Regularity of flows and optimal control of shear-thinning fluids

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Abstract

We study optimal control problems of systems describing the flow of incompressible shear-thinning fluids. Taking advantage of regularity properties of the flows, we derive necessary optimality conditions under a restriction on the optimal control.

Key words. Optimal control, steady Navier-Stokes equations, shear-thinning flows, strong solution, necessary optimality conditions.

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1 Introduction

This paper deals with optimal control problems associated with a viscous, incompressible fluid described by the following partial differential equations that generalize the Navier-Stokes system

$$\begin{cases} -\nabla\cdot(\tau(Dy)) + y\cdot\nabla y + \nabla\pi = u & \text{in } \Omega, \\ \nabla\cdot y = 0 & \text{in } \Omega, \\ y = 0 & \text{on } \Gamma, \end{cases} \tag{1.1}$$

where y is the velocity field, π is the pressure, τ is the extra stress tensor, $Dy = \frac{1}{2}\left(\nabla y + (\nabla y)^T\right)$ is the symmetric part of the velocity gradient ∇y, u is the given body force and $\Omega \subset \mathbb{R}^n$ ($n = 2$ or $n = 3$) is a bounded domain with boundary Γ. We assume that $\tau : \mathbb{S} \longrightarrow \mathbb{S}$ is a classical power law stress tensor of the form

$$\tau_R(\eta) = 2\nu\left(1 + |\eta|^2\right)^{\frac{\alpha-2}{2}}\eta \qquad \text{or} \qquad \tau_L(\eta) = 2\nu\left(1 + |\eta|\right)^{\alpha-2}\eta$$

where ν and α are positive constants. (Here $\mathbb{S}$ consists of all symetric $n\times n$-matrices.) We recall that a fluid is called shear-thickening if $\alpha > 2$ and shear-thinning if $1 < \alpha < 2$. For the special case $\tau(\eta) = 2\nu\eta$ ($\alpha = 2$), we recover the Navier-Stokes equation with viscosity coeficient $\nu > 0$.

The paper is concerned with the following optimal control problem

$$\begin{aligned} &\text{Minimize} \quad J(u,y) = \frac{1}{2}\int_\Omega |y - y_d|^2\, dx + \frac{\lambda}{2}\int_\Omega |u|^2\, dx \\ &\text{Subject to} \quad (u,y) \in U_{ad} \times W_0^{1,\alpha}(\Omega) \text{ satisfies (1.1) for some } \pi \in L^\alpha(\Omega) \end{aligned}$$

where y_d is some desired velocity field, λ is a positive constant, the set of admissible controls U_{ad} is a nonempty convex closed subset of $L^q(\Omega)$ with $q > n$ and $\frac{3n}{n+2} \le \alpha < 2$. Although the analysis of several results can be more general, in order to simplify the redaction, we will

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assume that $U_{ad} \subset \{v \in L^q(\Omega) \mid \|v\|_q \leq U\}$ for some $U > 0$. Throughout the paper, the optimal control problem corresponding to τ_R will be denoted by (P_R) and the optimal control problem corresponding to τ_L will be denoted by (P_L).

The partial differential equations describing the considered class of fluids were first proposed in [8], [9] and [10] as a modification of the Navier-Stokes system, and were similarly suggested in [11]. Existence of weak solutions in $W_0^{1,\alpha}(\Omega)$ was proved by both authors using compactness arguments and the theory of monotone operators for $\alpha \geq \frac{3n}{n+2}$.

This basic regularity may prove insufficient for deriving the necessary optimality conditions for control problems governed by these equations, especially when considering shear-thinning fluids. As a consequence of the combined effect of the convective term and the nonlinear stress tensor, the lack of regularity of the state variable creates some difficulties in connection with the local Lipschitz continuity (and thus with the Gâteaux differentiability) in adequate functional spaces of the control-to-state mapping and with the natural setting for the associated linearized equation and the adjoint state equation. These issues were overcome in the case of shear-thickening fluids treated in [2] and [4] by using a suitable functional setting involving weighted Sobolev spaces. The optimality conditions are obtained in both two-dimensional and three-dimensional cases, without assuming any further regularity on the state and without restraining the set of admissible controls. The only constraint concerns the optimal control.

The case of shear-thinning fluids is more delicate and the techniques in [2] and [4] do not apply. In [3], these difficulties were handled by introducing a family of smooth approximate control problems falling into the case $\alpha = 2$ and whose solutions converge towards a solution of the original problem. The properties of the approximate control-to-state mapping were carefully studied and the approximate optimality conditions established. Under a constraint concerning the size of the optimal control, the same that guarantees uniqueness of the corresponding state, the optimality conditions for the original problem were then established by passing to the limit. As expected, because of the reduced regularity of the corresponding state variable, the adjoint equation is to be understood in the sense of distribution and uniqueness of the adjoint state is not guaranteed.

These issues can be more easily managed if the velocity gradient is bounded. Nevertheless, despite the fact that system (1.1) was widely studied, higher global regularity of solutions is difficult to obtain in general and there are only few such results known up to nowadays. In the case of steady shear-thinning fluids and C^1 extra stress tensors, the most significant global regularity results up to the boundary have been obtained in [12] in the two-dimensional framework enabling the derivation of some optimality conditions in [14], though restricting all the admissible set of controls to guarantee uniqueness of the corresponding solution and differentiability of the control-to-state mapping.

In the present work we follow [1], where both two-dimensional and three-dimensional cases for C^1 and Lipschitz continuous extra stress tensors were treated, and identify a condition under which uniqueness and regularity of weak solutions are both guaranteed. Observing that the optimality conditions established in [3] apply to problem (P_R), we take advantage of the regularity results in [1] to prove that the corresponding adjoint equation can be interpreted in the weak sense and that the adjoint state is unique and more regular. This result is obtained under a restriction involving the L^q-norm of the optimal control, similar to the one imposed in [3].

Concerning problem (P_L), besides the difficulties induced by the nonlinearity of the extra stress tensor and the convective term, the non-regularity of the model has to be managed. Since τ_L is not differentiable at the origin, the optimality conditions of [3] cannot be used. To overcome this difficulty, we introduce a family of regularized problems that fall into the case of C^1 extra stress tensors. By exploiting some results of existence, uniqueness and regularity of solutions for the corresponding regularized state equation already established in [1], we derive the corresponding optimality systems and the optimality conditions for (P_L) are obtained by passing to the limit. A similar regularization approach was successfully used in [5] to study optimal control problems of systems governed by quasiliear elliptic equations with non differentiable coefficients at the origin and more recently in [6], for the treatment of a problem governed by the Bingham nonlinear mixed variational inequality.

The plan is as follows. Notation and some preliminary results are given in Section 2. Section 3 is devoted to existence, uniqueness and regularity results for the state equation and to the derivation of corresponding estimates. Section 4 deals with existence and uniqueness of weak solutions for the adjoint equation. In Section 5, we state and prove the necessary optimality conditions for problem (P_R). Section 6 is dedicated to the treatment of the control problem (P_L): we introduce a family of regularized control problems and derive the corresponding optimality conditions. Next, we establish some convergence results and by passing to the limit, we prove the optimality conditions for (P_L).

2 Notation and auxiliary algebraic lemmas

For $\eta, \zeta \in \mathbb{R}^{n\times n}$, we define the scalar product and the corresponding norm by

$$\eta : \zeta = \sum_{i,j=1}^{n} \eta_{ij}\zeta_{ij} \quad \text{and} \quad |\eta| = (\eta : \eta)^{\frac{1}{2}} .$$

For $\eta \in \mathbb{R}^{n\times n\times n\times n}$ and $\zeta \in \mathbb{R}^{n\times n}$, the scalar product $\eta : \zeta \in \mathbb{R}^{n\times n}$ is defined by

$$(\eta : \zeta)_{ij} = \sum_{k,\ell=1}^{n} \eta_{ijk\ell}\zeta_{k\ell} \qquad i,j = 1,\cdots,n,$$

and we can verify that for $\eta \in \mathbb{R}^{n\times n\times n\times n}$, $\zeta, \xi \in \mathbb{R}^{n\times n}$, we have

$$(\eta : \zeta) : \xi = (\xi : \eta) : \zeta,$$

where $\xi : \eta \in \mathbb{R}^{n\times n}$ is given by $(\xi : \eta)_{ij} = \sum_{k,\ell=1}^{n} \eta_{k\ell ij}\xi_{k\ell}$.

