Optimal control of shear-thinning fluids

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Abstract

The aim of this paper is to establish necessary optimality conditions for optimal control problems governed by steady, incompressible Navier-Stokes equations with shear-dependent viscosity. The main difficulty is related with the differentiability of the control-to-state mapping and is overcome by introducing a family of smooth approximate control problems, and by passing to the limit in the corresponding optimality conditions.

Key words. Optimal control, steady Navier-Stokes equations, shear-thinning flows, necessary optimality conditions.

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1 Introduction

This paper deals with optimal control problems associated with a viscous, incompressible fluid described by the following partial differential equations that generalize the Navier-Stokes system

$$\begin{cases} -\nabla\cdot(\tau(Dy)) + y\cdot\nabla y + \nabla\pi = u & \text{in } \Omega, \\ \nabla\cdot y = 0 & \text{in } \Omega, \\ y = 0 & \text{on } \Gamma, \end{cases} \tag{1.1}$$

where y is the velocity field, π is the pressure, τ is the extra stress tensor, $Dy = \frac{1}{2}\left(\nabla y + (\nabla y)^T\right)$ is the symmetric part of the velocity gradient ∇y, u is the given body force and $\Omega \subset I\!R^n$ ($n = 2$ or $n = 3$) is a bounded domain with boundary Γ. We assume that $\tau : I\!R^{n\times n}_{sym} \longrightarrow I\!R^{n\times n}_{sym}$ has a potential, i.e. there exists a function $\Phi \in C^2(I\!R^+_n, I\!R^+_n)$ with $\Phi(0) = 0$ such that

$$\tau_{ij}(\eta) = \frac{\partial\Phi(|\eta|^2)}{\partial\eta_{ij}} = 2\Phi'(|\eta|^2)\,\eta_{ij} \qquad \text{for all } \eta \in I\!R^{n\times n}_{sym}, \qquad \tau(0) = 0.$$

$\big($Here $I\!R^{n\times n}_{sym}$ consists of all symetric $(n\times n)$-matrices.$\big)$ Moreover, we assume that the following assumptions hold

A$_1$ - There exists a positive constant γ such that for all $i, j, k, \ell = 1, \cdots, n$

$$\left|\frac{\partial\tau_{k\ell}(\eta)}{\partial\eta_{ij}}\right| \le \gamma\left(1 + |\eta|^2\right)^{\frac{\alpha-2}{2}} \qquad \text{for all } \eta \in I\!R^{n\times n}_{sym}.$$

A$_2$ - There exists a positive constant ν such that

$$\tau'(\eta) : \zeta : \zeta = \sum_{ijk\ell} \frac{\partial\tau_{k\ell}(\eta)}{\partial\eta_{ij}}\zeta_{k\ell}\zeta_{ij} \ge \nu\left(1 + |\eta|^2\right)^{\frac{\alpha-2}{2}} |\zeta|^2 \quad \text{for all } \eta, \zeta \in I\!R^{n\times n}_{sym}.$$

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These assumptions are usually used in the literature and cover a wide range of non-Newtonian fluids. Typical prototypes of extra tensors used in applications are

$$\tau(\eta) = 2\nu \left(1 + |\eta|^2\right)^{\frac{\alpha-2}{2}} \eta \qquad \text{or} \qquad \tau(\eta) = 2\nu \left(1 + |\eta|\right)^{\alpha-2} \eta.$$

We recall that a fluid is called shear-thickening if $\alpha > 2$ and shear-thinning if $\alpha < 2$. For the special case $\tau(\eta) = 2\nu\eta$ ($\alpha = 2$), we recover the Navier-Stokes equation with viscosity coeficient $\nu > 0$.

The paper is concerned with the following optimal control problem

$$(P_\alpha) \qquad \begin{array}{ll} \text{Minimize} & J(u, y) = \frac{1}{2} \displaystyle\int_\Omega |y - y_d|^2 \, dx + \frac{\lambda}{2} \int_\Omega |u|^2 \, dx \\ \text{Subject to} & (u, y) \in U_{ad} \times W_0^{1,\alpha}(\Omega) \text{ satisfies (1.1) for some } \pi \in L^\alpha(\Omega) \end{array}$$

where y_d is some desired velocity field, λ is a positive constant, the set of admissible controls U_{ad} is a nonempty convex closed subset of $L^2(\Omega)$ and $\frac{3n}{n+2} \le \alpha \le 2$. Although the analysis of several results can be more general, in order to simplify the redaction, we will assume that $U_{ad} \subset \{v \in L^2(\Omega) \mid \|v\|_2 \le U\}$ for some $U > 0$.

The considered class of fluids is described by partial differential equations of the quasi-linear type. It was first proposed by Ladyzhenskaya in [17], [18] and [19] as a modification of the Navier-Stokes system (the viscosity depending on the shear-rate), and was similarly suggested by Lions in [20]. Existence of weak solutions was proved by both authors using compactness arguments and the theory of monotone operators. Since these pioneering results, much has been done and we emphasize the works by Nečas *et al.* who proved existence of weak and measure-valued solutions under the less restrictive assumption $\alpha > \frac{2n}{n+2}$ (see for example [25] and [11]).

In the absence of flow convection, optimal control problems governed by generalized Stokes systems can be studied following the ideas developed in [4] and [6] for problems governed by quasilinear elliptic equations. Similar underlying difficulties, consequence of the nonlinearity of the extra-stress tensor, are related with the differentiability of the control-to-state mapping. The corresponding analysis cannot be achieved in Sobolev spaces and the natural setting for the linearized equation and the adjoint state equation involves weighted Sobolev spaces. The lack of regularity of the state variable in the case of shear-thinnig fluids creates an additional difficulty that can be overcome by considering a family of approximate problems falling into the case $\alpha = 2$. Differentiability of the approximate control-to-state mapping can then be established, allowing to derive the appporoximate optimality conditions, and the optimality conditions by passing to the limit.

