On the exact and near-exact distributions of the product of generalized Gamma random variables and the generalized variance

Carlos A. Coelho[a,1], Barry C. Arnold[b]

[a]*Departamento de Matemática and Centro de Matemática e Aplicações, Faculdade de Ciências e Tecnologia, Universidade Nova de Lisboa, Portugal*
[b]*Statistics Department, Universit of California, Riverside, U.S.A.*

Abstract

In this paper the authors first obtain the exact distribution of the logarithm of the product of independent generalized Gamma r.v.'s (random variables) in the form of a Generalized Integer Gamma distribution of infinite depth, where all the rate and shape parameters are well identified. Then, by a routine transformation, simple and manageable expressions for the exact distribution of the product of independent generalized Gamma r.v.'s are derived. The method used also enables us to obtain quite easily very accurate, manageable and simple near-exact distributions in the form of Generalized Near-Integer Gamma distributions. Numerical studies are carried out to assess the precision of different approximations to the exact distribution and they show the high accuracy of the approximations provided by the near-exact distributions. As particular cases of the exact distributions obtained we have the distribution of the product of independent Gamma, Weibull, Frechet, Maxwell-Boltzman, Half-Normal, Raleigh and Exponential distributions, as well as the exact distribution of the generalized variance, the exact distribution of discriminants or Vandermonde determinants and the exact distribution of any linear combination of generalized Gumbel distributions, as well as yet the distribution of the product of any power of the absolute value of independent Normal r.v.'s.

Keywords: Characteristic functions, Generalized Integer Gamma distribution, Generalized Near-Integer Gamma distribution, Infinite sums, Near-exact distributions.

1. Introduction

There is a vast literature on the distribution of the product of independent Gamma and generalized Gamma r.v.'s and on the distribution of the product of independent chi-square r.v.'s. However, we think that these problems are still amenable to a different approach which may (i) lead to a simple and more manageable form for the exact distribution, (ii) enable an easy development of very well-fitting approximate distributions and (iii) yield an adequate unified approach for the distribution of the product of r.v.'s whose distribution may be seen as a particular case of the generalized Gamma distribution, as well as the distribution of linear combinations of generalized Gumbel r.v.'s.

Since the generalized Gamma distribution (Stacy, 1962; Amoroso, 1925; D'Addario, 1932) has as particular cases the Gamma, Chi-square, Exponential, Hal-Normal, Weibull, Frechet, Rayleigh, Maxwell-Boltzman distributions, as well as the distribution of any power of the absolute value of Normal r.v.'s (see App. A), the importance of the distribution of the product of independent generalized Gamma r.v.'s is due in part to the facts that:

i) if obtained in a general setting, it will have as particular cases the distribution of the product of any combination of independent r.v.'s with the above mentioned distributions which are particular cases of the generalized Gamma distribution, among which are the powers of the absolute value of Normal r.v.'s;

*Corresponding author: Faculdade de Ciências e Tecnologia, Universidade Nova de Lisboa, Departamento de Matemática, Quinta da Torre, 2829-516 Caparica, Portugal; phone: +351 21 294 8388, Fax: +351 21 294 8391; e-mail: cmac@fct.unl.pt.

Email addresses: `cmac@fct.unl.pt` (Carlos A. Coelho), `barry.arnold@ucr.edu` (Barry C. Arnold)

ii) the distribution of the discriminant, or square of the Vandermonde determinant, when sampling from Normal or Gamma populations, is the same as the distribution of the product of particular independent Gamma r.v.'s (Lu and Richards, 1993);

iii) the distribution of the generalized variance, which has been given considerable attention in the literature, is the distribution of the product of particular independent chi-squared r.v.'s;

iv) in its negative logarithm form, it yields

    i) the distribution of linear combinations of independent generalized Gumbel r.v.'s, which are the negative logarithm of a generalized Gamma r.v.;

    ii) as well as the distribution of the complete sufficient statistic for the shape parameter (r) in a Gamma distribution with unit or known rate parameter and the distribution of the maximum likelihood and the UMVU estimator of $\Psi(r)$, where $\Psi(\cdot)$ is the digamma function (see App. B).

Apart from the case that is usually referred to as 'the stochastic analog of the Gauss multiplication theorem' (see Theorem 2 of Gordon (1989) and Lemma 4.3 of Lu and Richards (1993)), in which case the product of particular independent Gamma r.v.'s has a Gamma distribution, the exact distribution of the product of independent Gamma or related r.v.'s does not have a simple representation.

Wells et al. (1962) obtained the exact distribution for the product of two central or non-central Chi-square r.v.'s, expressing the p.d.f. and c.d.f. through the use of modified Bessel functions of the second kind. Malik (1968) obtained expressions for the p.d.f. (and c.d.f. for a particular case) of the product of two independent generalized Gamma r.v.'s, both with the same power parameter, also in terms of modified Bessel functions of the second kind. Lomnicki (1967) obtained rather complicated expressions for the p.d.f. of the product of any number of independent standard Exponential or Weibull r.v.'s. However, for the product of Gamma r.v.'s the author only indicates a possible way to obtain the expressions for the p.d.f. saying that "for the Gamma distribution the situation is not so simple". Springer and Thompson (1970) obtained expressions for the p.d.f. of the product of independent Gamma r.v.'s in terms of the Meijer G function and Podolski (1972) obtained the distribution of the product of independent generalized Gamma r.v.'s all with the same power parameter also in terms of the Meijer G function. Mathai (1972a) obtains the distribution of the ratio of two products of independent generalized Gamma r.v.'s in terms of the Meijer G function. A particular case of this is the distribution of the product of independent generalized Gamma r.v.'s. This distribution assumes in this case the same form as that of the one obtained by Springer and Thompson (1970). Carter and Springer (1977) obtained the distribution of the product of independent H-function r.v.'s as H-functions. The H-function (Fox, 1961) alluded to here is a further generalization of the Meijer G function. Particular cases of H-function r.v.'s are, among others, the Gamma, Exponential, Chi-square, Weibull and Half-normal r.v.'s. More recently, Salo et al. (2006) obtained the exact p.d.f. and c.d.f. for the product of any number of independent Rayleigh r.v.'s in the form of Meijer G functions, providing series representations for the cases of three, four or five r.v.'s.

The representations and approaches using the Meijer G function and especially those involving the H function although very general, are not very helpful for obtaining a representation of the distribution which may be applied in practice to compute p-values or quantiles. This is because the G and H functions are indeed only alternative representations for Barnes type integrals, which then have to be solved numerically. One may argue that also the commonly used Gamma function is just a representation for an integral, which is indeed true. However, nowadays most available software packages capable of performing symbolic computations, are able to easily compute the Gamma function with a very high precision. This is not the case for the Meijer G function. This function is, at best, computable with moderate precision only for small values of its arguments, while the H function is usually not even implemented in such software.

The distribution of the generalized variance, which is a particular case of the distribution of the product of independent Gamma r.v.'s, has deserved some particular attention. Bagai (1965) obtained the exact distribution for the generalized variance for 5 through 10 variables involved, both for the non-central and central cases. However, the results are obtained case by case and not under a general formulation, even for the central case. Moreover the expressions obtained, besides involving infinite sums, also involve the generalized Gauss hypergeometric function. Mathai (1972b) obtained the exact non-central distribution of the generalized variance in terms of zonal polynomials, Psi functions, and Zeta functions. Hoel (1937) seems to have been the first to address the problem of approximating the exact distribution of the generalized variance. He suggests two approximations, one for cases where, in his own words, samples "are not too small", more precisely for cases where the sample size "is fairly large compared with" the square of the number of variables. In such a case he suggests approximating the distribution of the p-th root of the generalized variance by a Gamma distribution, where p is the number of variables in the multivariate Normal distribution that we are sampling from. This actually amounts to approximating the distribution of the generalized variance by a generalized Gamma distribution with power parameter equal to $1/p$. The other approximation Hoel suggests is a Normal approximation, for even larger samples. Gnanadesikan and Gupta (1970) studied Hoel's Gamma approximation and concluded that it worsens for increasing p. These authors then suggest a Normal approximation based on the Normal approximation to the distribution of the logarithm of a Chi-squared r.v. suggested by Bartlett and Kendall (1946). This approach is also followed by Regier (1976). Fujikoshi (1968) obtained a Normal distribution as the limit distribution for a normalized version of the non-central generalized variance when the non-centrality matrix is $O(n)$, where n represents the sample size. The same author also obtained a limit and an asymptotic expansion for the cumulative distribution function of a normalized version of the logarithm of the non-central generalized variance when the non-centrality matrix is constant up to terms of order $n^{-3/2}$, based on powers of the Normal c.d.f.. Sugiura and Nagao (1971) obtained an asymptotic expansion for the distribution of the generalized variance in the non-central case, when the non-centrality matrix is $O(n)$. Steyn (1978) suggests approximating the p.d.f. of a linear transformation of the generalized variance in the central case by the p.d.f. of a mixture of two standard Gamma distributions. However, this approximation does not perform appropriately since its supposed p.d.f. although integrating to one, when integrated from zero to infinity, only assumes positive values for values of the running variable larger than a given threshold. Consequently the integral between this threshold and infinity is much larger than one. Gordon (1989) obtained bounds, based on chi-square distributions, for the distribution of the generalized variance. Butler et al. (1992) obtained saddle-point approximations for the distribution of the generalized variance. Hao and Krishnamoorthy (2001) obtained a Normal approximation to the exact distribution of the logarithm of the generalized variance and another approximation similar to Hoel's approximation for the power $1/p$ of the generalized variance for the case of monotone missing data.