As already referred in the introduction, the extra stress tensor takes the regular form $\tau_R(\eta) = 2\nu\left(1+|\eta|^2\right)^{\frac{\alpha-2}{2}}\eta$ or the Lipschitz continuous form $\tau_L(\eta) = 2\nu\left(1+|\eta|\right)^{\alpha-2}\eta$. Standard arguments show that $\tau_R \in C^1(\mathbb{S})$ and $\tau_L \in C^1(\mathbb{S}\setminus\{0\})$ with

$$\frac{\partial(\tau_R(\eta))_{k\ell}}{\partial\eta_{ij}} = 2\nu(\alpha-2)\left(1+|\eta|^2\right)^{\frac{\alpha-4}{2}}\eta_{ij}\eta_{k\ell} + 2\nu\left(1+|\eta|^2\right)^{\frac{\alpha-2}{2}}\delta_{ik}\delta_{j\ell}, \tag{2.1}$$

$$\frac{\partial(\tau_L(\eta))_{k\ell}}{\partial\eta_{ij}} = 2\nu(\alpha-2)\left(1+|\eta|\right)^{\alpha-3}\frac{\eta_{ij}\eta_{k\ell}}{|\eta|} + 2\nu\left(1+|\eta|\right)^{\alpha-2}\delta_{ik}\delta_{j\ell}, \tag{2.2}$$

where $(\delta_{ij})_{ij}$ denotes the Kronecker tensor. To overcome the singularity of the derivative of τ_L at the origin, we introduce the regularized stress tensor $\tau_{L,\varepsilon} : \mathbb{S} \longrightarrow \mathbb{S}$ given by

$$\tau_{L,\varepsilon}(\eta) = 2\nu\left(1+r_\varepsilon\left(|\eta|\right)\right)^{\alpha-2}\eta \qquad 0 \le \varepsilon < 1,$$

where the regularizing function r_ε is given by $r_\varepsilon(s) = \sqrt{\varepsilon^2 + s^2}$. It is obvious that $\tau_{L,0} \equiv \tau_L$. Moreover, $\tau_{L,\varepsilon} \in C^1(\mathbb{S})$ for $\varepsilon > 0$ and

$$\frac{\partial(\tau_{L,\varepsilon}(\eta))_{k\ell}}{\partial\eta_{ij}} = 2\nu(\alpha-2)\left(1+r_\varepsilon(|\eta|)\right)^{\alpha-3}\frac{\eta_{ij}\eta_{k\ell}}{r_\varepsilon(|\eta|)} + 2\nu\left(1+r_\varepsilon(|\eta|)\right)^{\alpha-2}\delta_{ik}\delta_{j\ell}. \tag{2.3}$$

Throughout the paper we set

$$\tau'(\eta) : \zeta : \xi = \sum_{i,j=1}^{n}\sum_{k,\ell=1}^{n}\frac{\partial\tau_{k\ell}(\eta)}{\partial\eta_{ij}}\zeta_{k\ell}\xi_{ij} \qquad \eta,\zeta,\xi \in \mathbb{S} \quad (\eta \neq 0 \text{ if } \tau = \tau_L).$$

The next result deals with some useful properties of the tensors τ_R, τ_L and $\tau_{L,\varepsilon}$.

Lemma 2.1 *Let $\tau = \tau_R$ or $\tau = \tau_{L,\varepsilon}$ for $0 \le \varepsilon < 1$. Then*

For all $i,j,k,\ell = 1,\cdots,n$

$$\left|\frac{\partial\tau_{k\ell}(\eta)}{\partial\eta_{ij}}\right| \le 2\nu(3-\alpha)\left(1+|\eta|^2\right)^{\frac{\alpha-2}{2}} \qquad \textit{for all } \eta \in \mathbb{S} \quad (\eta \neq 0 \textit{ if } \varepsilon = 0)$$

and

$$\tau'(\eta) : \zeta : \zeta \ge \nu(\alpha-1)\left(1+|\eta|^2\right)^{\frac{\alpha-2}{2}}|\zeta|^2 \qquad \textit{for all } \eta,\zeta \in \mathbb{S} \quad (\eta \neq 0 \textit{ if } \varepsilon = 0).$$

Proof. Consider first $\tau = \tau_R$. Standard calculations together with (2.1) and the fact that $1 < \alpha < 2$ show that

$$\begin{aligned}
\frac{1}{2\nu}\left|\frac{\partial\tau_{k\ell}(\eta)}{\partial\eta_{ij}}\right| &\le (2-\alpha)\left(1+|\eta|^2\right)^{\frac{\alpha-4}{2}}|\eta|^2 + \left(1+|\eta|^2\right)^{\frac{\alpha-2}{2}} \\
&= \left(1+|\eta|^2\right)^{\frac{\alpha-4}{2}}\left((3-\alpha)|\eta|^2+1\right) \\
&\le (3-\alpha)\left(1+|\eta|^2\right)^{\frac{\alpha-2}{2}}
\end{aligned}$$

for all $\eta \in \mathbb{S}$. Moreover,

$$\begin{aligned}
\frac{1}{2\nu}\tau'(\eta) : \zeta : \zeta &= (\alpha-2)\left(1+|\eta|^2\right)^{\frac{\alpha-4}{2}}(\eta:\zeta)^2 + \left(1+|\eta|^2\right)^{\frac{\alpha-2}{2}}|\zeta|^2 \\
&\ge (\alpha-2)\left(1+|\eta|^2\right)^{\frac{\alpha-4}{2}}|\eta|^2|\zeta|^2 + \left(1+|\eta|^2\right)^{\frac{\alpha-2}{2}}|\zeta|^2 \\
&= \left(1+|\eta|^2\right)^{\frac{\alpha-4}{2}}\left((\alpha-1)|\eta|^2+1\right)|\zeta|^2 \\
&\ge (\alpha-1)\left(1+|\eta|^2\right)^{\frac{\alpha-2}{2}}|\zeta|^2
\end{aligned}$$

which gives the result. Similarly, considering $\tau = \tau_{L,\varepsilon}$ for $\varepsilon \in [0,1[$ and taking into account (2.2) and (2.3), we deduce that

$$\begin{aligned}
\frac{1}{2\nu}\left|\frac{\partial\tau_{k\ell}(\eta)}{\partial\eta_{ij}}\right| &\le (2-\alpha)\left(1+r_\varepsilon(|\eta|)\right)^{\alpha-3}\frac{|\eta|^2}{r_\varepsilon(|\eta|)} + \left(1+r_\varepsilon(|\eta|)\right)^{\alpha-2} \\
&\le (2-\alpha)\left(1+r_\varepsilon(|\eta|)\right)^{\alpha-3}|\eta| + \left(1+r_\varepsilon(|\eta|)\right)^{\alpha-2} \\
&\le (3-\alpha)\left(1+r_\varepsilon(|\eta|)\right)^{\alpha-2} \\
&\le (3-\alpha)\left(1+|\eta|\right)^{\alpha-2} \le (3-\alpha)\left(1+|\eta|^2\right)^{\frac{\alpha-2}{2}}
\end{aligned}$$

for all $\eta \in \mathbb{S}$ ($\eta \neq 0$ if $\varepsilon = 0$, i.e. if $\tau = \tau_L$). Moreover, since

$$\frac{(\eta:\zeta)^2}{r_\varepsilon(|\eta|)} \le \frac{|\eta|^2|\zeta|^2}{r_\varepsilon(|\eta|)} \le \frac{|\eta|}{r_\varepsilon(|\eta|)}r_\varepsilon(|\eta|)|\zeta|^2 \le r_\varepsilon(|\eta|)|\zeta|^2$$

we deduce that

$$\begin{aligned}
\tfrac{1}{2\nu}\tau'(\eta):\zeta:\zeta \quad &= (\alpha-2)\left(1+r_\varepsilon\left(|\eta|\right)\right)^{\alpha-3}\tfrac{(\eta:\zeta)^2}{r_\varepsilon(|\eta|)} + \left(1+r_\varepsilon\left(|\eta|\right)\right)^{\alpha-2}|\zeta|^2 \\
&\geq (\alpha-1)\left(1+r_\varepsilon\left(|\eta|\right)\right)^{\alpha-2}|\zeta|^2 \\
&\geq (\alpha-1)2^{\frac{\alpha-2}{2}}\left(1+\left(r_\varepsilon\left(|\eta|\right)\right)^2\right)^{\frac{\alpha-2}{2}}|\zeta|^2 \\
&\geq (\alpha-1)2^{\frac{\alpha-2}{2}}\left(1+\varepsilon^2\right)^{\frac{\alpha-2}{2}}\left(1+|\eta|^2\right)^{\frac{\alpha-2}{2}}|\zeta|^2 \\
&\geq (\alpha-1)2^{\alpha-2}\left(1+|\eta|^2\right)^{\frac{\alpha-2}{2}}|\zeta|^2 \\
&\geq \tfrac{\alpha-1}{2}\left(1+|\eta|^2\right)^{\frac{\alpha-2}{2}}|\zeta|^2
\end{aligned}$$

and the claimed result is proven. □

As a consequence of the previous lemma, we have the following standard continuity and monotonicity properties for τ_R, τ_L and $\tau_{L,\varepsilon}$.

