$$

Taking

$$f := (\Delta + \kappa^2)u_\kappa \in L^2(\Omega)$$

then

$$g_{\mathbf{n}} = \Lambda_\kappa(f) \in \mathrm{Range}\Lambda_\kappa.$$

□

3. Linear/affine classes of sources

Basic Hilbert spaces theory yields the orthogonal decomposition

$$L^2(\Omega) = \ker \Lambda_\kappa{}^\perp \oplus \ker \Lambda_\kappa.$$

Given $F \in L^2(\Omega)$ we define the *linear/affine* class $\mathcal{C}_F^\kappa$ as the level set

$$\mathcal{C}_F^\kappa := \pi_\kappa^{-1}(F) \tag{3}$$

where π_κ is the orthogonal projection of $L^2(\Omega)$ onto $\ker \Lambda_\kappa$. In other words,

$$\mathcal{C}_F^\kappa = \left\{ f = f_H + F \in L^2(\Omega) : \pi_\kappa(f) = F = (\Delta + \kappa^2)u, \;\; u \in H_0^2(\Omega) \right\}.$$

In order to analyse identification properties in these sets, we start by determine the orthogonal complement of $\ker \Lambda_\kappa$. Let

$$\mathcal{H}_\kappa := \left\{ u \in H^1(\Omega) : (\Delta + \kappa^2)u = 0 \right\}.$$

Lemma 2. *We have*

$$\mathcal{H}_\kappa^\perp = \ker \Lambda_\kappa$$

hence

$$L^2(\Omega) = \overline{\mathcal{H}_\kappa}^{L^2} \oplus \ker \Lambda_\kappa. \tag{4}$$

Proof. Let $v \in \mathcal{H}_\kappa^\perp$. Then,

$$\int_\Omega v\overline{w}dx = 0, \qquad \forall w \in \mathcal{H}_\kappa.$$

Let u be the unique element in $H^2(\Omega) \cap H_0^1(\Omega)$ such that $(\Delta + \kappa^2)u = v$. Notice that $\partial_{\mathbf{n}} u|_\Gamma \in H^{1/2}(\Gamma)$ and, in particular we can take the test function $w \in \mathcal{H}_\kappa$ such that $w|_\Gamma = \partial_{\mathbf{n}} u|_\Gamma$. Thus, substituting and integrating by parts yields

$$0 = \int_\Omega v\overline{w}dx = \int_\Omega (\Delta + \kappa^2)u\overline{w}dx = \int_\Gamma \partial_{\mathbf{n}} u \overline{\partial_{\mathbf{n}} u} d\sigma = ||\partial_{\mathbf{n}} u||^2_{L^2(\Gamma)}$$

hence $v \in (\Delta + \kappa^2)(H_0^2(\Omega)) = \ker \Lambda_\kappa$ and the inclusion $\mathcal{H}_\kappa^\perp \subset \ker \Lambda_\kappa$ follows.

To see the other inclusion, take $v = (\Delta + \kappa^2)u \in \ker \Lambda_\kappa$. Given $w \in \mathcal{H}_\kappa$ we have

$$\int_\Omega v\overline{w}dx = \int_\Omega (\Delta + \kappa^2)u\overline{w}dx = \int_\Omega u(\Delta + \kappa^2)\overline{w}dx = 0$$

and the result follows. □

It follows from the above Lemmas, that $\overline{\mathcal{H}_\kappa}^{L^2}$ can be identified with $H^{1/2}(\Gamma)$. We now show that, in some cases, these spaces are in fact homeomorphic.

Consider the *Bihelmholtz* boundary value problem

$$\begin{cases} (\Delta + \kappa)^2 u = F & \text{in } \Omega \\ u = g & \text{on } \Gamma \\ \partial_{\mathbf{n}} u = g_{\mathbf{n}} & \text{on } \Gamma \end{cases} . \tag{5}$$

Definition 3. *We say that $\kappa \geq 0$ is an admissible frequency for the Bihelmholtz BVP if, given a triplet*

$$(F, g, g_{\mathbf{n}}) \in L^2(\Omega) \times H^{3/2}(\Gamma) \times H^{1/2}(\Gamma),$$

the above Bihelmholtz problem (5) is well posed, with $u \in H^2(\Omega)$.

Notice that $\kappa = 0$ is an admissible frequency. In this case the above problem reduces to a BVP for the bilaplacian for which, the above well posedness condition holds true (eg. [8]).

Theorem 4. *If κ is an admissible frequency then $\overline{\mathcal{H}_\kappa}^{L^2}$ is homeomorphic to $H^{1/2}(\Gamma)$.*

Proof. Let $\eta_\kappa : \overline{\mathcal{H}_\kappa}^{L^2} \to H^{1/2}(\Gamma)$, be the linear map defined by

$$\eta_\kappa \circ \Pi_\kappa = \Lambda_\kappa \tag{6}$$

where Π_κ is the projection of $L^2(\Omega)$ onto $\overline{\mathcal{H}_\kappa}^{L^2}$. The map η_κ is clearly a continuous isomorphism. It remains to see that η_κ^{-1} is also continuous.