Concerning the generalized Gumbel distribution, a common application of this distribution is as the limit distribution of order statistics of distributions for which the p.d.f. decreases faster than any power law, for large values in the support of the r.v.. There is a wide range of areas where the distribution of the sum or linear combination of independent Gumbel r.v.'s finds applications (Loaiciga and Leipnik, 1999; Cetinkaya et al., 2001). Cetinkaya et al. (2001) approximate the distribution of the sum of two independent Gumbel r.v.'s by a single Gumbel distribution. Loaiciga and Leipnik (1999) obtained the exact c.d.f. for a linear combination of only two independent Gumbel r.v.'s as an infinite series. Nadarajah (2007) obtained the c.d.f. for a similar linear combination, where some restrictions are placed on the parameters, in terms of generalized hypergeometric functions and Nadarajah and Kotz (2008) obtain the exact p.d.f. and c.d.f. also in terms of generalized hypergeometric functions.

In this paper, by using a decomposition of the c.f. (characteristic function) of the logarithm of the product of independently distributed generalized Gamma r.v.'s induced by the use of a product representation for the Gamma function (see (4) ahead), the authors obtain a decomposition of the product of the r.v.'s. The bases of this decomposition are similar to the ones obtained by Kaluska and Krysicki (1997) for the generalized Gamma distribution and by Gordon (1989) and Bondesson (1978) for the common Gamma r.v.'s. However, by taking a completely different approach from the ones followed by other authors we are able to obtain expressions for both the p.d.f. and the c.d.f. of the distribution of the product of independent generalized Gamma r.v.'s which are quite simple and highly manageable. Furthermore these may be used to develop near-exact approximations.

Since the negative logarithm of a generalized Gamma r.v. is indeed a r.v. with a generalized Gumbel distribution, the approach followed will actually also enable us to obtain the distribution of linear combinations of any number of independent generalized Gumbel r.v.'s in a manageable form. This will allow the development of very well-fitting approximations, since this distribution is that of the negative logarithm of the product of independent generalized Gamma r.v.'s.

We defer discussion of the non-central case to a future paper. However, we remark that many of the results to be established in the present paper will be useful when considering the non-central case.

2. A different look at the distribution of the product of independent generalized Gamma r.v.'s

Let (see App. A for notation)

$$X_j \sim \Gamma(r_j, \lambda_j; \beta_j) \qquad j = 1, \dots, p \tag{1}$$

be p independent r.v.'s and let

$$W = \prod_{j=1}^{p} X_j \,. \tag{2}$$

Then the h-th moment of W will be

$$E\left(W^h\right) = \prod_{j=1}^{p} E\left(X_j^h\right) = \prod_{j=1}^{p} \frac{\Gamma(r_j + h/\beta_j)}{\Gamma(r_j)}\, \lambda_j^{-h/\beta_j}\,,$$

and thus the c.f. of

$$Z = -\log\, W\,,$$

using the notation $\Lambda = \prod_{j=1}^{p} \lambda_j^{1/\beta_j}$, is given by (for $t \in \mathbb{R}$)

$$\Phi_Z(t) = E\left(\exp(\mathrm{i}tZ)\right) = E\left(\exp(-\mathrm{i}t\, \log W)\right) = E\left(W^{-\mathrm{i}t}\right) = \prod_{j=1}^{p} \frac{\Gamma(r_j - \mathrm{i}t/\beta_j)}{\Gamma(r_j)}\, \lambda_j^{\mathrm{i}t/\beta_j} = \Lambda^{\mathrm{i}t} \prod_{j=1}^{p} \frac{\Gamma(r_j - \mathrm{i}t/\beta_j)}{\Gamma(r_j)}\,. \tag{3}$$

Then, using the relation

$$\Gamma(z) = \lim_{K\to\infty} \frac{\exp(-\gamma z)}{z} \prod_{k=1}^{K} \frac{k}{k+z}\, \exp(z/k)\,, \tag{4}$$

which is valid for any $z \in \mathbb{C}\backslash\{0, -1, -2, \dots\}$ (Alfors, 1979, § 2.4; Lang, 1999, Chap. XV, § 2), where γ is the Euler gamma constant, we may write

$$\begin{aligned}\Phi_Z(t) &= \Lambda^{\mathrm{i}t} \prod_{j=1}^{p} \left\{ \frac{r_j}{r_j - \mathrm{i}t/\beta_j}\, \exp\left\{-\gamma(r_j - \mathrm{i}t/\beta_j - r_j)\right\} \lim_{K\to\infty} \prod_{k=1}^{K} \frac{r_j + k}{r_j + k - \mathrm{i}t/\beta_j}\, \exp\left(\frac{r_j - \mathrm{i}t/\beta_j - r_j}{k}\right)\right\} \\ &= \left\{\Lambda\, \exp\left(\gamma \sum_{j=1}^{p} 1/\beta_j\right)\right\}^{\mathrm{i}t} \lim_{K\to\infty} \left[\exp\left(-\mathrm{i}t \sum_{j=1}^{p}\sum_{k=1}^{K} \frac{1}{k\beta_j}\right) \prod_{j=1}^{p} \left\{\prod_{k=0}^{K} \frac{\beta_j(r_j+k)}{\beta_j(r_j+k) - \mathrm{i}t}\right\}\right].\end{aligned} \tag{5}$$

This shows that Z has the same distribution as

$$\log\, \Lambda + \gamma \sum_{j=1}^{p} \frac{1}{\beta_j} + \sum_{k=0}^{\infty}\sum_{j=1}^{p} Y_{jk}^{*}$$

where (see App. C for notation), for $j = 1, \dots, p$,

$$\begin{cases} Y_{j0} & \sim \quad Exp\left(\beta_j r_j\right), \qquad \text{for } \beta_j > 0 \\ -Y_{j0} & \sim \quad Exp\left(-\beta_j r_j\right), \qquad \text{for } \beta_j < 0 \end{cases}$$

and, for $j = 1, \dots, p$ and $k = 1, 2, \dots$,

$$\begin{cases} Y_{jk} & \sim \quad Exp\left(\beta_j(r_j + k)| - \tfrac{1}{k\beta_j}\right), \qquad \text{for } \beta_j > 0 \\ -Y_{jk} & \sim \quad Exp\left(-\beta_j(r_j + k)| - \tfrac{1}{k\beta_j}\right), \qquad \text{for } \beta_j < 0\,. \end{cases}$$

We may note that this decomposition is actually similar to the one obtained in Theorem 6 of Kaluska and Krysicki (1997) and to the one in Theorem 1 of Gordon (1989). A similar construction, in the context of ordinary Gamma r.v.'s, was described by Bondesson (1978).

We may note that if we take, for $K = 0, 1, 2, \ldots$,

$$Z_K = \log \Lambda + \gamma \sum_{j=1}^{p} \frac{1}{\beta_j} + \sum_{k=0}^{K} \sum_{j=1}^{p} Y^*_{jk},$$

then as $K \to \infty$, we will have

$$Z_K \xrightarrow{d} Z.$$

Since the cases where we allow β_j's with different signs would pose problems that would increase the length of the paper to unreasonable limits, we prefer to address this problem in a future paper and consider here only the cases where all β_j's have the same sign, either positive or negative. As we will see in the sections ahead, even these simpler cases will cover a multitude of situations and distributions of major interest.

If all β_j are positive, the exact distribution of Z is thus a shifted GIG (Generalized Integer Gamma) distribution (see App. C) of infinite depth, i.e., the distribution of the sum of infinitely many independent shifted Gamma r.v.'s with integer shape parameters. We should note that the GIG distribution is still a legitimate distribution even when in (23) and (24) we let $p \to \infty$, since the p.d.f. and the c.d.f. in (23) and (24) are legitimate p.d.f.'s and c.d.f.'s for any positive integer value of p and as such still yield legitimate p.d.f.'s and c.d.f.'s even when we let $p \to \infty$. In the case in which all the β_j are negative, the r.v. $-Z$ will have the distribution described above.

In most cases the exact p.d.f. and c.d.f. of Z and W will actually have quite simple expressions in the form of a series (see the subsections ahead). In the subsections ahead we will analyze in more detail several particular cases that arise for different sets of values for the parameters involved.

2.1. The "simple" case

If all $\beta_j(r_j + k)$ $(j = 1, \ldots, p; k = 0, 1, 2, \ldots)$ are different, we will be facing what we call the "simple" case since then the exact distribution of $Z = -\log W$ will be a GIG distribution of infinite depth but with all the shape parameters equal to one and rate parameters $\beta_j(r_j + k)$ $(j = 1, \ldots, p; k = 0, 1, 2, \ldots)$. This distribution has relatively simple expressions for both the p.d.f. and the c.d.f., since if we take

$$h = kp + j, \quad h = 1, 2, \ldots; \text{ for } k = 0, 1, 2, \ldots; j = 1, \ldots, p,$$

and define

$$s_h = \beta_j(r_j + k), \tag{6}$$

then, from (23) and (24) in App. C, the exact p.d.f. of $Z = -\log W$ will be given by

$$f_Z(z) = \lim_{K\to\infty} C \sum_{h=1}^{p(K+1)} c_h \exp(-s_h(z - a)), \tag{7}$$

where

$$C = \prod_{h=1}^{p(K+1)} s_h, \qquad c_h = \prod_{\substack{j=1\\ j\neq h}}^{p(K+1)} \frac{1}{s_h - s_j}, \qquad a = \log \Lambda + \gamma \sum_{j=1}^{p} \frac{1}{\beta_j} + p \sum_{k=1}^{K} \frac{1}{k}. \tag{8}$$

The exact c.d.f. of $Z = -\log W$ will be given by

$$F_Z(z) = 1 - \lim_{K\to\infty} C \sum_{h=1}^{p(K+1)} \frac{c_h}{s_h} \exp(-s_h(z - a)), \tag{9}$$

Thus, the r.v. $W = \exp(-Z)$ will have p.d.f. and c.d.f. respectively given by

$$f_W(w) = \lim_{K\to\infty} C \sum_{h=1}^{p(K+1)} c_h\, (w \exp(a))^{s_h} \frac{1}{w}, \qquad \text{and} \qquad F_W(w) = \lim_{K\to\infty} C \sum_{h=1}^{p(K+1)} \frac{c_h}{s_h} (w \exp(a))^{s_h}.$$

In practice we will use for K some positive integer value.