Lemma 2.2 *Let $\tau=\tau_R$ or $\tau=\tau_{L,\varepsilon}$ for $0\leq\varepsilon<1$. Then,*

$$|\tau(\eta)| \leq 2\nu\left(1+|\eta|^2\right)^{\frac{\alpha-2}{2}}|\eta| \qquad \text{for all } \eta\in\mathbb{S},$$

$$(\tau(\eta)-\tau(\zeta)):(\eta-\zeta) \geq \nu(\alpha-1)\left(1+|\eta|^2+|\zeta|^2\right)^{\frac{\alpha-2}{2}}|\eta-\zeta|^2 \qquad \text{for all } \eta,\zeta\in\mathbb{S}.$$

Proof. The continuity condition is direct for $\tau=\tau_R$. For $\tau=\tau_{L,\varepsilon}$ with $\varepsilon\in[0,1[$, we easily see that

$$|\tau_{L,\varepsilon}(\eta)| \leq |\tau_L(\eta)| \leq |\tau_R(\eta)|\,.$$

The monotonicity condition is obviously satisfied for $\eta=\zeta$. If $\eta\neq\zeta$, by taking into account Lemma 2.1, we obtain

$$\begin{aligned}
(\tau(\eta)-\tau(\zeta)):(\eta-\zeta) \quad &= \int_0^1 \tau'(s\eta+(1-s)\zeta)\,ds:(\eta-\zeta):(\eta-\zeta) \\
&\geq \nu(\alpha-1)\int_0^1\left(1+|s\eta+(1-s)\zeta|^2\right)^{\frac{\alpha-2}{2}}\,ds\,|\eta-\zeta|^2 \\
&\geq \nu(\alpha-1)\int_0^1\left(1+s|\eta|^2+(1-s)|\zeta|^2\right)^{\frac{\alpha-2}{2}}\,ds\,|\eta-\zeta|^2 \\
&\geq \nu(\alpha-1)\left(1+|\eta|^2+|\zeta|^2\right)^{\frac{\alpha-2}{2}}|\eta-\zeta|^2
\end{aligned}$$

which gives the claimed result. □

Throughout the paper Ω is a bounded domain in $\mathbb{R}^n$ ($n=2$ or $n=3$). The boundary of Ω is denoted by Γ and is of class C^2. The space of infinitely differentiable functions with compact support in Ω will be denoted by $\mathcal{D}(\Omega)$. The standard Sobolev spaces are denoted by $W^{k,\alpha}(\Omega)$ ($k\in\mathbb{N}$ and $1<\alpha<\infty$), and their norms by $\|\cdot\|_{k,\alpha}$. We set $W^{0,\alpha}(\Omega)\equiv L^\alpha(\Omega)$, $\|\cdot\|_{L^\alpha}\equiv\|\cdot\|_\alpha$, $L^\alpha_0(\Omega)=\left\{v\in L^\alpha(\Omega)\mid\int_\Omega v(x)\,dx=0\right\}$ and $\alpha'=\frac{\alpha}{\alpha-1}$ (the dual exponent to α). We will also use the following notation

$$\begin{aligned}
(u,v) \quad &= \int_\Omega u(x)\cdot v(x)\,dx, \qquad && u\in L^\alpha(\Omega)^n,\ v\in L^{\alpha'}(\Omega)^n, \\
(\eta,\zeta) \quad &= \int_\Omega \eta(x):\zeta(x)\,dx, \qquad && \eta\in L^\alpha(\Omega)^{n\times n},\ \zeta\in L^{\alpha'}(\Omega)^{n\times n}.
\end{aligned}$$

Since many of the quantities occuring in the paper are vector-valued functions, the notation will be abreged for the sake of brevity and we will omit the space dimension n in the function space notation. (The meaning should be clear from the context.)

In order to eliminate the pressure in the weak formulation of the state equation, we will work in divergence-free spaces. Consider

$$\mathcal{V} = \{\varphi \in \mathcal{D}(\Omega) \mid \nabla \cdot \varphi = 0 \quad \text{in } \Omega\},$$

and denote by V_α the closure of $\mathcal{V}$ in the L^α-norm of gradients, i.e.

$$V_\alpha = \left\{\varphi \in W_0^{1,\alpha}(\Omega) \mid \nabla \cdot \varphi = 0 \quad \text{in } \Omega\right\}.$$

3 State equation

This section is devoted to existence, uniqueness and regularity results for the state equation and to derivation of some estimates useful for the subsequent analysis.

As already referred in the introduction, existence of weak solutions was proved by Ladyzhenskaya and Lions for $\alpha \geq \frac{3n}{n+2}$. The restriction on the exponent α ensures that the convective term belongs to L^1 when considering test functions in V_α. Multiplying equation (1.1) by test functions $\varphi \in V_\alpha$ and integrating, we obtain the following weak formulation.

Definition 3.1 *Let u be in $L^2(\Omega)$. A function $y \in V_\alpha$ is a weak solution of of the state equation* (1.1) *if*

$$(\tau(Dy), D\varphi) + b(y, y, \varphi) = (u, \varphi) \qquad \text{for all } \varphi \in V_\alpha,$$

where b is the trilinear form defined by $b(w, y, \varphi) = (w \cdot \nabla y, \varphi)$.

We recall that, having a solution satisfying the previous formulation, it is standard to construct the corresponding pressure $\pi \in L_0^\alpha(\Omega)$ such that

$$(\tau(Dy), D\varphi) + b(y, y, \varphi) - (\pi, \nabla \cdot y, \varphi) = (u, \varphi) \qquad \text{for all } \varphi \in W_0^{1,\alpha}(\Omega).$$

We will involve the pressure only in the statement of our results but not in the proofs, since it can always be reconstructed uniquely.

Besides existence of a weak solution for the state equation, results on uniqueness and regularity of such a solution are important in the treatment of control problems governed by shear-thinning fluids. In the present paper we follow [1], where both two-dimensional and three-dimensional cases for C^1 and Lipschitz continuous extra stress tensors were considered, and identify a condition under which uniqueness and regularity of weak solutions are both guaranteed.

Theorem 3.2 *Let $u \in L^q(\Omega)$ with $q > n$ and let $\tau = \tau_R$ with $\frac{3n}{n+2} \leq \alpha < 2$. Then problem* (1.1) *admits at least a weak solution $y_u \in V_\alpha$ and the following estimate holds*

$$\|Dy_u\|_\alpha^\alpha \leq \widetilde{C}\left(\frac{\|u\|_q}{\nu}\right)^{\alpha'} + |\Omega|, \tag{3.1}$$

where $\widetilde{C} \equiv \widetilde{C}(n, \alpha, q, \Omega)$. Moreover, there exists a positive constant κ depending only on n, α, q and Ω such that if

$$\kappa\left(\frac{\|u\|_q}{\nu} + \left(1 + \frac{\|u\|_q}{\nu}\right)^{\frac{2(2-\alpha)}{\alpha-1}} \frac{\|u\|_q}{\nu^2}\right) < 1, \tag{3.2}$$

then this solution is unique, belongs to $W^{2,q}(\Omega)$ *and the following estimate holds*

$$\|y_u\|_{2,q} \leq \widetilde{\kappa}\frac{\|u\|_q}{\nu} \tag{3.3}$$

with $\widetilde{\kappa} \equiv \widetilde{\kappa}(n, q, \Omega)$.