The case of problems governed by generalized Navier-Stokes equations is more delicate since another difficulty arises in connection with the convective term and the uniqueness of the state variable, guaranteed under some constraint on the data. It is similarly encountered when studying problems governed by the Navier-Stokes equations for which the necessary optimality conditions can be established by restraining all the admissible controls to satisfy this constraint (see for example [10] and [26]).

The difficulties related with the nonlinearity of the extra stress tensor and the convective term can be more easily handled (especially in the case of shear-thinning flows) if the gradient of the velocity is bounded. The corresponding viscosity, although non constant, is also bounded and the system can be studied as in the case of Navier-Stokes equations. These regularity results are few, difficult to obtain in general and do not seem to be available for the three-dimensional case. For the two-dimensional steady case, the boundedness of the gradient was proved by Kaplický *et al.* in [22] enabling Slawig to derive the corresponding optimality conditions in [27]. Similarly, Wachsmuth and Roubíček used the regularity results established in [21] to derive the optimality conditions for a two-dimensional unsteady system describing the flow of shear-tickening fluids (see [28]). Related to this aspect, we also mention Gunzburger and Trenchea who used the regularity results obtained in [14] to derive the optimality conditions for a problem governed by a three-dimensional modified Navier-Stokes system coupled with Maxwell equations (see [15]).

There are few works dealing with these problems when no higher regularity results are available. We mention the recent paper by De los Reyes [8], who considered a problem governed by the Bingham nonlinear mixed variational inequality. Besides difficulties induced by the nonlinearity of the viscoplastic and the convective terms, the non-regularity of the model has to be managed. By exploiting the specific structure of the non-differentiable term, a family of regularized problems is introduced, the corresponding optimality systems are derived and the optimality conditions for the original problem are obtained by passing to the limit. We also mention our work dealing with steady shear-thickening fluids where the restriction on the set of admissible controls has been relaxed and the optimality conditions obtained under a precise condition on the optimal control (see [2] and [3]).

In the case of shear-thinning fluids, the problems are more difficult to handle. The techniques developped in [2] cannot be directly applied because of the combined effect of the convective term and the nonlinear stress tensor. Moreover, unlike the case of problems governed by generalized Stokes systems and unless we restrict all the admissible controls, the differentiability of the approximate control-to-state mapping is not guaranteed, the approximate control problem does not fall in the case $\alpha = 2$ and further analysis is needed. Let us finally mention that in [2], [3] and in the present work, the considered potential is C^2. Nevertheless, the problems are still challenging especially in the case of shear-thinning flows. A further interesting aspect would be the adaptation of the techniques developed in [8] to the case of a less regular potential. (In this respect, see also the paper by Casas and Fernandez [5].)

In the present paper, we establish explicite estimates, carefully analyse the related equations and derive optimality conditions without restraining the set of admissible controls. The only constraint concerns the optimal control. The plan is as follows. Assumptions, notation and some preliminary results are given in Section 2. Section 3 is devoted to existence and uniqueness results for the state equation and to the derivation of corresponding estimates. Section 4 deals with existence of an optimal control while necessary optimality conditions are given in Section 5. In Section 6, we introduce a family of approximate control problems, study the properties of the corresponding control-to-state mapping and establish the approximate optimality conditions. By passing to the limit, we prove the optimality conditions for the control problem in Section 7.

2 Notation and preliminary results

Throughout the paper $\Omega \subset I\!R^n$ (n=2 or n=3) is a bounded domain with a boundary Γ of class C^2. Since many of the quantities occuring in the paper are vector-valued functions, we will use the same notation of norms for scalar, vector and matrix-valued functions for the sake of brevity.

2.1 Function spaces and classical inequalities

Let us define some useful function spaces. The space of infinitely differentiable functions with compact support in Ω will be denoted by $\mathcal{D}(\Omega)$. The standard Sobolev spaces are denoted by $W^{k,\alpha}(\Omega)$ ($k \in I\!N$ and $1 < \alpha < \infty$), and their norms by $\|\cdot\|_{k,\alpha}$. We set $W^{0,\alpha}(\Omega) \equiv L^\alpha(\Omega)$ and $\|\cdot\|_{L^\alpha} \equiv \|\cdot\|_\alpha$. In order to eliminate the pressure in the weak formulation of the state equation, we will work in divergence-free spaces. Consider

$$\mathcal{V} = \{\varphi \in \mathcal{D}(\Omega) \mid \nabla \cdot \varphi = 0\},$$

and denote by V_α the closure of $\mathcal{V}$ in the L^α-norm of gradients, i.e.

$$V_\alpha = \left\{\varphi \in W_0^{1,\alpha}(\Omega) \mid \nabla \cdot \varphi = 0\right\}.$$

Given $y \in W_0^{1,\alpha}(\Omega)$, we can associate two weighted Sobolev spaces V_α^y and H_α^y, where V_α^y is the set of functions $z \in V_2$ such that the norm $\|\cdot\|$ defined by

$$\|z\| = \left\|(1+|Dy|^2)^{\frac{\alpha-2}{4}} Dz\right\|_2$$

is finite, and H_α^y is the completion of $\mathcal{V}$ in V_α^y. It may be verified that V_α^y and H_α^y are Hilbert spaces and that $H_\alpha^y \subset V_\alpha^y$. Moreover, we have $V_2 \subset H_\alpha^y \subset V_\alpha$ if $\alpha \le 2$, with continuous injections. Weighted Sobolev Spaces of this type have been studied by Coffman et al. [7], Murthy and Stampacchia [24].

Let us now collect some useful auxiliary results. We begin by three classical inequalities.