Notice that η_κ^{-1} is defined by $\eta_\kappa^{-1}(g_{\mathbf{n}}) = \Pi_\kappa(f)$, where $f \in L^2(\Omega)$ is such that $\Lambda_\kappa(f) = g_{\mathbf{n}}$. Hence, given a sequence $(g_{\mathbf{n}})_k = \Lambda_\kappa(f_k)$ converging to 0 in $H^{1/2}(\Gamma)$ we have, by well posedness of (5), the convergence of a sequence v_k to 0, in $H^2(\Omega)$. Each function v_k is the unique solution of (5), for input data $(0, 0, \Lambda_\kappa(f_k))$. In particular,

$$(\Delta + \kappa^2)v_k \to 0 \text{ in } L^2(\Omega)$$

and the result follows from the fact that $(\Delta + \kappa^2)v_k = \Pi_\kappa(f_k)$.

□

3.1. Identification results from one single measurement

In this section, we shall assume that κ is arbitrarily fixed. We start with the following identification result for sources in $\mathcal{C}_F^\kappa$.

Theorem 5. *Let $f_i \in L^2(\Omega)$ be two source functions in $\mathcal{C}_F^\kappa$ such that $\Lambda_\kappa(f_1) = \Lambda_\kappa(f_2)$. Then*

$$f_1 = f_2$$

in the L^2 sense.

Proof. It is sufficient to show that $\ker \Lambda_\kappa \cap \mathcal{C}_0^\kappa = \{0\}$. But this is an immediate consequence of the identity

$$\mathcal{C}_0^\kappa = \ker \Lambda_\kappa{}^\perp .$$

□

In some particular cases, it is possible to identify sources from the knowledge of the corresponding level set $\mathcal{C}_F^\kappa$.

Theorem 6. *The set $\mathcal{C}_F^\kappa \cap H_0^1(\Omega)$ has at most one element. In particular, $H^1(\Omega)$ sources with support strictly contained in Ω can be identified from F.*

Proof. Since $\mathcal{C}_0^\kappa = \ker \Lambda_\kappa{}^\perp = \overline{\mathcal{H}_\kappa}$ then

$$\mathcal{C}_0^\kappa \cap H_0^1(\Omega) = \overline{\mathcal{H}_\kappa} \cap H_0^1(\Omega) = \{0\} .$$

□

Other affine classes can also be considered. For instance, in [14] and [5] the considered classes were

$$\mathcal{D}_F = \left\{ f \in L^2(\Omega) : (\Delta - \kappa^2) f = F \right\} . \tag{7}$$

As proved in [5], the decomposition (as linear spaces)

$$L^2(\Omega) = \left\{ f \in L^2(\Omega) : (\Delta - \kappa^2) f = 0 \right\} \oplus (\Delta + \kappa^2)(H_0^2(\Omega))$$

holds when κ^4 is not an eigenvalue for the bilaplacian. In particular, sources in $\mathcal{D}_F$ can be identified from one single boundary measurement.

However, orthogonal decomposition (4) provides the natural setting for full identification results.

3.2. Identification results - many boundary measurements

So far, we have considered only identification from one boundary measurement. In this section, we study identification taking many boundary measurements. It is well known that changing the input Dirichlet data and measuring the corresponding Neumann data, that is, having the Dirichlet to Neumann map, does not increase information regarding the source. Instead, as showed in [5], we have to consider measurements arising from several frequencies.

Let $\kappa \in \Upsilon \subseteq \mathbb{R}_0^+$ and consider the (eventually infinite) set of measurements

$$\mathcal{M}_f = \{\Lambda_\kappa(f) : \kappa \in \Upsilon\}.$$

Now define the following collections

$$\mathcal{H} := \bigcup_{\kappa\in\Upsilon} \mathcal{H}_\kappa \quad \text{and} \quad \mathcal{K} := \bigcap_{\kappa\in\Upsilon} \ker \Lambda_\kappa. \tag{8}$$

Notice that, since $\mathcal{H}_\kappa^\perp = \ker \Lambda_\kappa$ then

$$\mathcal{H}^\perp = \bigcap \mathcal{H}_\kappa^\perp = \mathcal{K}$$

hence,

$$\mathcal{K}^\perp = \overline{\text{span}\mathcal{H}}^{L^2}.$$

Denote by π the orthogonal projection of $L^2(\Omega) = \mathcal{K}^\perp \oplus \mathcal{K}$ onto $\mathcal{K}$. Following above notation we write

$$\mathcal{C}_F := \pi^{-1}(F).$$

The level set $\mathcal{C}_F$ is thus given by

$$\mathcal{C}_F = \left\{ f \in L^2(\Omega) : (\forall \kappa \in \Upsilon) \ \left(\exists v_\kappa \in H_0^2(\Omega)\right) \ \ s.t. \ (\Delta + \kappa^2)v_\kappa = F \right\}.$$

Next result generalizes Theorem 5 to any number of measurements. The proof can be established following the same ideas contained in the one measurement version and will be omitted.