Proof. By taking into account Lemma 2.1 and using standard arguments, we may prove existence of a weak solution $y_u \in V_\alpha$ of equation (1.1) (see, e.g., [13]) and the following estimate holds

$$\|Dy_u\|_\alpha^\alpha \leq C\left(\frac{\|u\|_2}{\nu}\right)^{\alpha'} + |\Omega|$$

with $C \equiv C(n, \alpha, \Omega)$ (see, e.g., Theorem 3.2 in [3]). Therefore,

$$\|Dy_u\|_\alpha^\alpha \leq C\left(|\Omega|^{\frac{1}{2}-\frac{1}{q}}\frac{\|u\|_q}{\nu}\right)^{\alpha'} + |\Omega|$$

and estimate (3.1) is proven. Moreover, due to Theorem 3.3 in [3] there exists a positive constant κ_1 depending only on n, α and Ω such that if

$$\kappa_1\left(1 + \frac{\|u\|_2}{\nu}\right)^{\frac{2(2-\alpha)}{\alpha-1}}\frac{\|u\|_2}{\nu^2} < 1, \tag{3.4}$$

then problem (1.1) admits a unique weak solution. On the other hand, due to Theorem 2.1 in [1], there exists a positive constant κ_2 depending only on n, q and Ω such that if

$$\kappa_2\left(\frac{\|u\|_q}{\nu} + \frac{\|u\|_q}{\nu^2}\right) < 1, \tag{3.5}$$

then problem (1.1) admits a strong solution $y_u \in W^{2,q}(\Omega)$ and (3.3) holds. Setting

$$\kappa = \max\left(\kappa_1|\Omega|^{\frac{1}{2}-\frac{1}{q}}\left(1 + |\Omega|^{\frac{1}{2}-\frac{1}{q}}\right)^{\frac{2(2-\alpha)}{\alpha-1}}, \kappa_2\right), \tag{3.6}$$

we can see that condition (3.2) implies (3.4) and (3.5) and the claimed result is proven. □

Theorem 3.3 *Let* $u \in L^q(\Omega)$ *with* $q > n$ *and let* $\tau = \tau_L$ *with* $\frac{3n}{n+2} \leq \alpha < 2$. *Then problem* (1.1) *admits at least a weak solution* $y_u \in V_\alpha$ *and estimate* (3.1) *holds. Moreover, if* u *satisfies condition* (3.2)*, then this solution is unique, belongs to* $W^{2,q}(\Omega)$ *and satisfies estimate* (3.3).

Proof. The proof follows exactly the same steps as in Theorem 3.2, using Theorem 2.2 instead of using Theorem 2.1 in [1]. □

Theorem 3.4 *Let* $u \in L^q(\Omega)$ *with* $q > n$ *and let* $\tau = \tau_{L,\varepsilon}$ *with* $\frac{3n}{n+2} \leq \alpha < 2$ *and* $0 < \varepsilon < 1$. *Then problem* (1.1) *admits at least a weak solution* $y_u^\varepsilon \in V_\alpha$ *and the following estimate holds*

$$\|Dy_u^\varepsilon\|_\alpha^\alpha \leq \widetilde{C}\left(\frac{\|u\|_q}{\nu}\right)^{\alpha'} + |\Omega|, \tag{3.7}$$

where $\widetilde{C} \equiv \widetilde{C}(n, \alpha, q, \Omega)$ *independent of* ε. *Moreover, if* u *satisfies condition* (3.2)*, then this solution is unique, belongs to* $W^{2,q}(\Omega)$ *and the following estimate holds*

$$\|y_u^\varepsilon\|_{2,q} \leq \widetilde{\kappa}\frac{\|u\|_q}{\nu} \tag{3.8}$$

with $\widetilde{\kappa} \equiv \widetilde{\kappa}(n, q, \Omega)$ *independent of* ε.

Proof. The proof follows exactly the same steps as in Theorem 3.2, using Theorem 4.1 instead of using Theorem 2.1 in [1]. $\square$

Remark 3.5 *i) Condition* (3.2) *is fulfilled if the term* $\frac{\|u\|_q}{\nu}$ *is "small enough", and can be interpreted either as a constraint on the size of* $\|u\|_q$ *(small body force* u*) or as a restriction on the viscosity parameter* ν *(large viscosity parameter* ν*).*

ii) Due to compactness results on Sobolev spaces, we deduce that a $W^{2,q}$ *solution belongs to* $C^{1,\delta}(\overline{\Omega})$ *for every* $\delta < 1 - \frac{n}{q}$*. Moreover, by taking into account* (3.3) *and* (3.8)*, we have*

$$\|y_u\|_{C^{1,\delta}(\overline{\Omega})} \leq \bar{\kappa}\frac{\|u\|_q}{\nu}, \tag{3.9}$$

$$\|y_u^\varepsilon\|_{C^{1,\delta}(\overline{\Omega})} \leq \bar{\kappa}\frac{\|u\|_q}{\nu}, \tag{3.10}$$

where $\bar{\kappa} \equiv \bar{\kappa}(n, q, \Omega)$ *is a positive constant independent of* ε*.*

4 Adjoint equation

We next investigate the following linear system

$$\begin{cases} -\nabla \cdot \left(\tau_R' \left(Dy_u\right)^T : Dp\right) + (\nabla y_u)^T p - y_u \cdot \nabla p + \nabla \pi = w & \text{in } \Omega, \\ \nabla \cdot p = 0 & \text{in } \Omega, \\ p = 0 & \text{on } \Gamma, \end{cases} \tag{4.1}$$

where $u \in L^q(\Omega)$ with $q > n$, $y_u \in V_\alpha$ is a corresponding solution of (1.1) and $w \in L^2(\Omega)$. Assuming that u satisfies condition (3.2), we deduce that $y_u \in W^{2,q}(\Omega)$ and we can consider the weak formulation

$$\begin{cases} \text{Find } p \in V_2 \text{ such that} \\ \left(\tau_R' \left(Dy_u\right) : D\varphi, Dp\right) + b\left(\varphi, y_u, p\right) + b\left(y_u, \varphi, p\right) = (w, \varphi) \qquad \text{for all } \varphi \in V_2. \end{cases} \tag{4.2}$$

Proposition 4.1 *Assume that* $\frac{3n}{n+2} \leq \alpha < 2$*. Let* $u \in L^q(\Omega)$ *(with* $q > n$*) satisfying* (3.2)*,* y_u *be the corresponding solution of* (1.1) *and* $w \in L^2(\Omega)$*. There exists a positive constant* $\widehat{\kappa}$ *depending only on* n*,* α*,* q *and* Ω *such that, if*

$$\widehat{\kappa}\left(1 + \frac{\|u\|_q}{\nu}\right)^{2-\alpha} \frac{\|u\|_q}{\nu^2} < 1 \tag{4.3}$$

then problem (4.1) *admits a unique weak solution* p_{uw} *in* V_2*. Moreover, the following estimate holds*

$$\|Dp_{uw}\|_2 \leq \mathcal{L}\left(\frac{\|u\|_q}{\nu}\right)\|w\|_2$$

with $\mathcal{L}(t) = \frac{\widehat{C}(1+t)^{2-\alpha}}{\nu - \widehat{\kappa}(1+t)^{2-\alpha}t}$ *and* $\widehat{C} \equiv \widehat{C}(n, \alpha, q, \Omega)$*.*

Proof. Consider $B : V_2 \times V_2 \longrightarrow \mathbb{R}$ the bilinear form defined by

$$B(p_1, p_2) = \left(\tau_R' \left(Dy_u\right) : Dp_2, Dp_1\right) + b\left(p_1, y_u, p_2\right) + b\left(y_u, p_1, p_2\right).$$

Recalling that

$$b\left(y_u, p, p\right) = 0$$

we get

$$\begin{aligned} B(p,p) &= \left(\tau_R'(Dy_u) : Dp, Dp\right) + b\left(p, y_u, p\right) + b\left(y_u, p, p\right) \\ &= \left(\tau_R'(Dy_u) : Dp, Dp\right) + b\left(p, y_u, p\right) \end{aligned}$$

for every $p \in V_2$. On the other hand, by using Lemma 2.1 and estimate (3.9) , we deduce that

$$\begin{aligned} \left(\tau_R'(Dy_u) : Dp, Dp\right) &\geq (\alpha-1)\nu \int_\Omega \left(1 + |Dy_u|\right)^{\alpha-2} |Dp|^2\, dx \\ &\geq (\alpha-1)\nu \left(1 + \|Dy_u\|_\infty\right)^{\alpha-2} \|Dp\|_2^2 \\ &\geq (\alpha-1)\nu\left(1+\bar\kappa\right)^{\alpha-2}\left(1 + \tfrac{\|u\|_q}{\nu}\right)^{\alpha-2} \|Dp\|_2^2. \end{aligned} \tag{4.4}$$