Lemma 2.1 (*Poincaré's inequality.*) *Let y be in $W_0^{1,\alpha}(\Omega)$ with $1 < \alpha \le 2$. Then the following estimate holds*

$$\|y\|_\alpha \le C_{P,\alpha} \|\nabla y\|_\alpha \qquad \text{with } C_{P,\alpha} = \begin{cases} \frac{n-1}{\sqrt{n}} |\Omega|^{\frac{1}{n}} & \text{if } \alpha = 2 \\ \frac{\alpha(n-1)}{2(n-\alpha)\sqrt{n}} |\Omega|^{\frac{1}{n}} & \text{if } \alpha < 2. \end{cases}$$

Proof. See for example [12], Chapter 2. $\square$

Lemma 2.2 (*Sobolev's inequality.*) *Let y be in $W_0^{1,\alpha}(\Omega)$ with $1 < \alpha < 2$. Then the following estimate holds*

$$\|y\|_{\frac{n\alpha}{n-\alpha}} \le \frac{\alpha(n-1)}{2(n-\alpha)\sqrt{n}} \|\nabla y\|_\alpha .$$

Proof. See for example [12], Chapter 2. $\square$

Lemma 2.3 (*Korn's inequality.*) *Let y be in $W_0^{1,\alpha}(\Omega)$ with $1 < \alpha \le 2$. Then there exists a positive constant $C_{K,\alpha} \le 1$ only depending on α and Ω such that*

$$C_{K,\alpha} \|\nabla y\|_{1,\alpha} \le \|Dy\|_\alpha .$$

Moreover, $C_{K,2} = \frac{1}{\sqrt{2}}$.

Proof. See for example [25] and [16]. $\square$

As a consequence, we have the following useful result.

Lemma 2.4 *Let w be in $L^2(\Omega)$ and y be in $W_0^{1,\alpha}(\Omega)$ with $\frac{2n}{n+2} < \alpha \le 2$. Then the following estimates hold*

$$|(w,y)| \le C_\alpha \|w\|_2 \|Dy\|_\alpha$$

with

$$C_\alpha = \begin{cases} \frac{\sqrt{2}(n-1)}{\sqrt{n}} |\Omega|^{\frac{1}{n}} & \text{if } \alpha = 2 \\ \frac{\alpha(n-1)}{2(n-\alpha)\sqrt{n}} \frac{1}{C_{K,\alpha}} |\Omega|^{\frac{(n+2)\alpha-2n}{2\alpha n}} & \text{if } \alpha < 2 \end{cases}$$

and where $C_{K,\alpha}$ is the constant of Korn.

Proof. For $\alpha = 2$, due to the Hölder, the Poincaré and the Korn inequalities, we have

$$|(w, y)| \leq \|w\|_2 \|y\|_2 \leq \frac{n-1}{\sqrt{n}} |\Omega|^{\frac{1}{n}} \|w\|_2 \|\nabla y\|_2 = \frac{\sqrt{2}(n-1)}{\sqrt{n}} |\Omega|^{\frac{1}{n}} \|w\|_2 \|Dy\|_2$$

wich gives the first estimate. Similarly, if $\alpha < 2$ (and since $\alpha > \frac{2n}{n+2}$) we have $L^2(\Omega) \hookrightarrow W_0^{1,\alpha}(\Omega)$ and by using the Hölder, the Sobolev and the Korn inequalities, we deduce that

$$\begin{aligned} |(w, y)| &\leq \|w\|_{\frac{n\alpha}{(n+1)\alpha - n}} \|y\|_{\frac{n\alpha}{n-\alpha}} \leq |\Omega|^{\frac{(n+2)\alpha - 2n}{2\alpha n}} \|w\|_2 \|y\|_{\frac{n\alpha}{n-\alpha}} \\ &\leq \frac{\alpha(n-1)}{2(n-\alpha)\sqrt{n}} |\Omega|^{\frac{(n+2)\alpha - 2n}{2\alpha n}} \|w\|_2 \|\nabla y\|_\alpha \leq C_\alpha \|w\|_2 \|Dy\|_\alpha \end{aligned}$$

and the second estimate is proven. □

Finally, we point out some notable facts related with the trilinear form b defined by

$$b(y_1, y_2, y_3) = (y_1 \cdot \nabla y_2, y_3) .$$

Lemma 2.5 *Let w, y and z be in $W_0^{1,\alpha}(\Omega)$ with $\frac{3n}{n+2} \leq \alpha \leq 2$. Then the following estimate holds*

$$|b(w, y, z)| \leq \kappa_1 \|Dw\|_\alpha \|Dy\|_\alpha \|Dz\|_\alpha$$

with

$$\kappa_1 = \begin{cases} \frac{2^{\frac{3}{2}}(n-1)}{n} |\Omega|^{\frac{1}{n(n-1)}} & \text{if } \alpha = 2 \\ \frac{1}{C_{K,\alpha}^3} \frac{(\alpha(n-1))^2}{4n(n-\alpha)^2} |\Omega|^{\frac{(n+2)\alpha - 3n}{n\alpha}} & \text{if } \alpha < 2 \end{cases}$$

and where $C_{K,\alpha}$ is the constant of Korn.

Proof. For $\alpha = 2$, due to Lemma 1.1, Chapter VIII in [12], we have

$$|b(w, y, z)| \leq \frac{n-1}{n} |\Omega|^{\frac{1}{n(n-1)}} \|\nabla w\|_2 \|\nabla y\|_2 \|\nabla z\|_2$$

and the conclusion follows by using the Korn inequality. If $\alpha < 2$, Hölder's and Sobolev's inequalities together with classical embedding results show that if $\frac{2\alpha}{\alpha-1} \leq \frac{n\alpha}{n-\alpha}$ (and thus $\alpha \geq \frac{3n}{n+2}$) then

$$\begin{aligned} |b(w, y, z)| &\leq \|w\|_{\frac{2\alpha}{\alpha-1}} \|\nabla y\|_\alpha \|z\|_{\frac{2\alpha}{\alpha-1}} \leq |\Omega|^{\frac{(n+2)\alpha - 3n}{n\alpha}} \|w\|_{\frac{n\alpha}{n-\alpha}} \|\nabla y\|_\alpha \|z\|_{\frac{n\alpha}{n-\alpha}} \\ &\leq \frac{(\alpha(n-1))^2}{4n(n-\alpha)^2} |\Omega|^{\frac{(n+2)\alpha - 3n}{n\alpha}} \|\nabla w\|_\alpha \|\nabla y\|_\alpha \|\nabla z\|_\alpha . \end{aligned}$$