Theorem 7. *If $f_1, f_2 \in L^2(\Omega)$ are such that*

$$\Lambda_\kappa(f_1) = \Lambda_\kappa(f_2), \ \ \forall \kappa \in \Upsilon$$

and $f_1, f_2 \in \mathcal{C}_F$ then

$$f_1 = f_2 \quad \textit{in the } L^2 \textit{ sense}.$$

In particular, a source $f \in \mathcal{C}_F$ can be fully identified from the boundary data $\mathcal{M}_f$.

Notice that, if $\Upsilon_1 \subset \Upsilon_2$ then the corresponding spaces $\mathcal{K}_1,\ \mathcal{K}_2$ satisfy

$$\mathcal{K}_2 \subset \mathcal{K}_1$$

This means, intuitively, that there is an information gain (regarding identification of f) by increasing the number of measurements. The limit case $\Upsilon = \mathbb{R}^+$ is of particular theoretical interest. In this case

$$\mathcal{K} = \{0\}$$

(cf. [5]). In other words, there is only one level set, namely $\mathcal{C}_0$. Hence,

Theorem 8. *An $L^2(\Omega)$ function can be fully identified from the set of measurements $\mathcal{M}_f$, for all frequencies $\kappa \in \mathbb{R}^+$.*

4. Numerical source reconstruction

We present two numerical methods for one measurement reconstructions. First, using an higher order problem, already implemented and tested in some previous papers (eg. [15], [14], [3]). The second, proposed by us, exploits directly the linearity of the problem and can be easily generalized for several measurements.

4.1. Reconstruction from one measurement using an higher order direct problem

Let $f \in \mathcal{C}_F^\kappa$ with $F \in L^2(\Omega)$ such that

$$G := (\Delta - \kappa^2)F \in L^2(\Omega)$$

and consider the 4th order direct problem

$$\left\{ \begin{array}{ll} (\Delta^2 - \kappa^4)w = G & \text{in } \Omega \\ w = 0 & \text{on } \Gamma \\ \partial_{\mathbf{n}} w = \Lambda_\kappa(f) & \text{on } \Gamma \end{array} \right. . \tag{9}$$

It is well known that (cf. [5])

$$f = (\Delta + \kappa^2)w.$$

Consider the following two cases:

First case: $G = 0$.

Suppose $f \in \mathcal{C}_0^\kappa$, ie., the 4th order equation (9) is homogeneous. As proposed in [3], the method of fundamental solutions is an efficient numerical method for this boundary value problem.

Recall that a fundamental solution for the operator $\Delta^2 - \kappa^4$ satisfies the equation

$$(\Delta^2 - \kappa^4)\Psi_\kappa = -\delta,$$

where δ is the Dirac delta distribution, centered at the origin. For $\kappa \neq 0$, we consider

$$\Psi_\kappa(x) = \frac{i}{8\kappa^2}\left(H_0^{(1)}(i\kappa|x|) + H_0^{(2)}(\kappa|x|)\right)$$

and for $\kappa = 0$,

$$\Psi_0(x) = -\frac{1}{8\pi}|x|^2 \log|x|.$$

Given $y \in \mathbb{R}^2 \setminus \overline{\Omega}$, we consider the point source function $\Psi_{\kappa,y}(x) := \Psi_\kappa(x-y)$. Notice that

$$(\Delta^2 - \kappa^4)\Psi_\kappa(x-y) = -\delta_y.$$

Let $\widehat{\Gamma}$ be an artificial boundary enclosing the domain $\overline{\Omega}$. The MFS approximation for problem (9) consists in taking (see [1] for other MFS choice of basis functions and theoretical results)

$$\tilde{w}(x) \approx \sum_{j=1}^{m_0} \alpha_j \Psi_\kappa(x-y_j) + \sum_{j=m_0+1}^{m} \alpha_j(\Delta - \kappa^2)\Psi_\kappa(x-y_j), \quad y_j \in \widehat{\Gamma}.$$

The coefficients, α_j, are computed in order to fit both boundary conditions at some collocation points $x_i \in \Gamma$.

The source f can now be approximated by

$$f(x) \approx (\Delta + \kappa^2)\tilde{w} = \sum_{j=1}^{m_0} \alpha_j(\Delta + \kappa^2)\Psi_\kappa(x-y_j).$$

Notice that the sum $\sum_{j=m_0+1}^{m} \alpha_j(\Delta - \kappa^2)\Psi_\kappa(x-y_j)$ vanishes when we apply $\Delta + \kappa^2$. This may lead to some undesired oscillations, as discussed in [3]. Moreover, the 4th order problem is well posed, with $w \in H^2(\Omega)$, so that $w \in H^{3/2}(\Gamma)$ and $\partial_{\mathbf{n}} w = \Lambda_\kappa(f) \in H^{1/2}(\Gamma)$. However, in practise, measurement errors are expected to occur. In particular, the measured data $\Lambda_\kappa(f)$ may not be in the appropriate functional framework.

Second case: $G \neq 0$.

In this nonhomogeneous framework, we can apply the domain method of fundamental solutions (MFS-D) (cf. [2]). This consists in taking as basis functions, fundamental solutions for Helmholtz equations. A fundamental solution for Helmholtz operator is

$$\Phi_\kappa(x) = \frac{i}{4}H_0^{(1)}(\kappa|x|), \quad \kappa > 0,$$

where $H_0^{(1)}$ is the Hankel function of first kind and zero order.