2.2. The "multiple" case

By the "multiple" case we mean the case where some of the s_h's in (6) are equal. In this case we will have to gather all the s_h that are equal to a given value as one only s_h and use the corresponding shape parameter equal to the number of occurrences of the value s_h (instead of being just equal to one as was the case with all the shape parameters in the previous subsection).

The fact that some of the shape parameters will be larger than one, will then force us to use for the p.d.f. and c.d.f. of both Z and W not just the expressions in the previous section but rather the general expressions in App. C for the GIG distribution.

2.3. The case of all equal power parameters

If all the power parameters β_j $(j = 1, \dots, p)$ in (1) are equal, with $\beta_j = \beta$ $(j = 1, \dots, p)$, the shift parameter a in (8) will then have a slightly simpler expression. In this case it is given by

$$a = \log\,\Lambda + \gamma\,\frac{p}{\beta} + p\sum_{k=1}^{K}\frac{1}{k}\,. \tag{10}$$

2.4. The case of all $\beta_j = 1$ and all $r_j = r$

In this case W is the product of regular Gamma r.v.'s all with the same shape parameter $r > 0$ and the c.f. of Z may be written as

$$\Phi_Z(t) = \{\Lambda\,\exp(\gamma p)\}^{\mathrm{i}t}\lim_{K\to\infty}\left[\exp\left(-\mathrm{i}t\,p\sum_{k=1}^{K}\frac{1}{k}\right)\left\{\prod_{k=0}^{K}\left(\frac{r+k}{r+k-\mathrm{i}t}\right)^{p}\right\}\right]$$

which shows that the distribution of Z is the same as the distribution of

$$\log\,\Lambda + \gamma\,p + Y_0 + \lim_{K\to\infty}\sum_{k=1}^{K}\left(Y_k + \frac{1}{k}\right)$$

where all the r.v.'s are independent, with

$$Y_k \sim \Gamma(p, r+k)\,, \quad k = 0, \dots, K\,.$$

Thus the exact p.d.f. and c.d.f. of Z may be respectively written as

$$f_Z(z) = \lim_{K\to\infty} f^{GIG}\Big(z - a \,\Big|\, \underbrace{p, \dots, p}_{K+1}; r, \dots, r+K; K+1\Big)$$

and

$$F_Z(z) = \lim_{K\to\infty} F^{GIG}\Big(z - a \,\Big|\, \underbrace{p, \dots, p}_{K+1}; r, \dots, r+K; K+1\Big)$$

where the shift parameter a is given by (10) above, for $\beta = 1$. The exact p.d.f. and c.d.f. of W are then, respectively

$$f_W(w) = \lim_{K\to\infty} f^{GIG}\Big(-\log\,w - a \,\Big|\, \underbrace{p, \dots, p}_{K+1}; r, \dots, r+K; K+1\Big)\,\frac{1}{w}$$

and

$$F_W(w) = 1 - \lim_{K\to\infty} F^{GIG}\Big(-\log\,w - a \,\Big|\, \underbrace{p, \dots, p}_{K+1}; r, \dots, r+K; K+1\Big).$$

If in this case we take all λ_j equal then W will be the product of i.i.d. Gamma r.v.'s.

2.5. The case of all $\beta_j = 1$ and all r_j differing by an integer quantity

Without any loss of generality let us suppose that

$$r_1 \leq r_2 \leq \ldots \leq r_p\,. \tag{11}$$

Then, if we suppose that all r_j ($j = 1, \ldots, p$) differ by an integer quantity, we will have, for any complex number c,

$$\Gamma(r_j + c) \;=\; \Gamma(r_p + c) \prod_{h=0}^{r_p-r_j-1} (r_j + c + h)^{-1}\,, \quad j = 1, \ldots, p-1\,,$$

where any empty product is to be taken to be equal to 1.

Consequently, with all $\beta_j = 1$ ($j = 1, \ldots, p$), the c.f. of Z may be written as

$$\Phi_Z(t) \;=\; \Lambda^{\mathrm{i}t} \left(\frac{\Gamma(r_p - \mathrm{i}t)}{\Gamma(r_p)}\right)^p \prod_{j=1}^{p-1} \prod_{h=0}^{r_p-r_j-1} \frac{r_j + h}{r_j + h - \mathrm{i}t}\,,$$

where, taking into account (11), we may write

$$\prod_{j=1}^{p-1} \prod_{h=0}^{r_p-r_j-1} \frac{r_j + h}{r_j + h - \mathrm{i}t} \;=\; \prod_{h=0}^{r_p-r_1-1} \left(\frac{r_1 + h}{r_1 + h - \mathrm{i}t}\right)^{u_h}$$

with

$$u_h = \#\, r_j{}'\mathrm{s} \;\leq\; r_1 + h\,, \quad h = 0, \ldots, r_p - r_1 - 1\,. \tag{12}$$

We may thus write

$$\Phi_Z(t) = \{\Lambda\, \exp(\gamma p)\}^{\mathrm{i}t} \left\{\prod_{h=0}^{r_p-r_1-1} \left(\frac{r_1 + h}{r_1 + h - \mathrm{i}t}\right)^{u_h}\right\} \lim_{K\to\infty} \left[\exp\left(-\mathrm{i}t\, p \sum_{k=1}^{K} \frac{1}{k}\right) \left\{\prod_{k=0}^{K} \left(\frac{r_p + k}{r_p + k - \mathrm{i}t}\right)^p\right\}\right],$$

which shows that in this case the distribution of Z is the same as the distribution of

$$\log\,\Lambda + \gamma\, p + \left(\sum_{h=0}^{r_p-r_1-1} Y_h^*\right) + Y_0 + \lim_{K\to\infty} \sum_{k=1}^{K} \left(Y_k + \frac{1}{k}\right)$$

where all the r.v.'s are independent, with

$$Y_h^* \;\sim\; \Gamma(u_h,\, r_1 + h)\,, \quad h = 0, \ldots, r_p - r_1 - 1$$

and

$$Y_k \;\sim\; \Gamma(p,\, r_p)\,, \quad k = 0, \ldots, K\,.$$

The exact p.d.f. and c.d.f. of Z may thus be respectively written as

$$f_Z(z) \;=\; \lim_{K\to\infty} f^{\,GIG}\Big(z - a\,\Big|\,\underbrace{u_0, \ldots, u_{r_p-r_1-1}}_{r_p-r_1}, \underbrace{p, \ldots, p}_{K+1}; \underbrace{r_1, \ldots, r_p - 1}_{r_p-r_1}, \underbrace{r_p, \ldots, r_p + K}_{K+1};\; r_p - r_1 + K + 1\Big)$$

and

$$F_Z(z) \;=\; \lim_{K\to\infty} F^{\,GIG}\Big(z - a\,\Big|\,\underbrace{u_0, \ldots, u_{r_p-r_1-1}}_{r_p-r_1}, \underbrace{p, \ldots, p}_{K+1}; \underbrace{r_1, \ldots, r_p - 1}_{r_p-r_1}, \underbrace{r_p, \ldots, r_p + K}_{K+1};\; r_p - r_1 + K + 1\Big)$$

with a given by (10) and $u_0, \ldots, u_{r_p-r_1-1}$ given by (12).

The exact p.d.f. and c.d.f. of W will then be respectively given by

$$f_W(w) = \lim_{K\to\infty} f^{GIG}\Big(-\log\, w - a \,\Big|\, \underbrace{u_0,\ldots,u_{r_p-r_1-1}}_{r_p-r_1}, \underbrace{p,\ldots,p}_{K+1};\; \underbrace{r_1,\ldots,r_p-1}_{r_p-r_1}, \underbrace{r_p,\ldots,r_p+K}_{K+1};\; r_p-r_1+K+1\Big)\frac{1}{w}$$

and

$$F_W(w) = 1 - \lim_{K\to\infty} F^{GIG}\Big(-\log\, w - a \,\Big|\, \underbrace{u_0,\ldots,u_{r_p-r_1-1}}_{r_p-r_1}, \underbrace{p,\ldots,p}_{K+1};\; \underbrace{r_1,\ldots,r_p-1}_{r_p-r_1}, \underbrace{r_p,\ldots,r_p+K}_{K+1};\; r_p-r_1+K+1\Big).$$

2.6. The exact distribution of the generalized variance

The generalized variance is the determinant of the sample variance-covariance matrix, which when sampling from the p-multivariate Normal distribution with mean vector $\underline{\mu} \in \mathbb{R}^p$ and $p{\times}p$ variance-covariance matrix Σ has, for a sample of size n, the same distribution as

$$W = \frac{|\Sigma|}{(n-1)^p}\prod_{j=1}^{p} X_j \tag{13}$$

where the X_j's are independent with distributions given by

$$X_j \sim \chi^2_{n-j} \equiv \Gamma\left(\frac{n-j}{2}, \frac{1}{2}\right), \qquad j = 1,\ldots,p\,. \tag{14}$$

The distribution of the generalized variance has been studied by several authors who have obtained it in a variety of more or less complicated forms. Bagai (1965) obtained the distribution for some values of p in the form of expressions involving hypergeometric functions.

What we propose here is to obtain expressions for the p.d.f. and the c.d.f. of the logarithm of the generalized variance in the form of GIG distributions of infinite depth and from these to obtain the expressions for the exact distribution of the generalized variance itself.