The conclusion follows by using the Korn inequality. □

Lemma 2.6 *Let w be in V_α and let y and z be in $W_0^{1,\alpha}(\Omega)$ with $\frac{3n}{n+2} \leq \alpha \leq 2$. Then*

$$b(w, y, z) = -b(w, z, y) \qquad \text{and} \qquad b(w, y, y) = 0.$$

2.2 The stress tensor

Let us recall that assumptions $\mathbf{A}_1$-$\mathbf{A}_2$ imply the following standard continuity and monotonicity properties for τ (see [25], Chapter 5)

$$|\tau(\eta)| \leq \frac{n^2\gamma}{\alpha-1} \left(1 + |\eta|^2\right)^{\frac{\alpha-2}{2}} |\eta|, \tag{2.1}$$

$$(\tau(\eta) - \tau(\zeta)) : (\eta - \zeta) \geq \nu \left(1 + |\eta^2| + |\zeta|^2\right)^{\frac{\alpha-2}{2}} |\eta - \zeta|^2 . \tag{2.2}$$

The next auxiliary results deal with properties related with the tensor τ.

Lemma 2.7 *Let* $1 < \alpha < 2$ *and let* $f \in L^{\frac{\alpha}{2-\alpha}}(\Omega)$, $g \in L^1(\Omega)$ *and* $h \in L^\alpha(\Omega)$ *be non negative functions satisfying*

$$h(x)^2 \le f(x)g(x) \qquad \textit{for a.e. } x \in \Omega.$$

Then,

$$\|h\|_\alpha^2 \le \|f\|_{\frac{\alpha}{2-\alpha}} \|g\|_1 .$$

Proof. Taking into account the condition satisfied by f, g, h, integrating and using the Hölder inequality, we obtain

$$\|h\|_\alpha^\alpha = \int_\Omega \left(h(x)^2\right)^{\frac{\alpha}{2}} dx \le \int_\Omega f(x)^{\frac{\alpha}{2}} g(x)^{\frac{\alpha}{2}} dx \le \left\|f^{\frac{\alpha}{2}}\right\|_{\frac{2}{2-\alpha}} \left\|g^{\frac{\alpha}{2}}\right\|_{\frac{2}{\alpha}} = \|f\|_{\frac{\alpha}{2-\alpha}}^{\frac{\alpha}{2}} \|g\|_1^{\frac{\alpha}{2}}$$

and the proof is complete. $\square$

Lemma 2.8 *Let* y *and* z *be in* $W_0^{1,\alpha}(\Omega)$ *with* $1 < \alpha \le 2$. *Then*

$$(\tau(Dy) - \tau(Dz), D(y-z)) \ge \frac{\nu\|D(y-z)\|_\alpha^2}{(|\Omega| + \|Dy\|_\alpha^\alpha + \|Dz\|_\alpha^\alpha)^{\frac{2-\alpha}{\alpha}}}.$$

Proof. If $\alpha = 2$, then the result is a direct consequence of the monotonicity condition (2.2). Assume then that $\alpha < 2$. Since y and z belong to $W_0^{1,\alpha}(\Omega)$, by setting

$$f = \left(1 + |Dy|^2 + |Dz|^2\right)^{\frac{2-\alpha}{2}}, \qquad g = \tfrac{1}{\nu}\left(\tau(Dy) - \tau(Dy)\right) : D(y-z), \qquad h = |D(y-z)|$$

and taking into account the monotonicity condition (2.2), we can see that the assumptions of Lemma 2.7 are fulfilled. Therefore

$$\begin{aligned} \|D(y-z)\|_\alpha^2 &\le \left\|(1 + |Dy|^2 + |Dz|^2)^{\frac{2-\alpha}{2}}\right\|_{\frac{\alpha}{2-\alpha}} \left\|\tfrac{1}{\nu}\left(\tau(Dy) - \tau(Dz)\right) : D(y-z)\right\|_1 \\ &\le \tfrac{1}{\nu}\left(|\Omega| + \|Dy\|_\alpha^\alpha + \|Dz\|_\alpha^\alpha\right)^{\frac{2-\alpha}{\alpha}} (\tau(Dy) - \tau(Dz), D(y-z)) \end{aligned}$$

and the result is proven. $\square$

Lemma 2.9 *Let* $1 < \alpha \le 2$ *and let* z *and* y *be in* $H_0^1(\Omega)$. *Then*

$$\left\|(1 + |Dy|^2)^{\frac{\alpha-2}{4}} Dz\right\|_2^2 \ge \frac{\|Dz\|_\alpha^2}{(|\Omega| + \|Dy\|_\alpha^\alpha)^{\frac{2-\alpha}{\alpha}}}.$$

Proof. The case $\alpha = 2$ is obvious. In case $\alpha < 2$, since y and z belong to $H_0^1(\Omega)$, by setting

$$f = \left(1 + |Dy|^2\right)^{\frac{2-\alpha}{2}}, \qquad g = |Dz|^2\left(1 + |Dy|^2\right)^{\frac{\alpha-2}{2}}, \qquad h = |Dz|,$$

we can see that the assumptions of Lemma 2.7 are satisfied and then

$$\begin{aligned} \|Dz\|_\alpha^2 &\le \left\|(1 + |Dy|^2)^{\frac{2-\alpha}{2}}\right\|_{\frac{\alpha}{2-\alpha}} \left\||Dz|^2(1 + |Dy|^2)^{\frac{\alpha-2}{2}}\right\|_1 \\ &\le \left(|\Omega| + \|Dy\|_\alpha^\alpha\right)^{\frac{2-\alpha}{\alpha}} \left\||Dz|^2(1 + |Dy|^2)^{\frac{\alpha-2}{2}}\right\|_1 \end{aligned}$$

which gives the result. $\square$

3 State equation

This section is devoted to existence and uniqueness results for the state equation and to derivation of explicite estimates useful for the subsequent analysis.

