On the product of independent Generalized Gamma random variables

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Abstract
The product of independent Generalized Gamma random variables arises in many problems and applications of the most different areas. Working with characteristic functions, which are the Fourier transforms of density functions, we study the structure of the exact distribution and, based on a truncation of the characteristic function of the negative logarithm of the product of independent Generalized Gamma random variables, a simple and accurate near-exact distribution is developed. The density and cumulative distribution functions of the near-exact distribution have manageable expressions allowing for the computation of $p-$values and quantiles. In the process, a flexible parameter, γ, is introduced in the representations of the exact and near-exact distributions which allows to choose the quality of the approximation developed. The numerical studies and simulations carried out show the accuracy of this approximation as well as its asymptotic properties.

Keywords
asymptotic approximation, characteristic function, Gamma distribution, logGamma distribution, Generalized Integer Gamma distribution, Generalized Near-Integer Gamma distribution, near-exact distributions.

AMS subject classifications: 62H10, 62E20, 62H05, 62E15

1 Introduction

The Generalized Gamma distribution was introduce in [20] and is of great interest, mainly due to its flexibility, in several areas of application, for example in Physics, Econometrics, Wireless Communications, Reliability Analysis, Hydrological Processes and Life Testing Models [1, 2, 12, 15, 19]. Another point of interest of this distribution is due to fact of having as particular cases some of the most important and known distributions such as the Gamma, Weibull and Raleigh distributions, for more details see for example [5]. Results on inference on the parameters of the Generalized Gamma distribution can be found in [8, 9, 10, 21, 7, 16]. The product of independent Generalized Gamma random variables arises naturally on problems posed in the above mentioned fields of research however the exact distribution does not have a simple expression which limits its use in practical terms. The exact distribution of the product of independent Generalized Gamma random variables was derived in [15], using the inverse Mellin transform and H-functions, but the hight computational investment required limits the practical usefulness of this result. The authors also present representations in terms of Meijer-G functions for two particular cases; when all the parameters β_j in (1) are equal or when these are rational numbers. At the same time in [17] the author also developed a representation in terms of Meijer-G function for the distribution of the product of independent Generalized Gamma random variables when all the β_j in (1) are equal. In [13] the exact density and cumulative distribution functions are obtained, for the case of the product of two Generalized Gamma random variables with equal shape parameters, using the modified Bessel function and the incomplete gamma function.

Using the characteristic function of the negative logarithm of the product of independent Generalized Gamma random variables, we give a different representation for the exact distribution and, based on this representation, we develop a simple and accurate near-exact distribution for the distribution of the product of independent Generalized Gamma random variables. In Section 2 we show that exact distribution of the negative logarithm of the product of independent Generalized Gamma random variables may be written as the distribution of the sum of two independent random variables, one with the distribution of the sum of independent logGamma distributions multiplied by a parameter and the other with the distribution of a shifted Generalized Integer Gamma distribution [3]. Based on this result we develop a simple but highly accurate near-exact distribution that is based on the shifted Generalized Near-Integer Gamma distribution [4] which has manageable expressions for the density and cumulative distribution functions. By construction, a new parameter, γ, is introduced on the previous representations which allows the improvement of the quality of the near-exact approximation developed by adjusting its value; we show that the approximation becomes more precise for high values of γ. In Section 3 we use a measure of proximity between characteristic functions, that is also an upper bound on the proximity between distribution functions, to assess the quality of the near-exact distribution proposed. In addition, simulations are carried out that show the accuracy of the near-exact quantiles and plots of the density and cumulative distribution functions are also presented.

2 The exact and near-exact distribution of the product of independent Generalized Gamma random variables

2.1 The exact distribution

Let X_j be a random variable with Gamma distribution with rate parameter $\lambda_j > 0$ and shape parameter $r_j > 0$, that is, $X_j \sim \Gamma(r_j, \lambda_j)$ with $j = 1, \ldots, p$. We say that the random variable $Y_j = X_j^{1/\beta_j}$, for $\beta_j \neq 0$, has a generalized Gamma distribution and we will denote this fact by

$$Y_j \sim G\Gamma(r_j, \lambda_j, \beta_j)\,. \tag{1}$$

The probability density function is given by

$$f_{Y_j}(y) = |\beta_j| \frac{\lambda_j^{r_j}}{\Gamma(r_j)} y^{\beta_j r_j - 1} \mathrm{e}^{-\lambda_j y^{\beta_j}} \quad , \quad y > 0$$

and the non central moments are

$$E\left[Y_j^h\right] = \int_{-\infty}^{\infty} y^h f_{Y_j}(y)\mathrm{d}y = \frac{\Gamma\left(r_j + h/\beta_j\right)}{\Gamma(r_j)}\,\lambda_j^{-h/\beta_j}\,. \tag{2}$$