We have to note that we are in the situation remarked in subsection 2.2. In this case, for a given value of $K = 0, 1, 2, \ldots$, we will have $2K + p$ different values for s_h, with

$$s_h = \frac{n-p+h-1}{2}\,, \quad h = 1,\ldots,2K+p\,,$$

with corresponding shape parameters u_h given by

$$u_h = \begin{cases} u_h = g_h & h = 1,2 \\ u_h = g_h + u_{h-2} & h = 3,\ldots,2K+p \end{cases}$$

with

$$g_h = \begin{cases} 1 & j = 1,\ldots,\min\left(2(K+1),p\right) \\ 0 & j = 1+\min\left(2(K+1),p\right),\ldots,\max\left(2(K+1),p\right) \\ -1 & j = 1+\max\left(2(K+1),p\right),\ldots,2K+p \end{cases}$$

or

$$g_h = (\#\text{ of elements in } \{2(K+1), p\} \geq h) - 1\,, \quad h = 1,\ldots,2K+p\,.$$

Then, according to the notation in App. C, for W in (13), the exact p.d.f. and c.d.f. of $Z = -\log W$ are respectively

$$f_Z^*(z) = \lim_{K\to\infty} f^{GIG}\left(z-a|u_1,\ldots,u_{2K+p};\frac{n-p}{2},\ldots,\frac{n-1+2K}{2};2K+p\right), \tag{15}$$

and

$$F_Z^*(z) = \lim_{K\to\infty} F^{GIG}\left(z-a|u_1,\ldots,u_{2K+p};\frac{n-p}{2},\ldots,\frac{n-1+2K}{2};2K+p\right),$$

for

$$a = -\log|\Sigma| + p\left(\log(n-1)+\gamma-\log 2-\sum_{k=1}^{K}\frac{1}{k}\right). \tag{16}$$

The exact p.d.f. and c.d.f. of $W = \exp(-Z)$ will then be

$$f_W(w) = \lim_{K\to\infty} f^{GIG}\left(-\log w-a\,\middle|\,u_1,\ldots,u_{2K+p};\frac{n-p}{2},\ldots,\frac{n-1+2K}{2};2K+p\right)\frac{1}{w},$$

and

$$F_W(w) = 1-\lim_{K\to\infty} F^{GIG}\left(-\log w-a\,\middle|\,u_1,\ldots,u_{2K+p};\frac{n-p}{2},\ldots,\frac{n-1+2K}{2};2K+p\right).$$

3. Near-exact distributions

Given the somewhat complicated structure of the exact distributions obtained, both for $Z = -\log W$ as well as for the r.v. W in (2), the development of near-exact distributions which will be simpler in structure but will still provide a good approximation to the distributions of interest becomes a worhtwhile goal.

Near-exact distributions for $Z = -\log W$ will be developed in the form of a shifted GNIG distribution (see App. C), by keeping, for a given value of $n^* \in \mathbb{N}$, unchanged the part of the c.f. of Z corresponding to the first $p*(n^*+1)$ terms (which correspond to a shifted GIG distribution of depth $p(n^*+1)$) and replacing the remaining part by the c.f. of a shifted Gamma r.v., in such a way that this near-exact distribution will match the first three exact moments of Z.

More precisely, since we may write, from (5),

$$\begin{aligned}\Phi_Z(t) = {}&\underbrace{\left\{\left\{\exp\left(p\gamma+\log\Lambda-p\sum_{k=1}^{p(n^*+1)}\frac{1}{k}\right)\right\}^{\mathrm{i}t}\prod_{j=1}^{p}\left\{\prod_{k=0}^{p(n^*+1)}(r_j+k)\,(r_j+k-\mathrm{i}t)^{-1}\right\}\right\}}_{\Phi_1(t)}\\ &\times\underbrace{\left\{\left\{\exp\left(p\gamma+\log\Lambda-p\sum_{k=1+\nu}^{\infty}\frac{1}{k}\right)\right\}^{\mathrm{i}t}\prod_{j=1}^{p}\left\{\prod_{k=1+\nu}^{\infty}(r_j+k)\,(r_j+k-\mathrm{i}t)^{-1}\right\}\right\}}_{\Phi_2(t)},\end{aligned} \tag{17}$$

where $\nu = p(n^*+1)$ and where $\Phi_1(t)$ will be left unchanged and $\Phi_2(t)$, given its structure, will be suitably replaced by

$$\Phi_2^*(t) = \exp(\mathrm{i}ta^*)\lambda^r(\lambda-\mathrm{i}t)^{-r},$$

which is the c.f. of a shifted Gamma distribution with shift parameter a^*, shape parameter r and rate parameter λ, in such a way that

$$\left.\frac{d^h}{dt^h}\Phi_2^*(t)\right|_{t=0} = \left.\frac{d^h}{dt^h}\Phi_2(t)\right|_{t=0}, \qquad \text{for } h = 1,2,3. \tag{18}$$

However, given the infinite product representation of $\Phi_2(t)$ in (17), which would make difficult the computation of its numerical derivatives, we will use instead $\Phi_Z(t)/\Phi_1(t)$, with $\Phi_Z(t)$ given by (3). Then the system of equations in (18) has the solutions

$$r = \frac{-4(\mu_1^2-\mu_2)^3}{(\mu^*)^2}, \qquad \lambda = \frac{-2(\mu_1^2-\mu_2)}{\mu^*}, \qquad a^* = \frac{\mu_1^2\mu_2-2\mu_2^2+\mu_1\mu_3}{\mu^*}$$

where

$$\mu_h = \mathrm{i}^{-h} \frac{d^h}{dt^h} \Phi_2(t)\bigg|_{t=0} \quad (h = 1, 2, 3) \quad \text{and} \quad \mu^* = 2\mu_1^3 - 3\mu_1\mu_2 + \mu_3 \,.$$

In this way the near-exact distribution obtained for Z will be a shifted GNIG distribution, which in the 'simple case' treated in subsection 2.1, will have depth $1 + p(n^* + 1)$ with p.d.f. and c.d.f. respectively given by (see App. C for notation)

$$f^{\,GNIG}\Big(z - a^{**}\,\big|\,\underbrace{1, \ldots, 1}_{p(n^*+1)}, r; s_1, \ldots, s_{p(n^*+1)}, \lambda; 1 + p(n^* + 1)\Big)$$

$$= C\,\lambda^r \sum_{h=1}^{p(n^*+1)} \exp(-s_h(z - a^{**}))\, c_h \,\frac{\Gamma\,(r, (\lambda - s_h)(z - a^{**}))}{\Gamma(r)}$$

and

$$F^{\,GNIG}\Big(z - a^{**}\,\big|\,\underbrace{1, \ldots, 1}_{p(n^*+1)}, r; s_1, \ldots, s_{p(n^*+1)}, \lambda; 1 + p(n^* + 1)\Big)$$

$$= \lambda^r \,\frac{\Gamma\,(r, \lambda(z - a^{**}))}{\Gamma(r)} - C\,\lambda^r \sum_{h=1}^{p(n^*+1)} \exp(-s_h(z - a^{**}))\, c_h \,\frac{\Gamma\,(r, (\lambda - s_h)(z - a^{**}))}{\Gamma(r)}$$

where $z\,(> 0)$ represents the running value of Z, s_h are given by (6), C and c_h ($h = 1, \ldots, p(n^* + 1)$) are given by (8), with K replaced by n^*, and $a^{**} = a + a^*$ where a is given by (8), with K replaced by n^*, and where we used the relation

$${}_1F_1(r, r + 1; -z) = \frac{r}{z^r}\,\Gamma(r, z)\,,$$

between the Kummer hypergeometric function and the incomplete Gamma function, with

$$\Gamma(r, z) = \int_0^z \exp(-x)\, x^{r-1}\, \mathrm{d}x\,.$$

Consequently $W = \exp(-Z)$ will have near-exact p.d.f. and c.d.f. respectively given by

$$f^{\,GNIG}\Big(-\log\, w - a^{**}\,\big|\,\underbrace{1, \ldots, 1}_{p(n^*+1)}, r; s_1, \ldots, s_{p(n^*+1)}, \lambda; 1 + p(n^* + 1)\Big)\frac{1}{w}$$

$$= C\,\lambda^r \sum_{h=1}^{p(n^*+1)} w^{s_h - 1}\, \exp(s_h a^{**})\, c_h \,\frac{\Gamma\,(r, (\lambda - s_h)(-\log\, w - a^{**}))}{\Gamma(r)}$$

and

$$1 - F^{\,GNIG}\Big(-\log\, w - a^{**}\,\big|\,\underbrace{1, \ldots, 1}_{p(n^*+1)}, r; s_1, \ldots, s_{p(n^*+1)}, \lambda; 1 + p(n^* + 1)\Big)$$

$$= 1 - \lambda^r \,\frac{\Gamma\,(r, \lambda(-\log\, w - a^{**}))}{\Gamma(r)} + C\,\lambda^r \sum_{h=1}^{p(n^*+1)} w^{s_h - 1}\, \exp(s_h a^{**})\, c_h \,\frac{\Gamma\,(r, (\lambda - s_h)(-\log\, w - a^{**}))}{\Gamma(r)}$$

For the general case, where some of the rate parameters may be equal, the near-exact distributions for $Z = -\log\, W$ will be GNIG distributions with depth smaller or equal to $1 + p(n^* + 1)$, with some of the integer shape parameters being larger than one.

For the negative logarithm of the generalized variance we will have as a near-exact distribution a shifted GNIG distribution of depth $1 + 2n^* + p$ with p.d.f.

$$f^{\,GNIG}\left(z - a^{**}\,\Big|\,u_1, \ldots, u_{2n^*+p}, r; \frac{n - p}{2}, \ldots, \frac{n - 1 + 2n^*}{2}, \lambda; 1 + 2n^* + p\right)$$

and c.d.f.

$$F^{GNIG}\left(z-a^{**}\,\middle|\,u_1,\ldots,u_{2n^*+p},r;\frac{n-p}{2},\ldots,\frac{n-1+2n^*}{2},\lambda;1+2n^*+p\right)$$

while the generalized variance itself will have as its near-exact p.d.f.

$$f^{GNIG}\left(-\log\,w-a^{**}\,\middle|\,u_1,\ldots,u_{2n^*+p},r;\frac{n-p}{2},\ldots,\frac{n-1+2n^*}{2},\lambda;1+2n^*+p\right)\frac{1}{w}$$

and c.d.f.

$$1-F^{GNIG}\left(-\log\,w-a^{**}\,\middle|\,u_1,\ldots,u_{2n^*+p},r;\frac{n-p}{2},\ldots,\frac{n-1+2n^*}{2},\lambda;1+2n^*+p\right).$$

4. Numerical Studies

The aim of this section is to show, through some simple examples, the behavior of different truncations of the exact distribution and to compare it with the behavior of the near-exact distributions. This will show the usefulness of these near-exact distributions in terms of precision. Moreover they also end up being simpler and more efficient in terms of computing time, given the fact that in order to attain a given precision they need to use less terms.