Moreover, Lemma 1.1, Chapter VIII in [7] together with the Korn inequality and estimate (3.9) show that

$$\begin{aligned} |b\left(p, y_u, p\right)| &\leq \tfrac{(n-1)}{n} |\Omega|^{\frac{1}{n(n-1)}} \|\nabla y_u\|_2 \|\nabla p\|_2^2 = \tfrac{2^{\frac{3}{2}}(n-1)}{n} |\Omega|^{\frac{1}{n(n-1)}} \|Dy_u\|_2 \|Dp\|_2^2 \\ &\leq C_1 \|Dy_u\|_\infty \|Dp\|_2^2 \leq C_1 \bar\kappa \tfrac{\|u\|_q}{\nu} \|Dp\|_2^2 \end{aligned} \tag{4.5}$$

with $C_1 = \frac{2^{\frac{3}{2}}(n-1)}{n} |\Omega|^{\frac{1}{n(n-1)}+\frac{1}{2}}$. Therefore, (4.4) and (4.5) yield

$$B(p,p) \geq \left((\alpha-1)\nu\left(1+\bar\kappa\right)^{\alpha-2}\left(1+\tfrac{\|u\|_q}{\nu}\right)^{\alpha-2} - C_1\bar\kappa\tfrac{\|u\|_q}{\nu}\right) \|Dp\|_2^2 \tag{4.6}$$

which shows that B is coercive on V_2 if u satisfies (4.3) with $\widehat\kappa = \frac{C_1\bar\kappa}{(\alpha-1)(1+\bar\kappa)^{\alpha-2}}$. Let us now prove that B is continuous. Similar arguments show that

$$\begin{aligned} \tfrac{1}{2\nu(3-\alpha)} \left|\left(\tau_R'(Dy_u) : Dp_2, Dp_1\right)\right| &\leq \int_\Omega \left(1+|Dy_u|\right)^{\alpha-2} |Dp_1||Dp_2|\, dx \\ &\leq \int_\Omega |Dp_1||Dp_2|\, dx \leq \|Dp_1\|_2 \|Dp_2\|_2 \end{aligned}$$

and

$$\begin{aligned} |b\left(p_1, y_u, p_2\right) + b\left(y_u, p_1, p_2\right)| &\leq 2C_1 \|Dy_u\|_\infty \|Dp_1\|_2 \|Dp_2\|_2 \leq 2C_1\bar\kappa\tfrac{\|u\|_q}{\nu}\|Dp_1\|_2\|Dp_2\|_2 \\ &\leq 2(\alpha-1)\nu\left(1+\bar\kappa\right)^{\alpha-2}\left(1+\tfrac{\|u\|_q}{\nu}\right)^{\alpha-2} \|Dp_1\|_2 \|Dp_2\|_2 \\ &\leq 2\nu(\alpha-1) \|Dp_1\|_2 \|Dp_2\|_2 \end{aligned}$$

for every $p_1, p_2 \in V_2$. Therefore,

$$B\left(p_1, p_2\right) \leq 4\nu \|Dp_1\|_2 \|Dp_2\|_2 .$$

The bilinear form B is then continuous and coercive on V_2. Applying the Lax-Milgram theorem, we deduce that problem (4.2) admits a unique solution p_{uw} in V_2. Taking into account (4.6), the Hölder, the Poincaré and the Korn inequalities, we obtain

$$\left((\alpha-1)\nu\left(1+\bar\kappa\right)^{\alpha-2}\left(1+\tfrac{\|u\|_q}{\nu}\right)^{\alpha-2} - C_1\bar\kappa\tfrac{\|u\|_q}{\nu}\right) \|Dp_{uw}\|_2^2 \leq B\left(p_{uw}, p_{uw}\right) = \left(w, p_{uw}\right)$$

$$\leq \|w\|_2 \|p_{uw}\|_2 \leq \tfrac{n-1}{\sqrt{n}} |\Omega|^{\frac{1}{n}} \|w\|_2 \|\nabla p_{uw}\|_2 = C_2 \|w\|_2 \|Dp_{uw}\|_2$$

which gives the estimate with $\widehat C = \frac{C_2(1+\bar\kappa)^{2-\alpha}}{\alpha-1}$ where $C_2 = \frac{2^{\frac{1}{2}}(n-1)}{\sqrt{n}}|\Omega|^{\frac{1}{n}}$. $\square$

Remark 4.2 *Let us notice that the regularity of the state is fundamental. As already referred in the introduction, in the absence of additional regularity on the weak solution, problem* (4.1) *is to be understood in the distribution sense*

$$\begin{cases} \text{Find } p \in V_\alpha \text{ such that} \\ \left(\tau_R'\left(Dy_u\right) : D\varphi, Dp\right) + b\left(\varphi, y_u, p\right) + b\left(y_u, \varphi, p\right) = (w, \varphi) \qquad \text{for all } \varphi \in \mathcal{V} \end{cases}$$

and there is no reason to assume uniqueness of solutions (*see, e.g.,* [3]).

The following proposition deals with very similar results related with the regularized adjoint equation.

Proposition 4.3 *Assume that* $\frac{3n}{n+2} \leq \alpha < 2$ *and* $0 < \varepsilon < 1$, *let* $u \in L^q(\Omega)$ (*with* $q > n$) *satisfying* (3.2), y_u^ε *be the corresponding solution of* (1.1) *and let* $w \in L^2(\Omega)$. *If* u *satisfies* (4.3), *then problem*

$$\begin{cases} -\nabla \cdot \left(\tau_{L,\varepsilon}'\left(Dy_u^\varepsilon\right)^T : Dp\right) + (\nabla y_u^\varepsilon)^T p - y_u^\varepsilon \cdot \nabla p + \nabla \pi^\varepsilon = w & \text{in } \Omega, \\ \nabla \cdot p = 0 & \text{in } \Omega, \\ p = 0 & \text{on } \Gamma, \end{cases} \tag{4.7}$$

admits a unique weak solution p_{uw}^ε *in* V_2. *Moreover, the following estimate holds*

$$\|Dp_{uw}^\varepsilon\|_2 \leq \mathcal{L}\left(\tfrac{\|u\|_q}{\nu}\right)\|w\|_2$$

with $\mathcal{L}$ *defined as in Proposition* 4.1 .

Proof. The proof follows exactly the same steps as in Proposition 4.1, using (3.10) instead of using (3.9). □

In order to simplify the statements of our main results and the corresponding proofs, we observe that the restrictions (3.2) and (4.3) on the control variable are very similar and can be summarized as

$$(\mathcal{CR}) \quad \begin{cases} \text{There exists a positive constant } \kappa^* \text{ depending only on } n, \alpha, q \text{ and } \Omega \text{ such that} \\ \qquad \kappa^*\left(\frac{\|u\|_q}{\nu} + \left(1 + \frac{\|u\|_q}{\nu}\right)^{\frac{2(2-\alpha)}{\alpha-1}} \frac{\|u\|_q}{\nu^2}\right) < 1 \end{cases}$$

by setting $\kappa^* = \max(\kappa, \widehat{\kappa})$. Hence, due to Theorem 3.2, Theorem 3.4, Proposition 4.1 and Proposition 4.3, we can see that imposing the restriction $(\mathcal{CR})$ on the control guarantees existence, uniqueness and regularity of the state, as well as existence and uniqueness of the adjoint state.

5 Necessary optimality conditions for (P_R)

Let us now formulate our first main result.