We are interested in studding the distribution of

$$Z = \prod_{j=1}^{p} Y_j$$

with $Y_j \overset{\text{ind.}}{\sim} G\Gamma(r_j, \lambda_j, \beta_j)$ $j = 1, \ldots, p$. The characteristic function of the random variable $W = -\log Z$ is defined as

$$\Phi_W(t) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}tw} f_W(w)\mathrm{d}w = E[\mathrm{e}^{\mathrm{i}tW}]$$

which, using the expression of the non central moments in (2), may be written as

$$\Phi_W(t) = E[Z^{-\mathrm{i}t}] = \prod_{j=1}^{p} E[Y_j^{-\mathrm{i}t}] = \prod_{j=1}^{p} \frac{\Gamma\left(r_j - \mathrm{i}t/\beta_j\right)}{\Gamma(r_j)}\,\lambda_j^{\mathrm{i}t/\beta_j}\,, \quad t \in \mathbb{R}. \tag{3}$$

Theorem 2.1 *The characteristic function of* $W = -\log \prod_{i=j}^{p} Y_j$ *with* $Y_j \overset{ind.}{\sim} G\Gamma(r_j, \lambda_j, \beta_j)$ *for* $r_j > 0$, $\lambda_j > 0$ *and* $\beta_j \neq 0$ *may be written as*

$$\Phi_W(t) = \underbrace{\left\{\prod_{j=1}^{p} \frac{\Gamma\left(r_j + \gamma - \mathrm{i}t/\beta_j\right)}{\Gamma(r_j + \gamma)}\right\}}_{\Phi_{W_1}(t)} \underbrace{\left\{\prod_{j=1}^{p}\prod_{k=0}^{\gamma-1}((r_j + k)\beta_j)((r_j + k)\beta_j - \mathrm{i}t)^{-1}\right\} \mathrm{e}^{\mathrm{i}t\sum\limits_{j=1}^{p}\log\lambda_j^{1/\beta_j}}}_{\Phi_{W_2}(t)} \tag{4}$$

Proof: See Appendix A.

From expression (4) and from the properties of the characteristic functions we may conclude that, when $\beta_j > 0$, the exact distribution of $W = -\log Z$ corresponds to the distribution of the sum of two independent random variables; W_1, with the distribution of the sum of independent logGamma distributions with parameters $r_j+\gamma$ and 1, multiplied by the parameter $1/\beta_j$ and, W_2, with the distribution of a shifted sum of $p \times \gamma$ independent Exponential distributions with parameters $(r_j + k)\beta_j$ for $j = 1, \ldots, p$ and $k = 0, \ldots, \gamma - 1$, and with shift parameter $\theta = \sum_{j=1}^{p} \log \lambda_j^{1/\beta_j}$. If we sum the Exponential distributions with the same parameter we may write $\Phi_{W_2}(t)$ in expression (4) as

$$\Phi_{W_2}(t) = \left\{\prod_{j=1}^{\ell} \alpha_j^{\delta_j}(\alpha_j - \mathrm{i}t)^{-\delta_j}\right\} \mathrm{e}^{\mathrm{i}t\theta} \tag{5}$$

where ℓ is the number of Exponential distributions with different parameters, α_j are the parameters of such Exponential distributions, and δ_j is the number of Exponential distributions with the same parameter α_j, for $j = 1, \ldots, \ell$. Thus we may say the distribution of

W_2 corresponds to a shifted Generalized Integer Gamma distribution (see [3]) with shape parameters δ_j, rate parameters α_j, for $j = 1, \ldots, \ell$, and with depth ℓ and shift parameter θ. We denote this fact by $W_2 \sim \text{SGIG}(\delta_1, \ldots, \delta_\ell\,;\, \alpha_1, \ldots, \alpha_\ell\,;\, \ell\,;\, \theta)$.