For Laplace ($\kappa = 0$),

$$\Phi_0(x) = -\frac{1}{2\pi}\log(|x|).$$

We consider the approximation

$$\tilde{w}(x) = \sum_{\lambda_k, y_j} \alpha_{\lambda_k, y_j} \Phi_{\lambda_k}(x - y_j), \quad y_j \in \widehat{\Gamma}$$

and compute the coefficients by fitting both boundary conditions at some points and the differential equation at some domain points, ie.,

$$(\Delta^2 - \kappa^4)\tilde{w}(x_i) = \sum_{\lambda_k, y_j} \alpha_{\lambda_k, y_j} (\lambda_k^4 - \kappa^4)\Phi_{\lambda_k}(x_i - y_j) = G(x_i), \quad x_i \in \Omega.$$

The source is approximated by

$$f(x) \approx \sum_{\lambda_k, y_j} \alpha_{\lambda_k, y_j} (-\lambda_k^2 + \kappa^2)\Phi_{\lambda_k}(x - y_j).$$

4.2. Reconstruction from one measurement using a direct approach

Due to the orthogonal decomposition (4), we can write, for sources in $\mathcal{C}_F^\kappa$,

$$f = f_\kappa + F.$$

Hence $f_\kappa = f - F = (\Delta + \kappa^2)(u - v)$ which gives, for $w := u - v \in H^2(\Omega)$,

$$\begin{cases} (\Delta + \kappa^2)w = f_\kappa & \text{in } \Omega \\ f_\kappa \in \overline{\mathcal{H}_\kappa} & \\ w = 0 & \text{on } \Gamma \\ \partial_{\mathbf{n}} w = \Lambda_\kappa(f) & \text{on } \Gamma \end{cases} \tag{10}$$

We start with the representation of f_κ. It is well known that, under some non resonance assumptions on the artificial bounded domain $\widehat{\Omega} \supset \overline{\Omega}$ such that $\widehat{\Gamma} = \partial\widehat{\Omega}$ (eg. [7]), the set of fundamental basis functions

$$\mathcal{S}_\Gamma^\kappa = \left\{ \Phi_{\kappa,y}|_\Gamma : y \in \widehat{\Gamma} \right\}$$

spans a dense subspace in $H^{3/2}(\Gamma)$. By well posedness, the set

$$\mathcal{S}_\Omega^\kappa = \left\{ \Phi_{\kappa,y}|_\Omega : y \in \widehat{\Gamma} \right\} \tag{11}$$

spans a dense subspace in $\overline{\mathcal{H}_\kappa}$. This justifies the approximation

$$f_\kappa \approx \sum_{y_j} \alpha_j \Phi_{\kappa, y_j}. \tag{12}$$

Now, for each Φ_{κ,y_j}, we solve

$$\begin{cases} (\Delta + \kappa^2) w_j = \Phi_{\kappa, y_j} & \text{in } \Omega \\ w_j = 0 & \text{on } \Gamma \end{cases}$$

from where we get

$$\mathcal{R} = \{\Lambda_\kappa(\Phi_{\kappa,y_1}), \ldots, \Lambda_\kappa(\Phi_{\kappa,y_n})\}.$$

We can now find the coefficients in (12) by computing the best fit

$$\sum_{y_j} \alpha_j \Lambda_\kappa(\Phi_{\kappa,y_j}) \in \operatorname{span}\mathcal{R}$$

to the available data $\Lambda_\kappa(f)$. Finally, the source can be retrieved by

$$f \approx \sum_{y_j} \alpha_j \Phi_{\kappa,y_j} + F.$$

In order to justify the proposed method, we establish the following properties.

Lemma 9. *The set of responses $\mathcal{R}$ is linearly independent in $\Gamma \setminus \{y_1, \ldots, y_n\}$, ($y_i \neq y_j$, $i \neq j$).*

Proof. Suppose that

$$\alpha_1 \Lambda_\kappa(\Phi_{\kappa,y_1}) + \ldots + \alpha_n \Lambda_\kappa(\Phi_{\kappa,y_n}) = 0.$$

Then,

$$u := \alpha_1 \Phi_{\kappa,y_1} + \ldots + \alpha_n \Phi_{\kappa,y_n}$$

belongs to $\ker \Lambda_\kappa$. On the other hand, $u \in \overline{\mathcal{H}_\kappa}$ and follows $u \in \overline{\mathcal{H}_\kappa} \cap \overline{\mathcal{H}_\kappa}^{\perp}$. We conclude that $u = 0$ in $\mathbb{R}^2 \setminus \{y_1, \ldots, y_n\}$ and the result is a consequence of the independence of $\mathcal{S}^\kappa_\Omega$ in $\mathbb{R}^2 \setminus \{y_1, \ldots, y_n\}$. □

Next result shows that, for some frequencies, any $H^{1/2}(\Gamma)$ function can be well approximated using basis functions

$$\mathcal{R}_\infty := \left\{ \Lambda_\kappa(\Phi_{\kappa,y}) : y \in \widehat{\Gamma} \right\}.$$