First mathematical investigations of (1.1) under conditions (2.1)-(2.2), were performed by J. L. Lions who proved existence of a weak solution for $\alpha \geq \frac{3n}{n+2}$ (see [20] for more details). The restriction on the exponent α ensures that the convective term belongs to L^1 when considering test functions in V_α (cf. Lemma 2.5).

Multiplying equation (1.1) by test functions $\varphi \in V_\alpha$ and integrating, we obtain the following weak formulation.

Definition 3.1 *Let $u \in L^2(\Omega)$. A function $y \in V_\alpha$ is a weak solution of (1.1) if*

$$(\tau(Dy), D\varphi) + b(y, y, \varphi) = (u, \varphi) \qquad \text{for all } \varphi \in V_\alpha.$$

Let us recall that, having a solution satisfying the formulation given in Definition 3.1, it is standard to construct the corresponding pressure $\pi \in L_0^\alpha(\Omega)$ such that

$$(\tau(Dy), D\varphi) + b(y, y, \varphi) - (\pi, \nabla \cdot y, \varphi) = (u, \varphi) \qquad \text{for all } \varphi \in W_0^{1,\alpha}(\Omega).$$

We will involve the pressure only in the formulations of the theorems and lemmas but not in the proofs, since it can always be reconstructed uniquely.

We begin by stating an existence result for the state equation and related useful estimates.

Theorem 3.2 *Assume that $\mathbf{A}_1$-$\mathbf{A}_2$ are fulfilled with $\frac{3n}{n+2} \leq \alpha \leq 2$. Then for $u \in L^2(\Omega)$, equation (1.1) admits at least a weak solution $y_u \in V_\alpha$. Moreover, the following estimates hold*

$$\|Dy_u\|_\alpha \leq \kappa_2 \left(1 + \frac{\|u\|_2}{\nu}\right)^{\frac{2-\alpha}{\alpha-1}} \frac{\|u\|_2}{\nu}, \tag{3.1}$$

$$\|Dy_u\|_\alpha^\alpha \leq \left(2^{\frac{2-\alpha}{2}} C_\alpha \frac{\|u\|_2}{\nu}\right)^{\frac{\alpha}{\alpha-1}} + |\Omega|, \tag{3.2}$$

where $\kappa_2 = C_\alpha \left(\frac{|\Omega|}{\alpha-1} + C_\alpha^{\frac{\alpha}{\alpha-1}}\right)^{\frac{2-\alpha}{\alpha}}$ with C_α defined in Lemma 2.4.

Proof. As already observed, existence of a weak solution for problem (1.1) with $\alpha \geq \frac{3n}{n+2}$ is well known. To establish the estimates, we split the proof into two steps.

Step 1. Let us set $\varphi = y_u$ in the weak formulation of (1.1) and use Lemma 2.8, Lemma 2.6 and Lemma 2.4 to obtain

$$\frac{\nu \|Dy_u\|_\alpha^2}{\left(|\Omega| + \|Dy_u\|_\alpha^\alpha\right)^{\frac{2-\alpha}{\alpha}}} \leq (\tau(Dy_u), Dy_u) = (u, y_u) \leq C_\alpha \|u\|_2 \|Dy_u\|_\alpha$$

If $\alpha = 2$, the estimate is direct. If $\alpha < 2$, we have

$$\|Dy_u\|_\alpha \leq \frac{C_\alpha \|u\|_2}{\nu} \left(|\Omega| + \|Dy_u\|_\alpha^\alpha\right)^{\frac{2-\alpha}{\alpha}}$$

and thus

$$\|Dy_u\|_\alpha^{\frac{\alpha}{2-\alpha}} \leq \left(\frac{C_\alpha \|u\|_2}{\nu}\right)^{\frac{\alpha}{2-\alpha}} \left(|\Omega| + \|Dy_u\|_\alpha^\alpha\right). \tag{3.3}$$

On the other hand, the Young inequality yields

$$\left(\frac{C_\alpha \|u\|_2}{\nu}\right)^{\frac{\alpha}{2-\alpha}} \|Dy_u\|_\alpha^\alpha \leq (2-\alpha) \|Dy_u\|_\alpha^{\frac{\alpha}{2-\alpha}} + (\alpha - 1) \left(\frac{C_\alpha \|u\|_2}{\nu}\right)^{\frac{\alpha}{(2-\alpha)(\alpha-1)}}. \tag{3.4}$$

Combining (3.3) and (3.4), we deduce that

$$(\alpha-1)\,\|Dy_u\|_\alpha^{\frac{\alpha}{2-\alpha}} \le \left(\frac{C_\alpha\|u\|_2}{\nu}\right)^{\frac{\alpha}{2-\alpha}} |\Omega| + (\alpha-1)\left(\frac{C_\alpha\|u\|_2}{\nu}\right)^{\frac{\alpha}{(2-\alpha)(\alpha-1)}}$$

and consequently

$$\begin{aligned}
\|Dy_u\|_\alpha &\le \left(\frac{|\Omega|}{\alpha-1} + \left(\frac{C_\alpha\|u\|_2}{\nu}\right)^{\frac{\alpha}{\alpha-1}}\right)^{\frac{2-\alpha}{\alpha}} \frac{C_\alpha\|u\|_2}{\nu} \\
&\le C_\alpha\left(\frac{|\Omega|}{\alpha-1} + C_\alpha^{\frac{\alpha}{\alpha-1}}\right)^{\frac{2-\alpha}{\alpha}} \left(1+\left(\frac{\|u\|_2}{\nu}\right)^{\frac{\alpha}{\alpha-1}}\right)^{\frac{2-\alpha}{\alpha}} \frac{\|u\|_2}{\nu} \\
&\le C_\alpha\left(\frac{|\Omega|}{\alpha-1} + C_\alpha^{\frac{\alpha}{\alpha-1}}\right)^{\frac{2-\alpha}{\alpha}} \left(1+\frac{\|u\|_2}{\nu}\right)^{\frac{2-\alpha}{\alpha-1}} \frac{\|u\|_2}{\nu}
\end{aligned}$$

which gives (3.1).