Using this representation, given by expressions (4) and (5), in the next subsection, we derive a simple but very accurate near-exact distribution for the distribution of the product of independent Generalized Gamma random variables.

2.2 Near-exact distribution for $W = -\log Z$ and for Z

One way of developing near-exact distributions is by using a factorization of the characteristic function of a random variable or of its logarithm and then approximating one the factors and leaving the remaining ones unchanged so that the resulting characteristic function corresponds to a known distribution and easy to use in practice. In our case, starting from the factorization of $\Phi_W(t)$ in (4) with $\Phi_{W_2}(t)$ in (5), we approximate the characteristic function $\Phi_{W_1}(t)$ by the characteristic function of a random variable with a shifted Gamma distribution and we leave $\Phi_{W_2}(t)$ unchanged. Thus, the near-exact characteristic function has the following structure

$$\Phi_{W_1^*}(t)\Phi_{W_2}(t) \tag{6}$$

where

$$\Phi_{W_1^*}(t) = \omega^\rho(\omega - \mathrm{i}t)^{-\rho}\mathrm{e}^{\mathrm{i}t\upsilon}, \quad t \in \mathbb{R}, \tag{7}$$

is the characteristic function of W_1^* with a logGamma distribution, with shape parameter ρ, rate parameter ω and shift parameter υ, we denote this fact by $W_1^* \sim \text{SGamma}(\rho, \omega, \upsilon)$. The parameters ρ, ω, and υ, are determined as the numerical solution of the system of equations

$$\left.\frac{\partial^j \Phi_{W_1^*}(t)}{\partial t^j}\right|_{t=0} = \left.\frac{\partial^j \Phi_{W_1}(t)}{\partial t^j}\right|_{t=0}, \qquad j = 1, 2, 3\,. \tag{8}$$

Theorem 2.2 *If we use as an asymptotic approximation of $\Phi_{W_1}(t)$ in (4) the characteristic function $\Phi_{W_1^*}(t)$ in (7), we obtain as near-exact distribution for $W = -\log\prod_{j=1}^p Y_j$ with $Y_j \overset{ind.}{\sim} G\Gamma(r_j, \lambda_j, \beta_j)$ for $r_j > 0$, $\lambda_j > 0$ and $\beta_j > 0$ $(j = 1, \ldots, p)$ a shifted Generalized Near-Integer Gamma distribution with integer shape parameters $\delta_1, \ldots, \delta_\ell$ and non-integer shape parameter ρ, with rate parameters $\alpha_1, \ldots, \alpha_\ell$ and ω, with depth $\ell + 1$ and shift parameter $\theta + \upsilon$, that we will denote by*

$$W_2 \sim \text{SGING}(\delta_1, \ldots, \delta_\ell\,, \rho\,;\, \alpha_1, \ldots, \alpha_\ell, \omega\,;\, \ell + 1\,;\, \theta + \upsilon)$$

where δ_j, α_j, θ and ℓ are the same as in (5) *and ρ, ω, and υ are obtained as the numerical solution of* (8)*.*

Proof: See Appendix A.

By simple transformation, using the notation of Appendix B in [14] for the GNIG distribution and considering the necessary adjustments due to the presence of the shift parameter, we obtain the near-exact probability density function of $Z = \prod_{j=1}^p Y_j$

$$f^{\text{GNIG}}\bigg(-\log(z) - (\theta + \upsilon)|\ \delta_1, \ldots, \delta_\ell, \rho\,;\, \alpha_1, \ldots, \alpha_\ell, \omega\,;\, \ell + 1\bigg)\frac{1}{z} \tag{9}$$

and cumulative distribution function

$$1-F^{\mathrm{GNIG}}\Big(-\log(z)-(\theta+\upsilon)\,|\,\delta_1,\dots,\delta_\ell,\rho\,;\,\alpha_1,\dots,\alpha_\ell,\omega\,;\,\ell+1\Big)\,. \tag{10}$$

In Section 3 we present plots for the probability density and cumulative distribution functions respectively given in (9) and (10). Modules for the implementation of the near-exact distribution proposed may be developed, for example, with software Mathematica or R, and may be obtained from the author.