A precise assessment of the closeness of each of the approximations presented to the exact distribution may be done through the use of the measures

$$\Delta_1=\frac{1}{2\pi}\int_{-\infty}^{+\infty}|\Phi_Z(t)-\Phi_n(t)|\,dt \qquad\text{and}\qquad \Delta_2=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\left|\frac{\Phi_Z(t)-\Phi_n(t)}{t}\right|dt\,, \tag{19}$$

where $\Phi_Z(t)$ and $\Phi_n(t)$ represent respectively the exact c.f. of Z and the c.f. corresponding to the approximation being considered. We should note that we have

$$\Delta_1\geq\max_{z\in\mathbb{R}}|f_Z(t)-f_n(t)| \qquad\text{and}\qquad \Delta_2\geq\max_{z\in\mathbb{R}}|F_Z(t)-F_n(t)| \tag{20}$$

where $f_Z(t)$ and $F_Z(t)$ represent respectively the exact p.d.f. and c.d.f. of Z and $f_n(t)$ and $F_n(t)$ represent the p.d.f. and c.d.f. that corresponds to the c.f. $\Phi_n(t)$. We should note that for Δ_2 it is possible to write for $W=\exp(-Z)$ a similar inequality to the one found in (20).

Further, if in (19) we take $n\equiv K$ and Z_K as the r.v. with c.f. $\Phi_K(t)$, we also have

$$\Delta_1 \underset{K\to\infty}{\longrightarrow} 0 \iff \Delta_2 \underset{K\to\infty}{\longrightarrow} 0 \iff Z_K \xrightarrow[K\to\infty]{d} Z\,.$$

4.1. A first example

In this first example we consider the product of three generalized Gamma distributions which are (see App. A for notation) respectively:

$$\Gamma\left(\frac{5}{6},\frac{10}{67};\frac{1}{5}\right),\quad \Gamma\left(\frac{5}{2},\frac{5}{28};1\right)\quad\text{and}\quad \Gamma\left(\frac{19}{5},\frac{5}{17};3\right).$$

In Table 1 we have the values of Δ_1 and Δ_2 for different truncations of the exact distribution and also near-exact distributions corresponding also to different numbers of exact terms kept in the distribution.

Table 1 – Values of Δ_1 and Δ_2 for different truncations of the exact distribution for Example 1, and also near-exact distributions with different numbers of exact terms kept.

	Δ_1	Δ_2
truncation of exact, $K=10$	1.70×10^{-2}	6.84×10^{-2}
truncation of exact, $K=20$	8.58×10^{-3}	7.95×10^{-2}
truncation of exact, $K=50$	3.45×10^{-3}	1.70×10^{-2}
truncation of exact, $K=100$	1.73×10^{-3}	8.58×10^{-3}
truncation of exact, $K=200$	8.64×10^{-4}	4.31×10^{-3}
near-exact, $n^*=0$	1.59×10^{-3}	3.17×10^{-3}
near-exact, $n^*=1$	5.76×10^{-4}	1.11×10^{-3}
near-exact, $n^*=2$	2.68×10^{-4}	5.01×10^{-4}
near-exact, $n^*=5$	5.51×10^{-5}	9.92×10^{-5}
near-exact, $n^*=10$	1.12×10^{-5}	1.98×10^{-5}
near-exact, $n^*=20$	1.81×10^{-6}	3.18×10^{-6}
near-exact, $n^*=50$	1.36×10^{-7}	2.38×10^{-7}

We may see that the near-exact distributions using truncations of the exact distribution with just a few terms, display a consistently high closeness to the exact distribution even higher than truncations of the exact distribution based on a few hundred terms. This may be easily seen by comparing for example the values of the measures Δ_1 and Δ_2 for the near-exact distribution with $n^* = 5$, where 6 exact terms are kept, and values of the same measures for the truncation of the exact with $K = 200$, where 201 exact terms are kept, with smaller values of Δ_1 and Δ_2 indicating a better performance, that is, a higher closeness to the exact distribution.

4.2. A second example

In this second example we consider the distribution of the product of only two generalized Gamma r.v.'s, both with a negative power parameter. The two r.v.'s considered have distributions

$$\Gamma\left(\frac{6}{5}, \frac{5}{17}; -1\right) \quad \text{and} \quad \Gamma\left(\frac{3}{10}, \frac{5}{28}; -1\right).$$

As it was done in Example 1, in Table 2 we have the values of Δ_1 and Δ_2 for different truncations of the exact distribution and also near-exact distributions corresponding also to different numbers of exact terms kept in the distribution.

Table 2 – Values of Δ_1 and Δ_2 for different truncations of the exact distribution for Example 2, and also near-exact distributions with different numbers of exact terms kept.

	Δ_1	Δ_2
truncation of exact, $K = 5$	2.01×10^{-2}	4.36×10^{-2}
truncation of exact, $K = 10$	1.05×10^{-2}	2.34×10^{-2}
truncation of exact, $K = 20$	5.35×10^{-3}	1.22×10^{-2}
truncation of exact, $K = 50$	2.17×10^{-3}	4.98×10^{-3}
near-exact, $n^* = 0$	1.27×10^{-3}	1.02×10^{-3}
near-exact, $n^* = 2$	8.43×10^{-5}	6.20×10^{-5}
near-exact, $n^* = 5$	1.22×10^{-5}	8.72×10^{-6}
near-exact, $n^* = 10$	2.10×10^{-6}	1.49×10^{-6}
near-exact, $n^* = 20$	3.12×10^{-7}	2.21×10^{-7}
near-exact, $n^* = 50$	2.22×10^{-8}	1.58×10^{-8}

Similar conclusions to those drawn for Example 1, may be drawn from the values in Table 2. Once again the near-exact distributions, using truncations of the exact distribution with just a few terms, display a consistently much higher closeness to the exact distribution. For example, the near-exact distribution based on just one exact term kept ($n^* = 0$) displays a better performance than the truncation with $K = 50$, where 51 exact terms are kept.

4.3. Generalized Variance

The aim of this subsection is to compare the performance of the near-exact distributions for the generalized variance, with that of other available approximations, namely Hoel's (1937) generalized gamma approximation, the Normal approximation from Gnanadesikan and Gupta (1970), Steyn (1978) Gamma approximation and the saddle-point approximation from Butler et al. (1992).

Since the saddle-point approximations only produce approximations for probabilities but do not yield an explicit distribution, we will not be able to use the measures Δ_1 and Δ_2. Instead we consider the computed quantiles for the different distributions. Then, by observing the evolution of these quantiles for different values of n^* for the near-exact distributions, we will be able to draw our conclusions.

In order to keep the values in Tables 3-5 within more conformable ranges, we decided to report the quantiles for the logarithm of the generalized variance, instead of for the generalized variance itself. All cases refer to situations where $\Sigma = I_p$.

From the results in Tables 3-5 we may see that Steyn's approximation has the poorest performance among all approximations. This is mainly due to the fact that it does not even yield a legitimate distribution for most situations. Although the function used to approximate the exact p.d.f. integrates to one, this function yields negative values for values of the running variable smaller than $(p-1)(p-2)(p/2(n-p)-1)/(4+(p-1)(p-2))$ and the values it gives for the approximate quantiles illustrate well the dangers we run when using approximations which may have not been well enough tested.

Table 3 – Quantiles for the logarithm of the generalized variance from the saddlepoint approximation and the near-exact distributions, for $p = 3$ different values of n.

	quantile			
	0.01	0.05	0.95	0.99
	$p = 3,\ n = 5$			
near-exact, $n^* = 0$	-3.0521739	-1.2479211	4.5815460	5.3958398
near-exact, $n^* = 5$	-3.0587769	-1.2478629	4.5813646	5.4110883
near-exact, $n^* = 10$	-3.0588701	-1.2478867	4.5812834	5.4113506
near-exact, $n^* = 20$	-3.0588883	-1.2478920	4.5812628	5.4114002
Hoel approx.	-3.4832801	-1.5970624	4.5263934	5.3935665
GG approx.	-2.2160081	-0.9922239	4.9152242	6.1390085
Steyn approx.	-0.7061300	-0.1590762	4.5010365	5.3177167
saddlepoint approx.	-3.0288383	-1.2220124	4.5837180	5.4129775
	$p = 3,\ n = 15$			
near-exact, $n^* = 0$	5.6811710	6.2416809	8.5663441	8.9760832
near-exact, $n^* = 5$	5.6792558	6.2421076	8.5664171	8.9788287
near-exact, $n^* = 10$	5.6789338	6.2421355	8.5663906	8.9792881
near-exact, $n^* = 20$	5.6788034	6.2421396	8.5663699	8.9794757
near-exact, $n^* = 50$	5.6787652	6.2421392	8.5663608	8.9795309
Hoel approx.	5.6429881	6.2144716	8.5685423	8.9863436
GG approx.	5.8043835	6.2868278	8.6156819	9.0981263
Steyn approx.	5.7597470	6.2866272	8.5533775	8.9633926
saddlepoint approx.	5.6792283	6.2425018	8.5664826	8.9796382
	$p = 3,\ n = 50$			
near-exact, $n^* = 0$	10.6877414	10.9506393	12.1256135	12.3501266
near-exact, $n^* = 5$	10.6875428	10.9506779	12.1256137	12.3503674
near-exact, $n^* = 10$	10.6874412	10.9506940	12.1256111	12.3504898
near-exact, $n^* = 20$	10.6873498	10.9507056	12.1256062	12.3505997
near-exact, $n^* = 50$	10.6872867	10.9507113	12.1256005	12.3506755
Hoel approx.	10.6832619	10.9476895	12.1266612	12.3524838
GG approx.	10.7189618	10.9624745	12.1379585	12.3814712
Steyn approx.	10.6940280	10.9549954	12.1233216	12.3477147
saddlepoint approx.	10.6872896	10.9507300	12.1256085	12.3507054

Hoel's approximation, although yielding the exact distribution for $p = 1$ and $p = 2$, has for $p > 5$ the problem of being only defined for values of n larger than $(p-1)(p-2)/2$ and has a much poorer performance than the near-exact and the saddle-point approximations. Its performance, although improving for larger sample sizes, worsens as p increases, exactly as Gnanadesikan and Gupta (1970) noticed. The Normal approximation suggested by these authors, seems to only start to show a better performance than Hoel's approximation for values of p larger than 10, as the authors admit in their paper.