Step 2. Let us now prove (3.2). Similar arguments together with (2.2) show that

$$\begin{aligned}
\|Dy_u\|_\alpha^\alpha &= \int_{\{x\,|\,|Dy_u(x)|\ge 1\}} |Dy_u(x)|^\alpha\,dx + \int_{\{x\,|\,|Dy_u(x)|<1\}} |Dy_u(x)|^\alpha\,dx \\
&\le \int_{\{x\,|\,|Dy_u(x)|\ge 1\}} \frac{|Dy_u(x)|^2}{|Dy_u(x)|^{2-\alpha}}\,dx + |\Omega| \\
&\le 2^{\frac{2-\alpha}{2}} \int_{\{x\,|\,|Dy_u(x)|\ge 1\}} \frac{|Dy_u(x)|^2}{(1+|Dy_u(x)|^2)^{\frac{2-\alpha}{2}}}\,dx + |\Omega| \\
&\le \frac{2^{\frac{2-\alpha}{2}}}{\nu} \int_{\{x\,|\,|Dy_u(x)|\ge 1\}} \tau(Dy_u(x)) : Dy_u(x)\,dx + |\Omega| \\
&\le \frac{2^{\frac{2-\alpha}{2}}}{\nu} (\tau(Dy_u), Dy_u) + |\Omega| = 2^{\frac{2-\alpha}{2}} \left(\frac{u}{\nu}, y_u\right) + |\Omega| \\
&\le 2^{\frac{2-\alpha}{2}} C_\alpha \frac{\|u\|_2}{\nu} \|Dy_u\|_\alpha + |\Omega|.
\end{aligned} \tag{3.5}$$

The Young inequality yields

$$2^{\frac{2-\alpha}{2}} C_\alpha \frac{\|u\|_2}{\nu} \|Dy_u\|_\alpha \le \frac{\alpha-1}{\alpha}\left(2^{\frac{2-\alpha}{2}} C_\alpha \frac{\|u\|_2}{\nu}\right)^{\frac{\alpha}{\alpha-1}} + \frac{1}{\alpha}\|Dy_u\|_\alpha^\alpha \tag{3.6}$$

and the claimed result follows by combining (3.5) and (3.6). □.

The next result deals with uniqueness of weak solutions.

Theorem 3.3 *Assume that* $\mathbf{A}_1$-$\mathbf{A}_2$ *are fulfilled with* $\frac{3n}{n+2} \le \alpha \le 2$ *and that* $u \in L^2(\Omega)$ *satisfies*

$$\bar\kappa\left(1+\frac{\|u\|_2}{\nu}\right)^{\frac{2(2-\alpha)}{\alpha-1}} \frac{\|u\|_2}{\nu^2} < 1, \tag{3.7}$$

where $\bar\kappa = \kappa_1\kappa_2\kappa_3$ *with* κ_1 *defined in Lemma* 2.5*,* κ_2 *defined in Theorem* 3.2 *and* $\kappa_3 = \left(3|\Omega| + 2^{\frac{(4-\alpha)\alpha}{2(\alpha-1)}} C_\alpha^{\frac{\alpha}{\alpha-1}}\right)^{\frac{2-\alpha}{\alpha}}$*. Then, equation* (1.1) *admits a unique weak solution* $y_u \in V_\alpha$*.*

Proof. Assume that y_u and χ_u are two weak solutions of (1.1) corresponding to u. Setting $\varphi = y_u - \chi_u$ in the corresponding weak formulation and taking into account Lemma 2.8 and Lemma 2.6, we obtain

$$\begin{aligned}
\frac{\nu\|D(y_u-\chi_u)\|_\alpha^2}{\left(|\Omega|+\|Dy_u\|_\alpha^\alpha+\|D\chi_u\|_\alpha^\alpha\right)^{\frac{2-\alpha}{\alpha}}} &\le \left(\tau(Dy_u)-\tau(D\chi_u), D\left(y_u-\chi_u\right)\right) \\
&= b\left(\chi_u,\chi_u,y_u-\chi_u\right) - b\left(y_u,y_u,y_u-\chi_u\right) \\
&= -b\left(y_u-\chi_u,\chi_u,y_u-\chi_u\right).
\end{aligned} \tag{3.8}$$

Lemma 2.5 and estimate (3.1) then yield

$$\begin{aligned} |b\left(y_u-\chi_u,\chi_u,y_u-\chi_u\right)| &\leq \kappa_1 \left\|D\left(y_u-\chi_u\right)\right\|_\alpha^2 \left\|D\chi_u\right\|_\alpha \\ &\leq \kappa_1\kappa_2 \left(1+\tfrac{\|u\|_2}{\nu}\right)^{\frac{2-\alpha}{\alpha-1}} \tfrac{\|u\|_2}{\nu} \left\|D\left(y_u-\chi_u\right)\right\|_\alpha^2 . \end{aligned} \tag{3.9}$$

On the other hand, by taking into account estimate (3.2), we have

$$\begin{aligned} \left(|\Omega|+\|Dy_u\|_\alpha^\alpha+\|D\chi_u\|_\alpha^\alpha\right)^{\frac{2-\alpha}{\alpha}} &\leq \left(3|\Omega|+2\left(2^{\frac{2-\alpha}{2}}C_\alpha\tfrac{\|u\|_2}{\nu}\right)^{\frac{\alpha}{\alpha-1}}\right)^{\frac{2-\alpha}{\alpha}} \\ &= \left(3|\Omega|+2\times 2^{\frac{(4-\alpha)\alpha}{2(\alpha-1)}}C_\alpha^{\frac{\alpha}{\alpha-1}}\left(\tfrac{\|u\|_2}{2\nu}\right)^{\frac{\alpha}{\alpha-1}}\right)^{\frac{2-\alpha}{\alpha}} \\ &\leq \kappa_3\left(1+\tfrac{\|u\|_2}{\nu}\right)^{\frac{2-\alpha}{\alpha-1}} . \end{aligned} \tag{3.10}$$