Theorem 5.1 *Assume that* $\frac{3n}{n+2} \leq \alpha < 2$. *Then problem* (P_R) *admits at least one solution* $(\bar{u}, \bar{y})$. *Moreover, if* $\bar{u}$ *satisfies* $(\mathcal{CR})$ *then there exists* $\bar{p} \in V_2$ *such that the following conditions hold*

$$\begin{cases} -\nabla \cdot (\tau_R(D\bar{y})) + \bar{y} \cdot \nabla \bar{y} + \nabla \bar{\pi} = \bar{u} & \text{in } \Omega, \\ \nabla \cdot \bar{y} = 0 & \text{in } \Omega, \\ \bar{y} = 0 & \text{on } \Gamma, \end{cases}$$

$$\begin{cases} -\nabla \cdot \big(\tau_R'(D\bar{y})^T : D\bar{p}\big) + (\nabla \bar{y})^T \bar{p} - \bar{y} \cdot \nabla \bar{p} + \nabla \widetilde{\pi} = \bar{y} - y_d & \textit{in } \Omega, \\ \nabla \cdot \bar{p} = 0 & \textit{in } \Omega, \\ \bar{p} = 0 & \textit{on } \Gamma, \end{cases} \tag{5.1}$$

$$(\bar{p} + \lambda \bar{u}, v - \bar{u}) \geq 0 \qquad \textit{for all } v \in U_{ad}. \tag{5.2}$$

Proof. Taking into account Lemma 2.1, we can see that the assumptions of Theorem 4.1 in [3] are fulfilled and existence of an optimal solution $(\bar{u}, \bar{y})$ for (P_R) is then guaranteed. Moreover, if $\bar{u}$ satisfies $(\mathcal{CR})$, then it satisfies (3.4) and due to Theorem 5.1 in [3], there exists $\bar{p} \in V_\alpha$ solution of (5.1) in the distribution sense

$$(\tau_R'(D\bar{y}) : D\varphi, D\bar{p}) + \big((\nabla \bar{y})^T \bar{p} - \bar{y} \cdot \nabla \bar{p}, \varphi\big) = (\bar{y} - y_d, \varphi) \qquad \text{for all } \varphi \in \mathcal{V}$$

and such that (5.2) holds. The conclusion follows by observing that due to Proposition 4.1, if $\bar{u}$ satisfies $(\mathcal{CR})$ then $\bar{p}$ is the unique weak solution of (5.1) in V_2. □

6 Necessary optimality conditions for (P_L)

In order to obtain necessary optimality conditions for (P_L) stated in Theorem 6.1 below, we introduce a family of problems $(P_{L,\varepsilon})_\varepsilon$ whose solutions converge towards a solution of (P_L). We derive the corresponding optimality conditions in Section 6.1, and we pass to the limit in these conditions in Section 6.2.

Theorem 6.1 *Assume that* $\frac{3n}{n+2} \leq \alpha < 2$. *Then problem* (P_L) *admits at least one solution* $(\bar{u}, \bar{y})$. *Moreover, if* $\bar{u}$ *satisfies* $(\mathcal{CR})$ *then there exists* $\bar{p} \in V_2$ *such that the following conditions hold*

$$\begin{cases} -\nabla \cdot (\tau_L(D\bar{y})) + \bar{y} \cdot \nabla \bar{y} + \nabla \bar{\pi} = \bar{u} & \textit{in } \Omega, \\ \nabla \cdot \bar{y} = 0 & \textit{in } \Omega, \\ \bar{y} = 0 & \textit{on } \Gamma, \end{cases}$$

$$-\nabla \cdot \big(\tau_L'(D\bar{y})^T : D\bar{p}\big) + (\nabla \bar{y})^T \bar{p} - \bar{y} \cdot \nabla \bar{p} + \nabla \widetilde{\pi} = \bar{y} - y_d \qquad \textit{in } \{x \in \Omega \mid |D\bar{y}(x)| > 0\}, \tag{6.1}$$

$$(\bar{p} + \lambda \bar{u}, v - \bar{u}) \geq 0 \qquad \textit{for all } v \in U_{ad}. \tag{6.2}$$

6.1 Regularized control problem

Taking into account Lemma 2.1, we can see that the arguments in the proof of Theorem 4.1 in [3] can be applied and existence of an optimal solution for (P_L) is then guaranteed. Let $(\bar{u}, \bar{y})$ be such a solution and assume that $\bar{u}$ satisfies $(\mathcal{CR})$ and $0 < \varepsilon < 1$. Introduce the cost functional

$$I(u, y) = J(u, y) + \tfrac{1}{2} \int_\Omega |u - \bar{u}|^2 \, dx$$

and the control problem

$$(P_{L,\varepsilon}) \quad \begin{cases} \text{minimize} \quad I(u, y^\varepsilon) \\ \text{subject to} \quad (u, y^\varepsilon) \in U_{ad} \times V_\alpha \text{ satisfies} \\ \qquad \begin{cases} -\nabla \cdot (\tau_{L,\varepsilon}(Dy)) + y \cdot \nabla y + \nabla \pi_\varepsilon = u & \text{in } \ \Omega, \\ \nabla \cdot y = 0 & \text{in } \ \Omega, \\ y = 0 & \text{on } \ \Gamma, \end{cases} \\ \qquad \text{for some } \pi_\varepsilon \in L_0^\alpha(\Omega). \end{cases} \tag{6.3}$$

The main result of this section deals with the necessary optimality conditions for the regularized problem $(P_{L,\varepsilon})$.

Theorem 6.2 *Assume that* $\frac{3n}{n+2} \leq \alpha < 2$. *For each* $0 < \varepsilon < 1$, *there exists at least one solution* $(\bar{u}^\varepsilon, \bar{y}^\varepsilon)$ *of* $(P_{L,\varepsilon})$. *Moreover, if* $\bar{u}^\varepsilon$ *satisfies* $(\mathcal{CR})$ *then there exists* $\bar{p}^\varepsilon \in V_2$ *such that*

$$\begin{cases} -\nabla \cdot \left(\tau'_{L,\varepsilon}\left(D\bar{y}^\varepsilon\right)^T : D\bar{p}^\varepsilon\right) + (\nabla \bar{y}^\varepsilon)^T \bar{p}^\varepsilon - \bar{y}^\varepsilon \cdot \nabla \bar{p}^\varepsilon + \nabla \widetilde{\pi}^\varepsilon = \bar{y}^\varepsilon - y_d & \text{in } \Omega, \\ \nabla \cdot \bar{p}^\varepsilon = 0 & \text{in } \Omega, \\ \bar{p}^\varepsilon = 0 & \text{on } \Gamma, \end{cases} \tag{6.4}$$

$$(\bar{p}^\varepsilon + (\lambda + 1)\bar{u}^\varepsilon - \bar{u}, v - \bar{u}^\varepsilon) \geq 0 \qquad \text{for all } v \in U_{ad}. \tag{6.5}$$

Proof. The proof follows exactly the same steps as in Theorem 5.1, using Proposition 4.3 instead of using Proposition 4.1. □

6.2 Convergence results

We first establish a useful convergence result.

Proposition 6.3 *Assume that* $\frac{3n}{n+2} \leq \alpha < 2$ *and let* $\left(v_\varepsilon, y^\varepsilon_{v_\varepsilon}\right)$ *be an admissible pair for* $(P_{L,\varepsilon})$. *There exists a subsequence* $(\varepsilon_k)_k$ *converging to zero and* (u, y), *an admissible pair for* (P_L), *such that*

$$v_{\varepsilon_k} \longrightarrow u \quad \text{weakly in } L^2(\Omega) \qquad \text{and} \qquad y^{\varepsilon_k}_{v_{\varepsilon_k}} \longrightarrow y \quad \text{strongly in } W^{1,\alpha}_0(\Omega).$$

Proof. The proof is split into three steps.

Step 1. To simplify the redaction, let us set $y^\varepsilon = y^\varepsilon_{v_\varepsilon}$. Since $(v_\varepsilon)_\varepsilon$ is uniformly bounded in the closed convex set U_{ad}, by taking into account estimate (3.7) we obtain

$$\|Dy^\varepsilon\|^\alpha_\alpha \leq \widetilde{C}\left(\frac{\|v_\varepsilon\|_q}{\nu}\right)^{\alpha'} + |\Omega| \leq \widetilde{C}\left(\frac{U}{\nu}\right)^{\alpha'} + |\Omega| \tag{6.6}$$

where $\widetilde{C}$ is independent of ε, and the sequence $(y^\varepsilon)_\varepsilon$ is then bounded in V_α. On the other hand, it is easy to see that

$$|\tau_{L,\varepsilon}(\eta)| \leq 2\nu|\eta|^{\alpha-1}$$

implying

$$\|\tau_{L,\varepsilon}(Dy^\varepsilon)\|^{\alpha'}_{\alpha'} \leq 2\nu\, \|Dy^\varepsilon\|^\alpha_\alpha$$

which together with (6.6) show that sequence $(\tau_{L,\varepsilon}(Dy^\varepsilon))_\varepsilon$ is uniformly bounded in $L^{\alpha'}(\Omega)$. There then exist a subsequence $(\varepsilon_k)_k$ converging to zero, $u \in U_{ad}$, $y \in V_\alpha$ and $\widetilde{\tau} \in L^{\alpha'}(\Omega)$ such that $(v_{\varepsilon_k})_k$ weakly converges to u in $L^2(\Omega)$, $(y^{\varepsilon_k})_k$ weakly converges to y in V_α and $(\tau_{L,\varepsilon_k}(Dy^{\varepsilon_k}))_k$ weakly converges to $\widetilde{\tau}$ in $L^{\alpha'}(\Omega)$. Moreover, since $\alpha > \frac{2n}{n+1}$, by using compactness results on Sobolev spaces, we deduce that $(y^{\varepsilon_k})_k$ strongly converges to y in $L^{\alpha'}(\Omega)$.