The saddle-point approximations have very good performances, but they fall quite short of the near-exact approximations in situations with very low sample sizes, namely when the number of variables involved gets larger and larger.

Table 4 – Quantiles for the logarithm of the generalized variance from the saddlepoint approximation and the near-exact distributions, for $p = 5$ different values of n.

	quantile			
	0.01	0.05	0.95	0.99
	$p = 5, \quad n = 7$			
near-exact, $n^* = 0$	-0.4801207	1.4091537	8.0218560	9.0409234
near-exact, $n^* = 5$	-0.4862904	1.4094239	8.0213513	9.0524435
near-exact, $n^* = 10$	-0.4863935	1.4094062	8.0212971	9.0526637
near-exact, $n^* = 20$	-0.4864151	1.4094018	8.0212827	9.0527095
Hoel approx.	-1.4047362	0.7550038	8.4351485	9.6214754
GG approx.	0.2575208	1.6391169	8.3083541	9.6899502
Steyn approx.	3.3823721	3.5253914	7.3439747	8.3334331
saddlepoint approx.	-0.4525408	1.4352117	8.0233086	9.0540166
	$p = 5, \quad n = 17$			
near-exact, $n^* = 0$	10.6338788	11.3072078	14.2038712	14.7357653
near-exact, $n^* = 5$	10.6324108	11.3074896	14.2038788	14.7376994
near-exact, $n^* = 10$	10.6321564	11.3075132	14.2038576	14.7380329
near-exact, $n^* = 20$	10.6320497	11.3075185	14.2038430	14.7381732
Hoel approx.	10.5078130	11.2159371	14.2405514	14.7963241
GG approx.	10.7494529	11.3501372	14.2497592	14.8504435
Steyn approx.	11.4471621	11.6938959	14.0706173	14.5855840
saddlepoint approx.	10.6322552	11.3077130	14.2039141	14.7382874
	$p = 5, \quad n = 52$			
near-exact, $n^* = 0$	18.2613579	18.5914905	20.0931899	20.3856881
near-exact, $n^* = 5$	18.2612049	18.5915161	20.0931867	20.3858654
near-exact, $n^* = 10$	18.2611264	18.5915272	20.0931835	20.3859560
near-exact, $n^* = 20$	18.2610555	18.5915356	20.0931792	20.3860379
near-exact, $n^* = 50$	18.2610059	18.5915401	20.0931747	20.3860949
Hoel approx.	18.2449403	18.5798109	20.1000075	20.3963598
GG approx.	18.2919839	18.6031546	20.1052371	20.4164078
Steyn approx.	18.3620914	18.6444261	20.0674678	20.3545597
saddlepoint approx.	18.2610030	18.5915507	20.0931794	20.3861165

Table 5 – Quantiles for the logarithm of the generalized variance from the saddlepoint approximation and the near-exact distributions, for $p = 10$ different values of n.

	quantile			
	0.01	0.05	0.95	0.99
	$p = 10, \quad n = 12$			
near-exact, $n^* = 0$	9.2827165	11.2891907	18.8907271	20.1713578
near-exact, $n^* = 5$	9.2769860	11.2896137	18.8905533	20.1801930
near-exact, $n^* = 10$	9.2768743	11.2896027	18.8905181	20.1803805
near-exact, $n^* = 20$	9.2768485	11.2895996	18.8905081	20.1804240
Hoel approx.	——	——	——	——
GG approx.	9.9021214	11.4848224	19.1248326	20.7075336
Steyn approx.	10.2389009	10.3198104	13.9855127	15.2587289
saddlepoint approx.	9.3117863	11.3132303	18.8921875	20.1815137
	$p = 10, \quad n = 22$			
near-exact, $n^* = 0$	24.4487928	25.2998417	29.0995616	29.8259320
near-exact, $n^* = 5$	24.4477940	25.3000060	29.0995401	29.8271375
near-exact, $n^* = 10$	24.4476107	25.3000240	29.0995254	29.8273577
near-exact, $n^* = 20$	24.4475294	25.3000296	29.0995161	29.8274555
Hoel approx.	——	——	——	——
GG approx.	24.5506559	25.3381685	29.1396473	29.9271599
Steyn approx.	25.5936768	25.6532168	27.7571480	28.4201865
saddlepoint approx.	24.4476093	25.3001215	29.0995533	29.8275278
	$p = 10, \quad n = 57$			
near-exact, $n^* = 0$	37.7086118	38.1562247	40.2294673	40.6411330
near-exact, $n^* = 5$	37.7085088	38.1562393	40.2294630	40.6412470
near-exact, $n^* = 10$	37.7084553	38.1562460	40.2294601	40.6413062
near-exact, $n^* = 20$	37.7084060	38.1562514	40.2294567	40.6413606
Hoel approx.	37.9279652	38.3926408	40.5405651	40.9667876
GG approx.	37.7378794	38.1674238	40.2409193	40.6704637
Steyn approx.	38.4533016	38.5158801	39.8996139	40.2776343
saddlepoint approx.	37.7083652	38.1562596	40.2294556	40.6414137

5. Conclusions

The exact distribution of the product of independent generalized Gamma r.v.'s was obtained in a form which not only embraces the distribution of the product of any r.v.'s whose distributions are particular cases of the generalized Gamma distribution, as well as any possible combination of such r.v.'s. Moreover, its form enabled us to be able to develop very well-performing near-exact approximations. Indeed, by considering negative or symmetric power parameters for the generalized Gamma r.v.'s we will also have the distribution of ratios of generalized Gamma r.v.'s.

Since simple truncations of the exact distribution, even when a very large number of terms is retained, do not usually exhibit adequate precision, these near-exact distributions emerge as very good approximations to the exact distribution. They are particularly suited for the computation of near-exact quantiles, to be used as replacement of the exact quantiles. Moreover, the near-exact distributions are easier to compute than simple truncations of the exact distribution, since for a considerably reduced number of terms they provide much better approximations.

Acknowledgment

This research was partially financially supported by CMA/FCT/UNL, under the project PEst-OE/MAT/ UI0297/ 2011.

Appendix A

The generalized Gamma distribution

Let X be a r.v. with a Gamma distribution with shape parameter r and rate parameter λ, that is, let X be a r.v. as in (22) and let

$$Y = X^{1/\beta}\,, \quad \text{for } \beta \in \mathbb{R} \setminus \{0\}\,.$$

We will say that the r.v. Y has a generalized Gamma distribution, with shape parameter r, rate parameter λ and power parameter β and we will denote this fact by

$$Y \sim \Gamma(r, \lambda; \beta)\,. \tag{21}$$

The r.v. Y has p.d.f

$$f_Y(y) = \frac{|\beta|\,\lambda^r}{\Gamma(r)}\,\exp(-\lambda y^\beta)\,y^{\beta r-1}\,, \quad (y > 0)$$

and its h-th moment is given by

$$E\left(Y^h\right) = E\left(X^{h/\beta}\right) = \frac{\Gamma(r + h/\beta)}{\Gamma(r)}\,\lambda^{-h/\beta}\,, \quad (h > -\beta r)\,.$$

This distribution has as particular cases several well-known distributions, which are listed in Table A.1.

A r.v. with a similar distribution was studied by Stacy (1962).

Table A.1 – Particular cases of the generalized Gamma distribution

parameters values	distribution	p.d.f.
$\beta > 0,\, r = 1$	Weibull	$f(x) = \beta\lambda\, e^{-\lambda x^{\beta}}\, x^{\beta-1}$
$\beta = 2,\ r = 1\ \lambda = 1/n\,(n \in \mathbb{N})$	Rayleigh	$f(x) = \frac{2}{n}\, e^{-\frac{x^2}{n}}\, x$
$\beta = 2,\ r = 1$	generalized Rayleigh	$f(x) = 2\lambda\, e^{-\lambda x^2}\, x$
$\beta = 2,\ r = 3/2$	Maxwell-Boltzmann	$f(x) = 4\frac{\lambda^{3/2}}{\sqrt{\pi}}\, e^{-\lambda x^{3/2}}\, x^2$
$\beta = 2,\ r = 1/2$	Half-Normal	$f(x) = 2\sqrt{\frac{\lambda}{\pi}}\, e^{-\lambda x^2}$
$\beta = 2,\ \lambda = 1/2,\ r = k/2\,(k \in \mathbb{N})$	Chi	$f(x) = \left(\frac{1}{2}\right)^{k/2-1} \frac{e^{-\frac{1}{2}x^2}}{\sqrt{\pi}}\, x^{k-1}$
$\beta < 0,\, r = 1$	Fréchet	$f(x) = -\beta\lambda\, e^{-\lambda x^{\beta}}\, x^{\beta-1}$
$\beta < 0,\, r = n\,(\in \mathbb{N})$	generalized Fréchet	$f(x) = \frac{-\beta\lambda^n}{(n-1)!}\, e^{-\lambda x^{\beta}}\, x^{n\beta-1}$
$\beta = -1$	inverse Gamma	$f(x) = \frac{\lambda^r}{\Gamma(r)}\, e^{-\frac{\lambda}{x}}\, x^{-r-1}$
$\beta = -1,\, r = 1/2$	Lévy≡Stable$(1/2, 1, 2\lambda, 0)$	$f(x) = \frac{\lambda^{1/2}}{\sqrt{\pi}}\, e^{-\frac{\lambda}{x}}\, x^{-3/2}$
$\beta = 1$	Gamma	$f(x) = \frac{\lambda^r}{\Gamma(r)}\, e^{-\lambda x}\, x^{r-1}$
$\beta = 1,\ \lambda = 1/2,\ r = k/2\,(k \in \mathbb{N})$	Chi-square	$f(x) = \frac{\left(\frac{1}{2}\right)^{k/2}}{\Gamma(k/2)}\, e^{-\frac{x}{2}}\, x^{k/2-1}$
$\beta = 1,\ r = 1$	Exponential	$f(x) = \lambda\, e^{-\lambda x}$

We should note that the Generalized Gamma distribution has some very interesting relations with other important distributions.