By combining (3.8), (3.9) and (3.10), we deduce that

$$\left(\frac{\nu}{\kappa_3\left(1+\frac{\|u\|_2}{\nu}\right)^{\frac{2-\alpha}{\alpha-1}}}-\kappa_1\kappa_2\left(1+\tfrac{\|u\|_2}{\nu}\right)^{\frac{2-\alpha}{\alpha-1}}\tfrac{\|u\|_2}{\nu}\right)\left\|D\left(y_u-\chi_u\right)\right\|_\alpha^2\leq 0$$

and thus $y_u=\chi_u$ if condition (3.7) is satisfied. □

Remark 3.4 *Notice that in the case of the Navier-Stokes equations* ($\alpha=2$)*, condition* (3.7) *reduces to*

$$\frac{\|u\|_2}{\nu^2}<\frac{\sqrt{n^3}}{4(n-1)^2|\Omega|^{\frac{1}{n-1}}}. \tag{3.11}$$

Remark 3.5 *Condition* (3.7) *is fulfilled if the term* $\frac{\|u\|_2}{\nu}$ *is "small enough", and can be interpreted either as a constraint on the size of* $\|u\|_2$ *(small body force* u*) or as a restriction on the viscosity parameter* ν *(large viscosity parameter* ν*).*

4 Existence of an optimal control

Theorem 4.1 *Assume that* **A**1*-***A**2 *are fulfilled with* $\frac{3n}{n+2}\leq\alpha\leq 2$*. Then problem* (P_α) *admits at least a solution.*

Proof. The proof is split into three steps.

Step 1. Considering a minimizing sequence $(u_k,y_k)_k\subset U_{ad}\times V_\alpha$, let us establish related estimates and preliminary convergence results. Since $(u_k)_k$ is uniformly bounded in the closed convex set U_{ad}, by taking into account (3.2) we obtain

$$\|Dy_k\|_\alpha^\alpha\leq\left(2^{\frac{2-\alpha}{2}}C_\alpha\tfrac{\|u_k\|_2}{\nu}\right)^{\frac{\alpha}{\alpha-1}}+|\Omega|\leq\left(2^{\frac{2-\alpha}{2}}C_\alpha\tfrac{U}{\nu}\right)^{\frac{\alpha}{\alpha-1}}+|\Omega| \tag{4.1}$$

and the sequence $(y_k)_k$ is then bounded in V_α. On the other hand, the continuity condition (2.1) implies that for $\eta\neq 0$

$$|\tau(\eta)|\leq\tfrac{n^2\gamma}{\alpha-1}\left(1+|\eta|^2\right)^{\frac{\alpha-2}{2}}|\eta|\leq\tfrac{n^2\gamma}{\alpha-1}|\eta|^{\alpha-2}|\eta|=\tfrac{n^2\gamma}{\alpha-1}|\eta|^{\alpha-1}.$$

The previous inequality is also valid for $|\eta| = 0$ and implies

$$\|\tau(Dy_k)\|_{\frac{\alpha}{\alpha-1}}^{\frac{\alpha}{\alpha-1}} \leq \frac{n^2\gamma}{\alpha-1}\|Dy_k\|_\alpha^\alpha$$

which together with (4.1) show that sequence $(\tau(Dy_k))_k$ is uniformly bounded in $L^{\frac{\alpha}{\alpha-1}}(\Omega)$. There then exist a subsequence, still indexed by k, $u \in U_{ad}$, $y \in V_\alpha$ and $\widetilde{\tau} \in L^{\frac{\alpha}{\alpha-1}}(\Omega)$ such that $(u_k)_k$ weakly converges to u in $L^2(\Omega)$, $(y_k)_k$ weakly converges to y in V_α and $(\tau(Dy_k))_k$ weakly converges to $\widetilde{\tau}$ in $L^{\frac{\alpha}{\alpha-1}}(\Omega)$. Moreover, since $\alpha > \frac{2n}{n+1}$, by using compactness results on Sobolev spaces, we deduce that $(y_k)_k$ strongly converges to y in $L^{\frac{\alpha}{\alpha-1}}(\Omega)$.

Step 2. Let us now prove that (u, y) is an admissible pair for (P_α). Taking into account the convergence results obtained in Step 1, we deduce that for every $\varphi \in \mathcal{V}$, we have

$$\begin{aligned} |b(y_k, y_k, \varphi) - b(y, y, \varphi)| &\leq |b(y_k - y, y_k, \varphi)| + |b(y, y_k - y, \varphi)| \\ &= |b(y_k - y, y_k, \varphi)| + |b(y, \varphi, y_k - y)| \\ &\leq \left(\|\nabla y_k\|_\alpha \|\varphi\|_\infty + \|y\|_\alpha \|\nabla\varphi\|_\infty\right)\|y_k - y\|_{\frac{\alpha}{\alpha-1}} \\ &\longrightarrow 0 \qquad \text{when } k \to +\infty. \end{aligned} \tag{4.2}$$

Moreover, by passing to the limit in the weak formulation corresponding to y_k, we obtain

$$(\widetilde{\tau}, D\varphi) + b(y, y, \varphi) = (u, \varphi) \qquad \text{for all } \varphi \in \mathcal{V}$$

and by using the fact that $\mathcal{V}$ is dense in V_α and that $y \in L^{\frac{2\alpha}{\alpha-1}}(\Omega)$ if $\alpha \geq \frac{3n}{n+2}$, it follows that

$$(\widetilde{\tau}, D\varphi) + b(y, y, \varphi) = (u, \varphi) \qquad \text{for all } \varphi \in V_\alpha. \tag{4.3}$$