Theorem 10. *Suppose κ is an admissible frequency for the bihelmholtz and that κ^2 is not an eigenvalue for the Helmholtz-Dirichlet problem in a bounded regular domain $\widehat{\Omega}$, with $\overline{\Omega} \subset \widehat{\Omega}$. Then, the set $\mathcal{R}_\infty$ spans a dense subspace in $H^{1/2}(\Gamma)$.*

Proof. The density result follows from Theorem 4 and density of fundamental solutions in $\overline{\mathcal{H}_\kappa}$. In fact, given $g_{\mathbf{n}} \in H^{1/2}(\Gamma)$, let f be the unique element in $\overline{\mathcal{H}_\kappa}$ such that

$$g_{\mathbf{n}} = \eta_\kappa(f),$$

where η_κ is the homeomorphism defined in (6). Since $\mathcal{S}_\Omega^\kappa$ is dense in $\widehat{\overline{\mathcal{H}_\kappa}}$, there exists a sequence of fundamental solutions $\Phi_{\kappa,y_k} \in \mathcal{H}_\kappa$, with $y_k \in \Gamma$ such that

$$\sum_{j=0}^{n} \alpha_j \Phi_{\kappa,y_j} \to f \text{ in } L^2(\Omega).$$

Notice that, on one hand

$$\eta_\kappa \left(\sum_{j=0}^{n} \alpha_j \Phi_{\kappa,y_j} \right) = \sum_{j=0}^{n} \alpha_j \Lambda_\kappa \left(\Phi_{\kappa,y_j} \right)$$

and on the other,

$$\eta_\kappa \Big(\sum_{j=0}^{n} \alpha_j \Phi_{\kappa,y_j} \Big) \to \eta_\kappa(f) \quad \text{in } H^{1/2}(\Gamma).$$

This concludes the proof. □

We finish this section with a stability result. We show that our numerical method applied to the reconstruction of admissible sources is stable in the $H^{1/2}(\Gamma)$ sense.

Theorem 11. *Suppose that κ is an admissible frequency and let $g_{\mathbf{n}} = \Lambda_\kappa(f) \in H^{1/2}(\Gamma)$ with $f \in \mathcal{C}_0^\kappa$. Suppose that exists a sequence*

$$v_k := \sum_{j=1}^{k} \alpha_j \Phi_{\kappa,y_j} \in \mathcal{H}_\kappa$$

such that $\Lambda_\kappa(v_k)$ converges to $g_{\mathbf{n}}$ in $H^{1/2}(\Gamma)$. Then, the sequence v_k converges to f in $L^2(\Omega)$.

Proof. The above convergence condition, implies, by continuity of η_κ^{-1},

$$\eta_\kappa^{-1} \left(\Lambda_\kappa(v_k) \right) \to \eta_\kappa^{-1} \left(g_{\mathbf{n}} \right) \text{ in } L^2(\Omega).$$

The result follows from the identities (recall that $v_k, f \in \overline{\mathcal{H}_\kappa}$)

$$\eta_\kappa^{-1} \left(\Lambda_\kappa(v_k) \right) = v_k \quad \text{and} \quad \eta_\kappa^{-1} \left(g_{\mathbf{n}} \right) = f.$$

□

Remark 12. *The above results rely on density results concerning fundamental solutions basis functions. Similar results can be obtain by considering other basis functions with similar density properties. For instance, one can take plane waves as basis functions. Recall that a plane wave function is defined by*

$$\Phi_{\kappa,d}(x) := e^{i\kappa x.d}, \quad d \in S^1$$

and that $\Phi_{\kappa,d} \in \mathcal{H}_\kappa$. The following density results are well known

$$\overline{span\{\Phi_{\kappa,d}|_\Gamma : d \in S^1\}}^{L^2} = L^2(\Gamma)$$

and

$$\overline{span\{\Phi_{\kappa,d}|_\Omega : \kappa \in \mathbb{R}^+,\ d \in S^1\}}^{L^2} = L^2(\Omega).$$

This makes plane waves basis functions also suited for the proposed method. The main advantage of these basis functions is that they are much faster to evaluate than Hankel basis functions $H_0^{(1)}$.

4.3. Several boundary measurements

Proposition 8 states that a source can be fully identified from the infinite set of boundary data

$$\mathcal{M}_f = \left\{\Lambda_\mu(f) : \mu \in \mathbb{R}_0^+\right\}.$$

On the other hand, the set (see (11) for definition)

$$\bigcup_{\kappa \in \mathbb{R}_0^+} \mathcal{S}_\Omega^\kappa$$

spans a dense subspace in $L^2(\Omega)$. In particular, we can consider the approximation

$$f \approx \widetilde{f} := \sum_{\kappa,y} \alpha_{\kappa,y} \Phi_{\kappa,y}. \tag{13}$$

Each basis function $\Phi_{\kappa,y}$ generates the boundary data $\Lambda_\mu(\Phi_{\kappa,y})$ and, as in the previous section, the coefficients can be computed by fitting the available data.