For $\alpha \in \mathbb{R} \setminus \{0\}$, any α-power of the absolute value of a Normal random variable with null expected value has a Generalized Gamma distribution. More precisely,

$$Y \sim N(0, \sigma^2) \implies |Y|^{\alpha} \sim \Gamma\left(\frac{1}{2}, \frac{1}{2\sigma^2}, \frac{2}{\alpha}\right), \quad \text{for} \quad \alpha \in \mathbb{R} \setminus \{0\},$$

which for $\alpha = 1$ yields the Half-Normal distribution (see Table A.1).

Also, there is an important relation between the Generalized Gamma distribution and the Gumbel distribution. Let Y have a Generalized Gamma distribution as in (21) and consider the r.v.

$$W = a - \log(Y).$$

The r.v. W has p.d.f.

$$f_W(w) = \frac{|\beta|\lambda^r}{\Gamma(r)} \exp\{-\lambda \exp(-\beta(w-a))\} \exp\{-\beta r(w-a)\}, \quad w \in \mathbb{R}, \beta \in \mathbb{R} \setminus \{0\}; r, \lambda > 0$$

which shows that W has a Generalized Gumbel distribution, since for $r = 1$ and $\lambda = 1$ we have

$$f_W(w) = |\beta| \exp\{-\exp\{-\beta(w-a)\}\} \exp\{-\beta(w-a)\}, \quad w \in \mathbb{R}$$

which for $\beta > 0$ is the p.d.f. of a Gumbel distribution with c.d.f.

$$F_W(w) = \exp\{-\exp\{-\beta(w-a)\}\}, \quad w \in \mathbb{R}.$$

Appendix B

The complete sufficient statistic for the shape parameter (r) in a Gamma distribution with unit or known rate parameter and the UMVU estimator of $\Psi(r)$

Let $X_1, \dots, X_n$ be i.i.d. r.v.'s with $X_i \sim \Gamma(r, \lambda)$, that is, let X_i $(i = 1, \dots, n)$ have p.d.f.

$$f_{X_i}(x) = \frac{\lambda^r}{\Gamma(r)} \exp(-\lambda x)\, x^{r-1} ,$$

where λ is assumed known.

Then the likelihood function is

$$L(r; x_1, x_2, \dots, x_n) = \frac{\lambda^{rn}}{(\Gamma(r))^n} \exp\left(-\lambda \sum_{i=1}^{n} x_i\right)\left(\prod_{i=1}^{n} x_i\right)^{r-1}$$

and the log-likelihood is

$$\mathcal{L}(r; x_1, x_2, \dots, x_n) = nr \log\, \lambda - n \log \Gamma(r) - \lambda \sum_{i=1}^{n} x_i + (r-1) \sum_{i=1}^{n} \log\, x_i ,$$

with

$$\frac{\partial \mathcal{L}}{\partial r} = n \log\, \lambda - n\Psi(r) + \sum_{i=1}^{n} \log\, x_i ,$$

where $\Psi(\,\cdot\,)$ is the digamma function. Consequently the MLE of $\Psi(r)$ is given by

$$\widehat{\Psi(r)} = \frac{1}{n} \sum_{i=1}^{n} \log\, X_i + \log\, \lambda ,$$

and the MLE of $\exp(\Psi(r))$ is

$$\widehat{\exp(\Psi(r))} = \lambda \prod_{i=1}^{n} X_i^{1/n} .$$

On the other hand we may write

$$f_{X_i}(x) = \exp\left(-\lambda x + (r-1) \log\, x\right) \frac{\lambda^r}{\Gamma(r)} ,$$

which shows that $\sum_{i=1}^{n} \log X_i$ is the complete sufficient statistic for r.

Since $E(\log X_i) = \Psi(r) - \log\, \lambda$, we have $E\left[\widehat{\Psi(r)}\right] = \Psi(r)$, so that the MLE, being unbiased and a function of the complete sufficient statistic is also UMVU.

Appendix C

The shifted Gamma, GIG (Generalized Integer Gamma) and GNIG (Generalized Near-Integer Gamma) distributions

We will use this Appendix to establish some notation concerning distributions used in the paper, as well as to give the expressions for the p.d.f.'s (probability density functions) and c.d.f.'s (cumulative distribution functions) of the GIG (Generalized Integer Gamma) and GNIG (Generalized Near-Integer Gamma) distributions.

C.1 Unshifted versions

We will say that the r.v. X has a Gamma distribution with rate parameter $\lambda > 0$ and shape parameter $r > 0$, if its p.d.f. may be written as

$$f_X(x) = \frac{\lambda^r}{\Gamma(r)}\, \exp(-\lambda x)\, x^{r-1}\,, \quad (x > 0)$$

and we will denote this fact by

$$X \sim \Gamma(r, \lambda)\,. \tag{22}$$

We may note that the c.f. (characteristic function) of X is

$$\Phi_X(t) = \lambda^r\, (\lambda - \mathrm{i}t)^{-r}\,,$$

and that

$$\Gamma(1, \lambda) \equiv Exp(\lambda)\,.$$

Let

$$X_j \;\sim\; \Gamma(r_j, \lambda_j) \qquad j = 1, \ldots, p$$

be p independent r.v.'s with Gamma distributions with shape parameters $r_j \in \mathbb{N}$ and rate parameters $\lambda_j > 0$, with $\lambda_j \neq \lambda_{j'}$, for all $j, j' \in \{1, \ldots, p\}$. We will say then that the r.v.

$$Y \;=\; \sum_{j=1}^{p} X_j$$

has a GIG (Generalized Integer Gamma) distribution of depth p, with shape parameters r_j and rate parameters λ_j, $(j = 1, \ldots, p)$, and we will denote this fact by

$$Y \;\sim\; GIG(r_j, \lambda_j;\; p)\,.$$

The p.d.f. and c.d.f. (cumulative distribution function) of Y are respectively given by (Coelho, 1998)

$$f_Y(y) = f^{GIG}(y|r_1, \ldots, r_p; \lambda_1, \ldots, \lambda_p; p) \;=\; C \sum_{j=1}^{p} P_j(y)\, \exp(-\lambda_j\, y)\,, \qquad (y > 0) \tag{23}$$

and

$$F_Y(y) = F^{GIG}(y|r_1, \ldots, r_j; \lambda_1, \ldots, \lambda_p; p) \;=\; 1 - C \sum_{j=1}^{p} P_j^*(y)\, \exp(-\lambda_j\, y)\,, \qquad (y > 0) \tag{24}$$

where

$$C = \prod_{j=1}^{p} \lambda_j^{r_j}\,, \qquad P_j(y) = \sum_{k=1}^{r_j} c_{j,k}\, y^{k-1} \tag{25}$$

and

$$P_j^*(y) = \sum_{k=1}^{r_j} c_{j,k}\, (k-1)! \sum_{i=0}^{k-1} \frac{y^i}{i!\, \lambda_j^{k-i}}$$

with

$$c_{j,r_j} = \frac{1}{(r_j - 1)!} \prod_{\substack{i=1\\ i\neq j}}^{p} (\lambda_i - \lambda_j)^{-r_i}\,, \qquad j = 1, \ldots, p\,, \tag{26}$$

and

$$c_{j,r_j-k} = \frac{1}{k} \sum_{i=1}^{k} \frac{(r_j - k + i - 1)!}{(r_j - k - 1)!}\, R(i, j, p)\, c_{j,r_j-(k-i)}\,, \qquad (k = 1, \ldots, r_j - 1;\; j = 1, \ldots, p) \tag{27}$$

where

$$R(i,j,p) = \sum_{\substack{k=1\\k\neq j}}^{p} r_k \left(\lambda_j - \lambda_k\right)^{-i} \quad (i = 1,\ldots,r_j - 1)\,. \tag{28}$$

The GNIG (Generalized Near-Integer Gamma) distribution of depth $p+1$ (Coelho, 2004) is the distribution of the r.v.

$$Z = Y_1 + Y_2$$

where Y_1 and Y_2 are independent, Y_1 having a GIG distribution of depth p and Y_2 with a Gamma distribution with a non-integer shape parameter r and a rate parameter $\lambda \neq \lambda_j$ $(j = 1,\ldots,p)$. The p.d.f. (probability density function) of Z is given by

$$f^{GNIG}(z|r_1,\ldots,r_p,r;\,\lambda_1,\ldots,\lambda_p,\lambda;p+1) =$$
$$C\lambda^r \sum_{j=1}^{p} \exp(-\lambda_j z) \sum_{k=1}^{r_j} \left\{ c_{j,k} \frac{\Gamma(k)}{\Gamma(k+r)} z^{k+r-1}\, {}_1F_1(r,k+r,-(\lambda-\lambda_j)z) \right\}, \quad (z>0) \tag{29}$$

and the c.d.f. (cumulative distribution function) given by

$$\begin{aligned} F^{GNIG}(z|r_1,\ldots,r_p,r;\,\lambda_1,\ldots,\lambda_p,\lambda;p+1) &= \frac{\lambda^r z^r}{\Gamma(r+1)}\, {}_1F_1(r,r+1,-\lambda z) \\ &-C\lambda^r \sum_{j=1}^{p} \exp(-\lambda_j z) \sum_{k=1}^{r_j} c^*_{j,k} \sum_{i=0}^{k-1} \frac{z^{r+i}\lambda_j^i}{\Gamma(r+1+i)}\, {}_1F_1(r,r+1+i,-(\lambda-\lambda_j)z) \quad (z>0) \end{aligned} \tag{30}$$

where C is still given by (25) and

$$c^*_{j,k} = \frac{c_{j,k}}{\lambda_j^k}\Gamma(k)$$

with $c_{j,k}$ given by (26) through (28) above. In the above expressions ${}_1F_1(a,b;z)$ is the Kummer confluent hypergeometric function. This function has usually very good convergence properties and is nowadays easily handled by a number of software packages.