Step 2. Let us now prove that (u, y) is an admissible pair for (P_L). Taking into account the convergence results obtained in Step 1, we deduce that for every $\varphi \in \mathcal{V}$, we have

$$\begin{aligned} |b(y^{\varepsilon_k}, y^{\varepsilon_k}, \varphi) - b(y, y, \varphi)| &\leq |b(y^{\varepsilon_k} - y, y^{\varepsilon_k}, \varphi)| + |b(y, y^{\varepsilon_k} - y, \varphi)| \\ &= |b(y^{\varepsilon_k} - y, y^{\varepsilon_k}, \varphi)| + |b(y, \varphi, y^{\varepsilon_k} - y)| \\ &\leq \left(\|\nabla y^{\varepsilon_k}\|_\alpha \|\varphi\|_\infty + \|y\|_\alpha \|\nabla\varphi\|_\infty\right) \|y^{\varepsilon_k} - y\|_{\alpha'} \\ &\longrightarrow 0 \qquad \text{when } k \to +\infty. \end{aligned} \tag{6.7}$$

Moreover, by passing to the limit in the weak formulation corresponding to y^{ε_k}, we obtain

$$(\widetilde{\tau}, D\varphi) + b\,(y, y, \varphi) = (u, \varphi) \qquad \text{for all } \varphi \in \mathcal{V}$$

and by using the fact that $\mathcal{V}$ is dense in V_α and that $y \in L^{\frac{2\alpha}{\alpha-1}}(\Omega)$ if $\alpha \geq \frac{3n}{n+2}$, it follows that

$$(\widetilde{\tau}, D\varphi) + b\,(y, y, \varphi) = (u, \varphi) \qquad \text{for all } \varphi \in V_\alpha. \tag{6.8}$$

In particular, since $b(y, y, y) = 0$, we have

$$(\widetilde{\tau}, Dy) = (\widetilde{\tau}, Dy) + b\,(y, y, y) = (u, y)\,. \tag{6.9}$$

On the other hand, the monotonicity property in Lemma 2.2 implies

$$\left(\tau_{L,\varepsilon_k}\left(Dy^{\varepsilon_k}\right) - \tau_{L,\varepsilon_k}\left(D\varphi\right), Dy^{\varepsilon_k} - D\varphi\right) \geq 0 \qquad \text{for all } \varphi \in V_\alpha. \tag{6.10}$$

Since $\left(\tau_{L,\varepsilon_k}\left(Dy^{\varepsilon_k}\right), Dy^{\varepsilon_k}\right) = (v_{\varepsilon_k}, y^{\varepsilon_k})$, by substituing in (6.10), we obtain

$$(v_{\varepsilon_k}, y^{\varepsilon_k}) - \left(\tau_{L,\varepsilon_k}\left(Dy^{\varepsilon_k}\right), D\varphi\right) - \left(\tau_{L,\varepsilon_k}\left(D\varphi\right), Dy^{\varepsilon_k} - D\varphi\right) \geq 0 \tag{6.11}$$

for all $\varphi \in V_\alpha$. Observing that

$$\begin{aligned} |\tau_{L,\varepsilon}(\eta) - \tau_L(\eta)| &= \left|(1 + r_\varepsilon\left(|\eta|\right))^{\alpha-2} - (1 + |\eta|)^{\alpha-2}\right| |\eta| \\ &\leq \left(r_\varepsilon\left(|\eta|\right) - |\eta|\right) |\eta| = \tfrac{\varepsilon^2}{r_\varepsilon(|\eta|) + |\eta|}\, |\eta| \leq \varepsilon^2 \end{aligned}$$

it follows that

$$\lim_{k\to\infty} \left\|\tau_{L,\varepsilon_k}(D\varphi) - \tau_L(D\varphi)\right\|_{\alpha'} = 0$$

and by passing to the limit in (6.11), we get

$$(u, y) - (\widetilde{\tau}, D\varphi) - \left(\tau_L\left(D\varphi\right), Dy - D\varphi\right) \geq 0 \qquad \text{for all } \varphi \in V_\alpha.$$

This inequality together with (6.9) then yields

$$\left(\widetilde{\tau} - \tau_L\left(D\varphi\right), Dy - D\varphi\right) \geq 0 \qquad \text{for all } \varphi \in V_\alpha$$

and by setting $\varphi = y - t\psi$ with $t > 0$, we obtain

$$\left(\widetilde{\tau} - \tau_L\left(Dy - tD\psi\right), D\psi\right) \geq 0 \qquad \text{for all } \psi \in V_\alpha.$$

Letting t tend to zero and using the continuity of τ_L, we deduce that

$$\left(\widetilde{\tau} - \tau_L\left(Dy\right), D\psi\right) \geq 0 \qquad \text{for all } \psi \in V_\alpha$$

and thus

$$(\widetilde{\tau}, D\psi) = \left(\tau_L\left(Dy\right), D\psi\right) \qquad \text{for all } \psi \in V_\alpha. \tag{6.12}$$

Combining (6.8) and (6.12) gives

$$\left(\tau_L\left(Dy\right), D\varphi\right) + b\,(y, y, \varphi) = (u, \varphi) \qquad \text{for all } \varphi \in V_\alpha$$

showing that

$$y^{\varepsilon_k} \longrightarrow y \qquad \text{weakly in } W_0^{1,\alpha}(\Omega)$$

and that (u,y) is admissible for (P_L).

Step 3. Let us now prove the strong convergence of $(y^{\varepsilon_k})_k$ to y in $W_0^{1,\alpha}(\Omega)$. Lemma 2.2 together with Lemma 2.8 in [3] and estimate (6.6) yield

$$\begin{aligned}\left(\tau_{L,\varepsilon_k}\left(Dy^{\varepsilon_k}\right)-\tau_{L,\varepsilon_k}\left(Dy\right),D\left(y^{\varepsilon_k}-y\right)\right) &\geq \frac{\nu(\alpha-1)\|D(y^{\varepsilon_k}-y)\|_\alpha^2}{\left(|\Omega|+\|Dy^{\varepsilon_k}\|_\alpha^\alpha+\|Dy\|_\alpha^\alpha\right)^{\frac{2-\alpha}{\alpha}}}\\ &\geq \frac{\nu(\alpha-1)\|D(y^{\varepsilon_k}-y)\|_\alpha^2}{\left(3|\Omega|+2\widetilde{C}\left(\frac{U}{\nu}\right)^{\alpha'}\right)^{\frac{2-\alpha}{\alpha}}}.\end{aligned}$$

Therefore, by taking into account the previous convergence results, we deduce that

$$\begin{aligned}&\frac{\nu(\alpha-1)}{\left(3|\Omega|+2\widetilde{C}\left(\frac{U}{\nu}\right)^{\alpha'}\right)^{\frac{2-\alpha}{\alpha}}}\limsup_k\|D(y^{\varepsilon_k}-y)\|_\alpha^2\\ &\leq \limsup_k\left(\tau_{L,\varepsilon_k}\left(Dy^{\varepsilon_k}\right)-\tau_{L,\varepsilon_k}\left(Dy\right),D\left(y^{\varepsilon_k}-y\right)\right)\leq\limsup_k\left(\tau_{L,\varepsilon_k}\left(Dy^{\varepsilon_k}\right),D\left(y^{\varepsilon_k}-y\right)\right)\\ &\leq \limsup_k\left((v_{\varepsilon_k},y^{\varepsilon_k})-\left(\tau_{L,\varepsilon_k}\left(Dy_{\varepsilon_k}\right),Dy\right)\right)\leq(u,y)-(\widetilde{\tau},Dy)=0\end{aligned}$$

and the claimed result is proven □

Next, we prove that the solutions of problems $(P_{L,\varepsilon})$ form an approximating family for (P_L).