In practise, however, is not feasible to have access to all this measured data. Therefore, we have to consider a finite set of measurements

$$\mathcal{M}_f = \{\Lambda_{\kappa_1}(f), \ldots, \Lambda_{\kappa_n}(f)\}$$

and the approximation (13) taking only a finite set of basis functions. As in remark 12, we can also consider

$$\widetilde{f}(x) = \sum_{p=1}^{l} \sum_{q=1}^{m} \alpha_{p,q} e^{i\kappa_p x \cdot d_q}.$$

5. Numerical Examples

In this section we present some numerical results in order to illustrate the feasibility of the proposed method. In the following, we considered the domain of propagation

$$\Omega = \left\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 4\right\}$$

and used plane waves as basis functions.

Example 1. In this example we retrieved the $f_{\mathcal{H}} \in \mathcal{H}_\kappa$ part of a source function, from one single measurement. We considered the source function (see Fig. 1)

$$f = f_{\mathcal{H}} + f_{\mathcal{H}^\perp}$$

where

$$f_{\mathcal{H}}(X) = \frac{i}{4} H_0^{(1)}\left(1.3|X + (0, 2.2)|\right) + 2\frac{i}{4} H_0^{(1)}\left(1.3|X - (2.1, 0)|\right) + 3e^{-\frac{1.3}{\sqrt{2}} iX\cdot(1,1)} \in \mathcal{H}_{1.3}$$

and

$$f_{\mathcal{H}^\perp}(X) = -1.3\,(X \cdot X)^2 - 5.6X \cdot X + 11.2 \in \ker \Lambda_{1.3}.$$

Notice that the imaginary part of $f_{\mathcal{H}}$ is the imaginary part of f. The real part of $f_{\mathcal{H}}$ is plotted in Fig. 2. The corresponding boundary velocity (measured data), $\Lambda_{1.3}(f)$, was considered at 80 uniformly distributed boundary points. For the approximation, we took

$$\widetilde{f_{\mathcal{H}}}(X) = \sum_{j=1}^{n} \alpha_k e^{1.3iX.d_j}$$

with $d_1, \ldots, d_n$ uniformly distributed points over the two dimensional unit sphere S^1. The coefficients $\alpha_k \in \mathbb{C}$ were computed by fitting, in a least squares sense, the set of responses

$$\mathcal{R} = \left\{\Lambda_{1.3}(e^{1.3iX.d_1}), \ldots, \Lambda_{1.3}(e^{1.3iX.d_n})\right\}$$

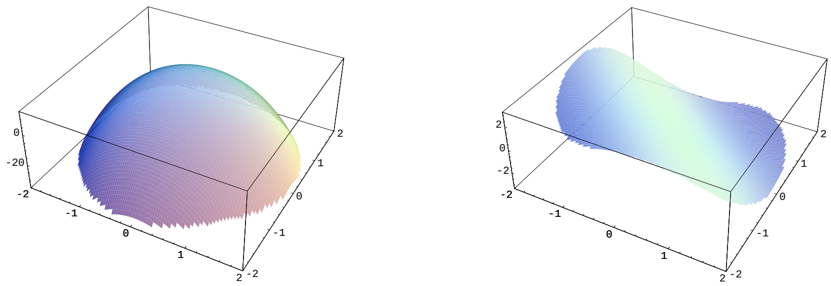
to the measured data, that is, by minimizing

$$\left\| \left(\Lambda_{1.3}(f)(x_1) - \sum_{j=1}^{n} \alpha_j \Lambda_{1.3}(e^{1.3iX.d_j})(x_1), \ldots, \Lambda_{1.3}(f)(x_{80}) - \sum_{j=1}^{n} \alpha_j \Lambda_{1.3}(e^{1.3iX.d_j})(x_{80}) \right) \right\|_2$$

where $x_1, \ldots, x_{80}$ are the observation points.

We start by reconstructing $f_{\mathcal{H}}$ using $n = 5$ basis functions and assuming no noise in the measured data. In this case we obtained a reconstruction result with L^2 norm error of 6.2, that corresponds to a relative error of 5.9%. By increasing the number of basis functions to 10 and then 20 we obtained better results. The evolution of the corresponding L^2 norm errors are summarized in Table 5. The absolute componentwise error for $n = 10$ basis functions is plotted in Fig. 3.

The robustness of the method was also tested by retrieving the source from data with noise. We considered data with up to 5 %, 10% and 15 % of noise. Table 5 summarizes the evolution of the relative error for these levels of noise and in Fig. 4 we present the reconstruction for 10 % of noise.



(a) Real part (b) Imaginary part

Figure 1: Source function considered in Example 1.