C.2 Shifted versions

We will say that the r.v. $X^* = X+a$ has a shifted Gamma distribution (or a three parameter Gamma distribution) if the r.v. X has the Gamma distribution in (22). The r.v. X^* has p.d.f.

$$f_{X^*}(x) = \frac{\lambda^r}{\Gamma(r)} \exp\left(-\lambda(x-a)\right)\,(x-a)^{r-1}\,, \quad (x>a)$$

and c.f.

$$\Phi_{X^*}(t) = E\left(\exp\{\mathrm{i}t(X+a)\}\right) = \exp(\mathrm{i}ta)\; E\left(\exp(\mathrm{i}tX)\right) = \exp(\mathrm{i}ta)\;\Phi_X(t) = \exp(\mathrm{i}ta)\;\lambda^r\,(\lambda-\mathrm{i}t)^{-r}\,.$$

We will denote the fact that X^* has this shifted Gamma distribution by

$$X^* \sim \Gamma(r,\lambda\,|\,a)\,,$$

with

$$\Gamma(1,\lambda\,|\,a) \equiv Exp(\lambda\,|\,a)\,.$$

Let

$$X_j^* \sim \Gamma(r_j,\lambda_j\,|\,a_j) \qquad j = 1,\ldots,p$$

be p independent r.v.'s with shifted Gamma distributions with shape parameters $r_j \in \mathbb{N}$ and rate parameters $\lambda_j > 0$, with $\lambda_j \neq \lambda_{j'}$, for all $j, j' \in \{1,\ldots,p\}$, and let

$$Y^* = \sum_{j=1}^{p} X_j^*\,.$$

Then the c.f. of Y^* will be, for $a = \sum_{j=1}^{p} a_j$,

$$\Phi_{Y^*}(y) = \prod_{j=1}^{p} \exp(\mathrm{i}ta_j)\, \lambda_j^{r_j}\, (\lambda_j - \mathrm{i}t)^{-r_j} = \exp(\mathrm{i}ta) \prod_{j=1}^{p} \lambda_j^{r_j}\, (\lambda_j - \mathrm{i}t)^{-r_j}\,,$$

so that the distribution of Y^* will be what we call a shifted GIG distribution of depth p, with shape parameters r_j, rate parameters λ_j, $(j = 1, \ldots, p)$ and shift parameter $a = \sum_{j=1}^{p} a_j$, and we will denote this fact by

$$Y^* \sim GIG(r_j, \lambda_j; p\,|\,a)\,.$$

The p.d.f. and c.d.f. of Y^* are respectively given by

$$f_{Y^*}(y) = f^{GIG}(y - a\,|\,r_1, \ldots, r_p; \lambda_1, \ldots, \lambda_p; p)$$

and

$$F_{Y^*}(y) = F^{GIG}(y - a\,|\,r_1, \ldots, r_p; \lambda_1, \ldots, \lambda_p; p)\,.$$

If Y_1^* and Y_2^* are two independent r.v.'s, with

$$Y_1^* \sim GIG(r_j, \lambda_j; p\,|\,a_1) \qquad \text{and} \qquad Y_2^* \sim \Gamma(r, \lambda\,|\,a_2)\,,$$

then the distribution of

$$Z^* = Y_1^* + Y_2^*$$

will be what we call a shifted GNIG distribution of depth $p+1$, with rate parameters $\lambda_1, \ldots, \lambda_p$ and λ, shape parameters $r_1, \ldots, r_p$ and r and shift parameter $a = a_1 + a_2$. The r.v. Z^* will have c.f.

$$\Phi_{Z^*}(t) = \exp\{\mathrm{i}t(a_1 + a_2)\}\; \lambda^r\, (\lambda - \mathrm{i}t)^{-r} \prod_{j=1}^{p} \lambda_j^{r_j}\, (\lambda_j - \mathrm{i}t)^{-r_j}\,,$$

and p.d.f. and c.d.f. respectively given by

$$f_{Z^*}(z) = f^{GNIG}(z - a\,|\,r_1, \ldots, r_p, r;\; \lambda_1, \ldots, \lambda_p, \lambda; p+1)$$

and

$$F_{Z^*}(z) = F^{GNIG}(z - a\,|\,r_1, \ldots, r_p, r;\; \lambda_1, \ldots, \lambda_p, \lambda; p+1)\,.$$

References

Amoroso, L. (1925). Ricerche intorno alla curva dei redditi. *Ann. Mat. Pura Appl.*, Ser. 4, 21:123-159.

Bagai, O. P. (1965). The Distribution of the Generalized Variance. *The Annals of Mathematical Statistics* 36:120-130.

Bartlett, M. S., Kendall, D. G. (1946). The statistical analysis of variance heterogeneity and the logarithmic transformation. *J. Roy. Stat. Soc. Suppl.* 8:128-138.

Bondesson, L. (1978). On infinite divisibility of powers of a gamma variable. *Scand. Actuar. J.* 61:48-61.

Butler, R. W., Huzurbazar, S., Booth, J. G. (1992). Saddlepoint approximations for the generalized variance and Wilks' statistic. *Biometrka* 79:157-169.

Carter, B. D., Springer, M. D. (1977). The Distribution of Products, Quotients and Powers of Independent H-Function Variates. *SIAM Journal on Applied Mathematics* 33:542-558.

Cetinkaya, C., Kanodia, V., Knightly, E. W. (2001). Scalable services via egress admission control. *IEEE Transactions on Multimedia* 3:69-81.

Coelho, C. A. (1998). The Generalized Integer Gamma distribution – a basis for distributions in Multivariate Statistics. *J. Multivariate Analysis* 64:86-102.

Coelho, C. A. (2004). The Generalized Near-Integer Gamma distribution: a basis for 'near-exact' approximations to the distributions of statistics which are the product of an odd number of independent Beta random variables. *Journal of Multivariate analysis* 89:191-218.

D'Addario, R. (1932). Intorno alla curva dei redditi di Amoroso. *Riv. Italiana Statist. Econ. Fnanza*, anno 4, No. 1.

Fox, C. (1961). The G and H-functions as symmetrical Fourier kernels. *Transactions of the American Mathematical Society* 98:395-429.

Gnanadesikan, M., Gupta, S. S. (1970). A selection procedure for multivariate Normal distributions in terms of the generalized variances. *Technometrics* 12:103-117.

Gordon, L. (1989). Bounds for the distribution of the generalized variance. *Annals of Statistics* 17:1684-1692.

Hao, J., Krishnamoorty, K. (2001). Inferences on a Normal covariance matrix and generalized variance with monotone missing data. *Journal of Multivariate Analysis* 78:62-82.

Hoel, P. G. (1937). A significance test for component analysis. *Annals of Mathematical Statistics* 8:149-158.

Kaluska, M., Krysicki, W. (1997). On decompositions of some random variables. *Metrika* 46:159-175.

Lang, S. (1999). *Complex Analysis*, 4th ed., Springer-Verlag, New York.

Loiaciga, H. A., Leipnik, R. B. (1999). Analysis of extreme hydrologic events with Gumbel distributions: marginal and additive cases. *Stochastic Environmental Research and Risk Environment* 13:251-259.

Lomnicki, Z. A. (1967). On the distribution of products of random variables. *Journal of the Royal Statistical Society*, Ser. B 29:51-524.

Lu,I.-L., Richards, D. (1993). Random discriminants. *The Annals of Statistics* 21:1982-2000.

Malik, H. J. (1968). Exact distribution of the product of independent generalized Gamma variables with the same shape parameter. *Annals of Mathematical Statistics* 39:1751-1752.

Mathai, A. M. (1972a). Products and ratios of generalized Gamma variates. *Skandinavisk Aktuarietidskrift*, 1972:193-198.

Mathai, A. M. (1972b). The exact non-central distribution of the generalized variance. *Annals of the Institute of Mathematical Statistics* 24:53-65.

Nadarajah, S. (2005). On the product and ratio of Laplace and Bessel random variables. *Journal of Applied Mathematics* 4:393-402.

Nadarajah, S. (2007). Linear combination of Gumbel random variables. *Stochastic Envirnmental Research and Rik Assessment* 21:283-286.

Nadarajah, S., Kotz, S. (2008). Comments on ”Scalable services via egress admission control”. *IEEE Transactions on Multimedia* 10:160-161.

Podolski, H. (1972). The distribution of a product of n independent random variables with generalized Gamma distribution. *Demonstratio Mathematica* IV:119-123.

Regier, M. H. (1976). Simplified selection procedures for multivariate Normal populations. *Technometrics* 18:483-489.

Salo, J., El-Sallabi, H. M., Vainikainen, P. (2006). The distribution of the product of independent Rayleigh random variables. *IEEE Transactions on Antennas and Propagation* 54:639-643.

Springer, M. D., Thompson, W. E. (1970). The Distribution of Products of Beta, Gamma and Gaussian Random Variables. *SIAM Journal on Applied Mathematics* 18:721-737.

Stacy, E. W. (1962). A generalization of the Gamma distribution. *Annals of Mathematical Statistics* 33:1187-1192.

Steyn, H. S. (1978). On approximations for the central and noncentral distribution of the generalized variance. *Journal of the American Statistical Association* 73:670-675.

Sugiura, N., Nagao, H. (1971). Asymptotic expansion of the distribution of the generalized variance for noncentral Wishart matrix, when $\Omega = O(n)$. *Annals of the Institute of Statistical Mathematics* 23:469-475.

Wells, W. T., Anderson, R. L., Cell, J. W. (1962). The Distribution of the Product of Two Central or Non-Central Chi-Square Variates. *The Annals of Mathematical Statistics* 33: 1016-1020.