Proposition 6.4 *Assume that $\frac{3n}{n+2}\leq\alpha<2$. Let $(\bar u^\varepsilon,\bar y^\varepsilon)$ be a solution of $(P_{L,\varepsilon})$. There exists a subsequence $(\varepsilon_k)_k$ converging to zero such that*

$$\lim_{k\to+\infty}\|\bar u^{\varepsilon_k}-\bar u\|_2=0,\qquad \lim_{k\to+\infty}\|\bar y^{\varepsilon_k}-\bar y\|_{1,\alpha}=0,\qquad \lim_{k\to+\infty}I(\bar u^{\varepsilon_k},\bar y^{\varepsilon_k})=J(\bar u,\bar y).$$

Proof. Setting $v_\varepsilon=\bar u$ for all $\varepsilon>0$ and $v_\varepsilon=\bar u^\varepsilon$ and applying Proposition 6.3, we deduce that there exists a subsequence $(\varepsilon_k)_k$ such that $(y_{\bar u}^{\varepsilon_k})_k$ converges in $W_0^{1,\alpha}(\Omega)$ to $\bar y$ (the unique solution of (1.1) corresponding to $\bar u$), $(\bar u^{\varepsilon_k})_k$ weakly converges in $L^2(\Omega)$ to some u and $(\bar y^{\varepsilon_k})_k$ converges in $W_0^{1,\alpha}(\Omega)$ to y (a solution of (1.1) corresponding to u). Using the lower semicontinuity of I and the admissibility of $(\bar u,y_{\bar u}^{\varepsilon_k})$ for (P_{L,ε_k}), we obtain

$$\begin{aligned}\tfrac12\|y-y_d\|_2^2+\tfrac{\lambda}{2}\|u\|_2^2+\tfrac12\|u-\bar u\|_2^2&\leq\liminf_k I(\bar u^{\varepsilon_k},\bar y^{\varepsilon_k})\leq\limsup_k I(\bar u^{\varepsilon_k},\bar y^{\varepsilon_k})\\ &\leq\lim_k I(\bar u,y_{\bar u}^{\varepsilon_k})=\tfrac12\|\bar y-y_d\|_2^2+\tfrac{\lambda}{2}\|\bar u\|_2^2\end{aligned}$$

and consequently

$$J(u,y)+\tfrac12\|u-\bar u\|_2^2\leq J(\bar u,\bar y).$$

Since $(\bar u,\bar y)$ is solution of (P_L), we have $J(\bar u,\bar y)\leq J(u,y)$ and thus $u=\bar u$. Recalling that $\bar u$ satisfies $(\mathcal{CR})$, we deduce that $y=\bar y$ and thus

$$\lim_{k\to+\infty}I(\bar u^{\varepsilon_k},\bar y^{\varepsilon_k})=J(\bar u,\bar y).$$

Finally, observing that

$$\begin{aligned}\tfrac12\limsup_k\|\bar u^{\varepsilon_k}-\bar u\|_2^2&=\limsup_k\left(I(\bar u^{\varepsilon_k},\bar y^{\varepsilon_k})-\tfrac12\|\bar y^{\varepsilon_k}-y_d\|_2^2-\tfrac{\lambda}{2}\|\bar u^{\varepsilon_k}\|_2^2\right)\\ &\leq J(\bar u,\bar y)-\tfrac12\|\bar y-y_d\|_2^2-\tfrac{\lambda}{2}\liminf_k\|\bar u^{\varepsilon_k}\|_2^2\\ &=\tfrac{\lambda}{2}\|\bar u\|_2^2-\tfrac{\lambda}{2}\liminf_k\|\bar u^{\varepsilon_k}\|_2^2\leq0\end{aligned}$$

we conclude that $(\bar u^{\varepsilon_k})_k$ converges to $\bar u$ strongly in $L^2(\Omega)$. □

6.3 Proof of Theorem 6.1

Let $(\bar{u}^{\varepsilon_k}, \bar{y}^{\varepsilon_k})$ be the solution of (P_{L,ε_k}) given in Proposition 6.4. Since $\bar{u}$ satisfies $(\mathcal{CR})$, we deduce that there exists $k_1 \in \mathbb{N}$ such that $\bar{u}^{\varepsilon_k}$ also satisfies $(\mathcal{CR})$ for every $k > k_1$. Due Theorem 6.2, there exists $\bar{p}^{\varepsilon_k} \in V_2$ such that

$$\begin{cases} -\nabla \cdot \left(\tau'_{L,\varepsilon_k}(D\bar{y}^{\varepsilon_k})^T : Dp\right) + (\nabla \bar{y}^{\varepsilon_k})^T p - \bar{y}^{\varepsilon_k} \cdot \nabla p + \nabla \widetilde{\pi}^{\varepsilon_k} = \bar{y}^{\varepsilon_k} - y_d, \\ \nabla \cdot p = 0, \\ p_{|\Gamma} = 0, \end{cases} \tag{6.13}$$

$$(\bar{p}^{\varepsilon_k} + (\lambda+1)\bar{u}^{\varepsilon_k} - \bar{u}, v - \bar{u}^{\varepsilon_k}) \geq 0 \qquad \text{for all } v \in U_{ad}. \tag{6.14}$$

Moreover, due to (3.8) and Proposition 4.3 the following estimates hold

$$\|\bar{y}^{\varepsilon_k}\|_{2,q} \leq \widetilde{\kappa} \frac{\|\bar{u}^{\varepsilon_k}\|_q}{\nu} \leq \widetilde{\kappa} \frac{U}{\nu},$$

$$\|D\bar{p}^{\varepsilon_k}\|_2 \leq \mathcal{L}\left(\frac{\|\bar{u}^{\varepsilon_k}\|_q}{\nu}\right) \|\bar{y}^{\varepsilon_k} - y_d\|_2 \leq \mathcal{L}\left(\frac{U}{\nu}\right) \|\bar{y}^{\varepsilon_k} - y_d\|_2$$

where $\widetilde{\kappa} \equiv \widetilde{\kappa}(n, q, \Omega)$ is a positive constant independent of k. Therefore, the sequences $(\bar{y}^{\varepsilon_k})_k$ and $(\bar{p}^{\varepsilon_k})_k$ are uniformly bounded in $W^{2,q}(\Omega)$ and V_2, respectively. There then exist a subsequence, still indexed by k, and $\bar{p} \in V_2$ such that $(\bar{p}^{\varepsilon_k})_k$ weakly converges to $\bar{p}$ in V_2. Moreover, since $q > n$, by compactness results on Sobolev spaces, we deduce that $(\bar{y}^{\varepsilon_k})_k$ strongly converges to $\bar{y}$ in $C^{1,\delta}(\overline{\Omega})$ with $\delta < 1 - \frac{n}{q}$ and $(\bar{p}^{\varepsilon_k})_k$ strongly converges to $\bar{p}$ in $L^2(\Omega)$. It follows that

$$\lim_{k \to +\infty} \left((\nabla \bar{y}^{\varepsilon_k})^T \bar{p}^{\varepsilon_k} - \bar{y}^{\varepsilon_k} \cdot \nabla \bar{p}^{\varepsilon_k}, \varphi\right) = \left((\nabla \bar{y})^T \bar{p} - \bar{y} \cdot \nabla \bar{p}, \varphi\right) \tag{6.15}$$

for every $\varphi \in \mathcal{V}$ and

$$\lim_{k \to +\infty} \left(\tau'_{L,\varepsilon_k}(D\bar{y}^{\varepsilon_k}) : D\varphi, D\bar{p}^{\varepsilon_k}\right) = (\tau'_L(D\bar{y}) : D\varphi, D\bar{p}) \tag{6.16}$$

for every $\varphi \in \mathcal{V}(\Omega_0) = \{\varphi \in \mathcal{D}(\Omega_0) \mid \nabla \cdot \varphi = 0 \quad \text{in } \Omega_0\}$ with $\Omega_0 = \{x \in \Omega \mid |D\bar{y}(x)| > 0\}$. Taking into account (6.15) and (6.16), and passing to the limit in (6.13) and (6.14), we obtain

$$(\tau'_L(D\bar{y}) : D\varphi, D\bar{p}) + \left((\nabla \bar{y})^T \bar{p} - \bar{y} \cdot \nabla \bar{p}, \varphi\right) = (\bar{y} - y_d, \varphi) \quad \text{for all } \varphi \in \mathcal{V}(\Omega_0),$$

and

$$(\bar{p} + \lambda \bar{u}, v - \bar{u}) \geq 0 \qquad \text{for all } v \in U_{ad}$$

which gives (6.1) and (6.2) and the claimed result is proven. □

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