3 Numerical Studies and simulations

In this section we study the quality of the near-exact distribution developed for the distribution of the product of independent Generalized Gamma random variables. For this purpose we consider, in next tables and figures, the following cases:

Case I - $r_j=\{2,3,5\}$, $\lambda_j=\{1,2,10\}$ and $\beta_j=\{5,6,7\}$;

Case II - $r_j=\{\frac{1}{4},\frac{9}{5},1\}$, $\lambda_j=\{\sqrt{2},\pi,3\}$ and $\beta_j=\{\frac{2}{7},\frac{2}{11},\sqrt{3}\}$;

Case III - $r_j=\{1,2\}$, $\lambda_j=\{5,6\}$ and $\beta_j=\{3,4\}$.

In Table 1 we computed the exact quantiles for the product of independent Generalized Gamma random variables using the inversion formulas in [6] and we compared them with simulated quantiles (denoted by "simul.") evaluated from samples of size 10 000 000 and with the near-exact quantiles (denoted by "near") obtained, for $\gamma=30$, using the near-exact cumulative distribution function in (10). From Table 1 we conclude that the values obtained for the near-exact quantiles are very close to exact ones matching at least 5 decimal places and are more reliable than the quantiles that were obtained by simulation.

Table 1: Comparison between exact, simulated and near-exact quantiles

	Case - I			Case - II			Case - III		
prob.	exact	simul.	near	exact	simul.	near	exact	simul.	near
0.90	1.291245	1.291189	1.291246	1.696369	1.695775	1.696369	0.588087	0.588144	0.588087
0.95	1.369168	1.368910	1.369169	2.527685	2.524854	2.527687	0.657770	0.657572	0.657771
0.99	1.517857	1.517909	1.517854	4.691656	4.692405	4.691657	0.794812	0.794915	0.794811

Now we consider the measure

$$\Delta=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\left|\frac{\Phi_W(t)-\Phi^*(t)}{t}\right|\,\mathrm{d}t\,,$$

where $\Phi_W(t)$ and $\Phi^*(t)$ are respectively the exact and approximate characteristic functions of W, and $F_W(w)$ and $F^*(w)$ the corresponding cumulative distribution functions. This measure is also an upper bound on the proximity between distribution functions, since

$$\sup_{w\in\mathbb{R}}|F_W(w)-F^*(w)|\le\Delta\,.$$

For more details on this measure see for example [14]. In Table 2 we present values of the measure Δ for the three cases considered. In this table, the values of the measure are low which indicates that the approximation is adequate and precise, and we may also note that the values are even lower for increasing values of γ.

Table 2: Values of Δ for the SGNIG proposed considering different values of γ; using the near-exact characteristic function in (6) and the exact characteristic function in (3)

γ	Case - I	Case - II	Case - III
4	1.7×10^{-4}	3.3×10^{-5}	1.2×10^{-4}
10	2.0×10^{-5}	2.1×10^{-6}	1.1×10^{-5}
50	2.4×10^{-7}	1.6×10^{-8}	1.0×10^{-7}
500	2.7×10^{-10}	1.6×10^{-11}	1.1×10^{-10}

In Figure 1 we present examples of plots of the near-exact cumulative distribution and density functions, obtained form expressions (10) and (9) respectively, when $\gamma = 30$ and for the three cases considered in this paper.