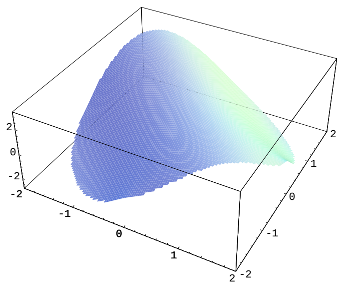


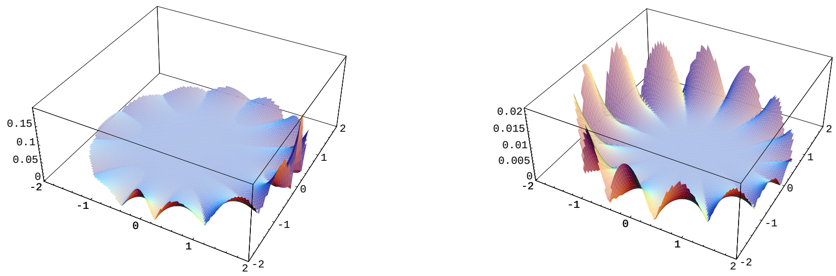
Figure 2: Real part of $f_{\mathcal{H}}$ (Example 1).

Number of basis functions	L^2 error
5	6.16
10	0.062
20	0.031

Table 1: Evolution of the L^2 norm error with the number of basis functions (Example 1).

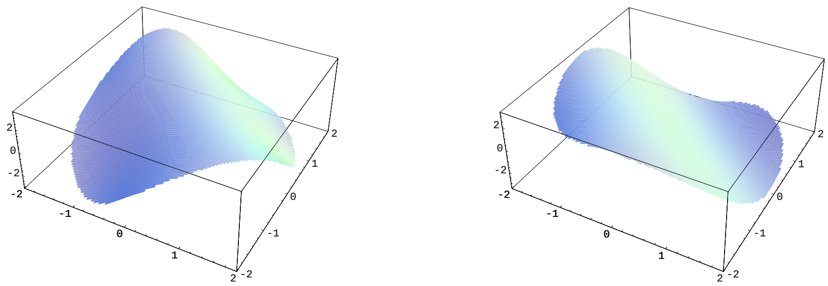
Noise	L^2 relative error
5%	4.4%
10%	7.6%
15%	14.1%

Table 2: Evolution of the relative error with the noise in the data using 10 basis functions (Example 1).



(a) $|\Re(f_{\mathcal{H}} - \widetilde{f}_{\mathcal{H}})|$ (b) $|\Im(f_{\mathcal{H}} - \widetilde{f}_{\mathcal{H}})|$

Figure 3: Pointwise error with 10 basis functions.



(a) Real part (b) Imaginary part

Figure 4: Reconstruction with 10 basis functions for data with 10% of random noise.

Example 2. In this example we tested full source reconstruction. We considered the compactly supported source (see Fig. 5)

$$f(X) = -0.5h(1 - ||X - (-0.3, 1)||_2) + 2h_1(1 - ||X - (1.1, -0.2)||_1)$$

where

$$h(x) = \frac{ct(x)}{ct(x) + ct(1.1 - x)}, \qquad h_1(x) = \frac{ct(x)}{ct(x) + ct(1.5 - x)}$$

and ct is the smooth cut-off function $ct(x) = e^{-1/x}$ if $x > 0$ and $ct(x) = 0$, otherwise.

The source was approximated by

$$f(X) \approx \sum_{j=1}^{6} \sum_{k=1}^{15} \alpha_{j,k} e^{i(2j-1)X.d_k}, \quad d_k \in S^1$$

and the coefficients were computed by fitting the measured data at 80 boundary (observation) points. We tested with $m = 1, 2, 3$ and 7 measurements. The associated frequen-

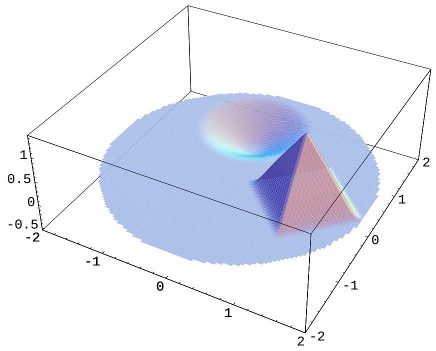


Figure 5: Source considered in Example 2.

cies were $\kappa = 1.5$, $\kappa = 1.5, 3.5$, $\kappa = 1.5, 3.5, 5.5$ and $\kappa = 1.5, 3.5, 5.5, 7.5, 9.5, 11.5, 13.5$, respectively. Numerical results, for data without noise, are presented in Fig. 6.

We can observe that the number and location of the source peaks are detected with three measurements (see also the contour plot in Fig. 8). Taking seven measurements the shape of the peaks is also retrieved.

Last test concerns numerical reconstruction from noisy data. In Fig. 7 we considered data with up to 5 % of noise and several measurements (two and three).

6. Conclusions

In this work we proposed a direct method for the reconstruction of heat/acoustic sources from boundary measurements. The main advantage of the proposed method over other higher order approaches is that it only needs to collect responses from certain source basis functions. This process is independent of the source that we are recovering. Hence, once the library of data is computed, it can be used to retrieve any source, for instance, by data fitting. The method was theoretically justified and tested for several examples. Numerical results shows that the method is feasible and robust. Possible extensions to source reconstruction from partial data are straightforward. In particular, the reconstruction of sources with support strictly contained in Ω with many (partial) boundary measurements does not requires any *a priori* data completion method. This is also an advantage over several reconstruction methods based on Green's formula (cf. [5]).