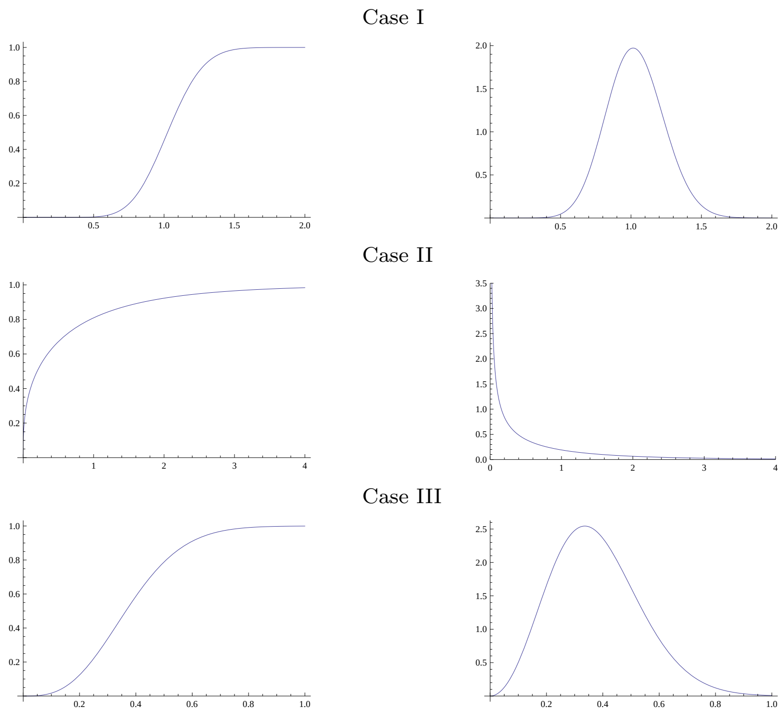

Figure 1: Near-exact distribution and density functions for the product of independent Generalized Gamma random variables when $\gamma = 30$

4 Conlusions

The procedure used in this work enabled the development of a simple and accurate near-exact distribution for the distribution of the product of independent Generalized Gamma random variables. This near-exact distribution is based on a shifted Generalized Near-Integer Gamma distribution which has manageable and easy to implement density and cumulative distribution functions. The numerical studies developed show the accuracy and adequability of the near-exact distribution developed as well as its good asymptotic properties for increasing values of the parameter γ.

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A Proof of Theorem 2.1 and 2.2

Proof of Theorem 2.1
Using the expression of the characteristic function of W in (3) we may do the following algebraic manipulation that leads to the result of Theorem 2.1

$$\begin{aligned}
\Phi_W(t) &= \prod_{j=1}^{p} \frac{\Gamma\left(r_j - \mathrm{i}t/\beta_j\right)}{\Gamma(r_j)}\, \lambda_j^{\mathrm{i}t/\beta_j} \\
&= \left\{ \prod_{j=1}^{p} \frac{\Gamma(r_j+\gamma-\mathrm{i}t/\beta_j)}{\Gamma(r_j+\gamma)} \frac{\Gamma(r_j+\gamma)}{\Gamma(r_j+\gamma-\mathrm{i}t/\beta_j)} \frac{\Gamma\left(r_j-\mathrm{i}t/\beta_j\right)}{\Gamma(r_j)} \right\} \mathrm{e}^{\mathrm{i}t\sum_{j=1}^{p}\log\lambda_j^{1/\beta_j}} \\
&= \prod_{j=1}^{p} \frac{\Gamma(r_j+\gamma-\mathrm{i}t/\beta_j)}{\Gamma(r_j+\gamma)} \left\{ \prod_{j=1}^{p}\prod_{k=0}^{\gamma-1} \left((r_j+k)\beta_j\right)\left((r_j+k)\beta_j-\mathrm{i}t\right)^{-1} \right\} \mathrm{e}^{\mathrm{i}t\sum_{j=1}^{p}\log\lambda_j^{1/\beta_j}} .
\end{aligned}$$

□

Proof of Theorem 2.2
It is enough to note that

$$\begin{aligned}
\Phi_{W_1^*}(t)\Phi_{W_2}(t) &= \underbrace{\omega^{\rho}(\omega-\mathrm{i}t)^{-\rho}\mathrm{e}^{\mathrm{i}t\upsilon}}_{\Phi_{W_1^*}(t)} \underbrace{\left\{ \prod_{j=1}^{\ell} \alpha_j^{\delta_j}(\alpha_j-\mathrm{i}t)^{-\delta_j} \right\} \mathrm{e}^{\mathrm{i}t\theta}}_{\Phi_{W_2}(t)} \\
&= \left(\left\{ \prod_{j=1}^{\ell} \alpha_j^{\delta_j}(\alpha_j-\mathrm{i}t)^{-\delta_j} \right\} \omega^{\rho}(\omega-\mathrm{i}t)^{-\rho} \right) \mathrm{e}^{\mathrm{i}t(\upsilon+\theta)}
\end{aligned}$$

where δ_j, α_j, θ and ℓ are the same as in (5) and ρ, ω, and υ are obtained as the numerical solution of (8). □

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