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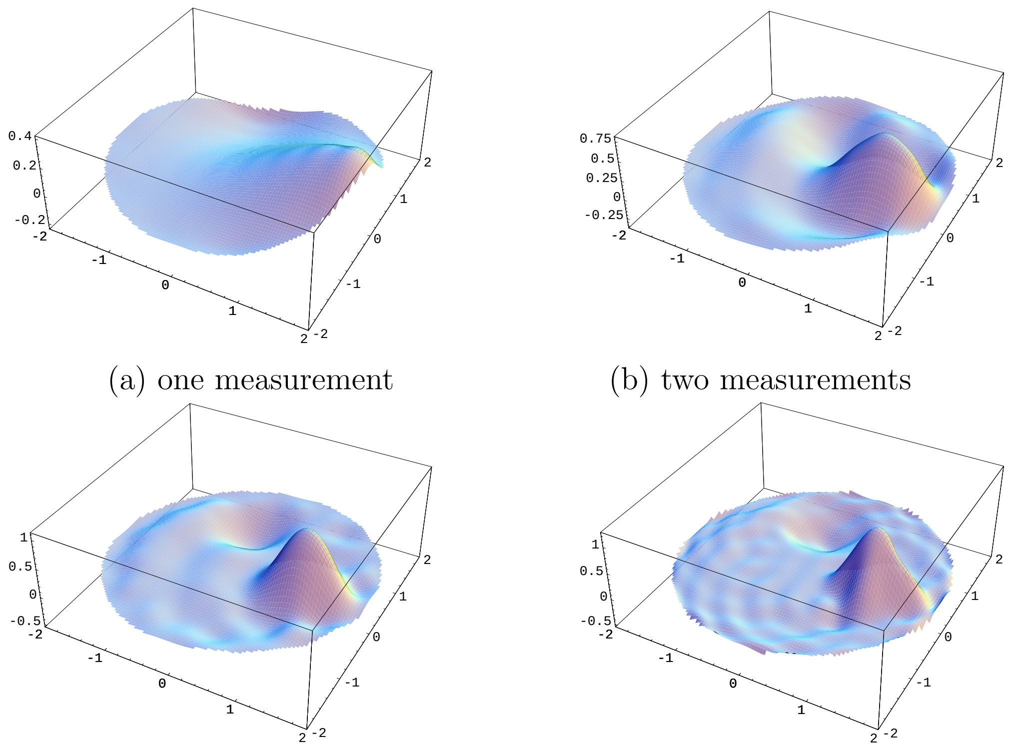

(a) one measurement (b) two measurements

(c) three measurements (d) seven measurements

Figure 6: Source reconstruction from several boundary measurements (data without noise).

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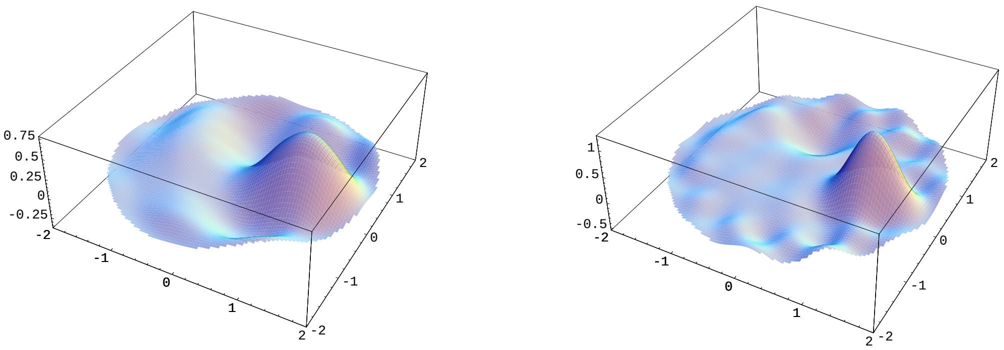
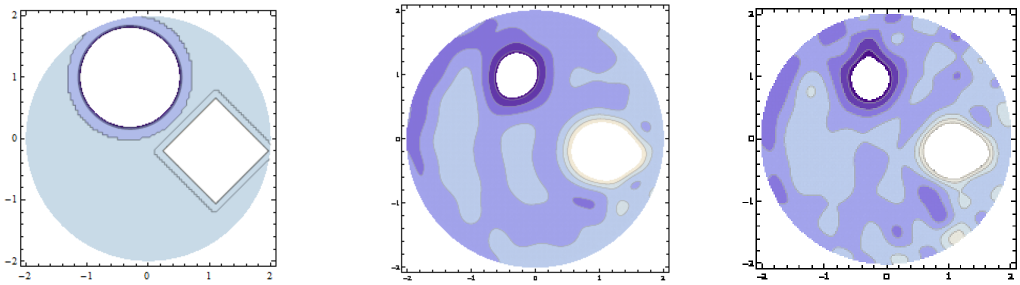


(a) two measurements (b) three measurements

Figure 7: Source reconstruction for several boundary noisy measurements with up to 5 % of noise.



(a) source (b) without noise (c) with 5% noise

Figure 8: Contour plot of the reconstructed source from three boundary measurements without noise (plot (b)) and with noise (plot (c)).

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