In particular, by taking into account Lemma 2.6, we have

$$(\widetilde{\tau}, Dy) = (\widetilde{\tau}, Dy) + b(y, y, y) = (u, y). \tag{4.4}$$

On the other hand, the monotonicity assumption (2.2) implies

$$(\tau(Dy_k) - \tau(D\varphi), Dy_k - D\varphi) \geq 0 \qquad \text{for all } \varphi \in V_\alpha. \tag{4.5}$$

Since $(\tau(Dy_k), Dy_k) = (u_k, y_k)$, by substituing in (4.5), we obtain

$$(u_k, y_k) - (\tau(Dy_k), D\varphi) - (\tau(D\varphi), Dy_k - D\varphi) \geq 0 \qquad \text{for all } \varphi \in V_\alpha$$

and by passing to the limit, we get

$$(u, y) - (\widetilde{\tau}, D\varphi) - (\tau(D\varphi), Dy - D\varphi) \geq 0 \qquad \text{for all } \varphi \in V_\alpha.$$

This inequality together with (4.4) then yields

$$(\widetilde{\tau} - \tau(D\varphi), Dy - D\varphi) \geq 0 \qquad \text{for all } \varphi \in V_\alpha$$

and by setting $\varphi = y - t\psi$ with $t > 0$, we obtain

$$(\widetilde{\tau} - \tau(Dy - tD\psi), D\psi) \geq 0 \qquad \text{for all } \psi \in V_\alpha.$$

Letting t tend to zero and using the continuity of τ, we deduce that

$$(\widetilde{\tau} - \tau(Dy), D\psi) \geq 0 \qquad \text{for all } \psi \in V_\alpha$$

and thus

$$(\widetilde{\tau}, D\psi) = (\tau(Dy), D\psi) \qquad \text{for all } \psi \in V_\alpha. \tag{4.6}$$

Combining (4.3) and (4.6), we deduce that

$$(\tau(Dy), D\varphi) + b(y, y, \varphi) = (u, \varphi) \qquad \text{for all } \varphi \in V_\alpha$$

showing that

$$y_k \longrightarrow y \qquad \text{weakly in } W_0^{1,\alpha}(\Omega)$$

and that (u, y) satisfies (1.1).

Step 3. Finally, from the convexity and continuity of J, it follows the lower semicontinuity of J in the weak topology and

$$J(u, y) \leq \liminf_k J(u_k, y_k) = \inf(P_\alpha),$$

showing that (u, y) is a solution for (P_α). □

5 Statement of the necessary optimality conditions

In order to obtain the necessary optimality conditions for (P_α) stated in Theorem 5.1 below, a family of problems $(P_\alpha^\varepsilon)_\varepsilon$ whose solutions converge towards a solution of (P_α) is introduced and the corresponding optimality conditions are derived in Section 6. We pass to the limit in these conditions in Section 7.

Let us now formulate our main result.

Theorem 5.1 *Assume that* $\mathbf{A}_1$-$\mathbf{A}_2$ *are fullfilled with* $\frac{3n}{n+2} \leq \alpha \leq 2$. *Let* $\bar{u}$ *be a solution of* (P_α) *satisfying condition* (3.7) *and let* $\bar{y} \in V_\alpha$ *be the associated state. There then exists* $\bar{p} \in V_\alpha$ *such that the following conditions hold*

$$\begin{cases} -\nabla \cdot (\tau(D\bar{y})) + \bar{y} \cdot \nabla \bar{y} + \nabla \bar{\pi} = \bar{u} & \textit{in } \Omega, \\ \nabla \cdot \bar{y} = 0 & \textit{in } \Omega, \\ \bar{y} = 0 & \textit{on } \Gamma, \end{cases}$$

$$\begin{cases} -\nabla \cdot (\tau'(D\bar{y}) : D\bar{p}) - \bar{y} \cdot \nabla \bar{p} + (\nabla \bar{y})^T \bar{p} + \nabla \widetilde{\pi} = \bar{y} - y_d & \textit{in } \Omega, \\ \nabla \cdot \bar{p} = 0 & \textit{in } \Omega, \\ \bar{p} = 0 & \textit{on } \Gamma, \end{cases} \tag{5.1}$$

$$(\bar{p} + \lambda \bar{u}, v - \bar{u}) \geq 0 \qquad \textit{for all } v \in U_{ad}. \tag{5.2}$$

Moreover, $\bar{p}$ *satisfies*

$$(\tau'(D\bar{y}) : D\bar{p}, D\bar{p}) + (\bar{p} \cdot \nabla \bar{y}, \bar{p}) \leq (\bar{y} - y_d, \bar{p}). \tag{5.3}$$

Notice that the optimality conditions for (P_α) are obtained under a constraint on the optimal control, the same that guarantees uniqueness of the corresponding state. This result seems interesting in the sense that we do not need to impose any other constraint on the admissible set of controls. Notice also that for $\alpha < 2$, condition (5.3) implies that $\bar{p}$ belongs to $V_\alpha^{\bar{y}}$ and not necessarily to $H_\alpha^{\bar{y}}$. Therefore, the adjoint equation is to be understood in the distributional sense

$$(\tau'(D\bar{y}) : D\bar{p}, D\varphi) + \left((\nabla \bar{y})^T \bar{p} - \bar{y} \cdot \nabla \bar{p}, \varphi\right) = (\bar{y} - y_d, \varphi) \qquad \text{for all } \varphi \in \mathcal{V}.$$

Let us finish this section by considering the case of the Navier-Stokes equations. It corresponds to $\alpha = 2$ and $V_\alpha^{\bar y} \equiv H_\alpha^{\bar y} \equiv V_2$. The first order optimality conditions we obtain in this case are less restrictive than the ones obtained in [9], [10], [26] where all the admissible controls are subject to a condition that ensures the uniqueness of the corresponding states. Condition (3.7) reduces to (3.11) and guarantees uniqueness of both optimal state and optimal adjoint state. It implies that the set U_{ad} of admissible controls satisfies the property (C), introduced by Gunzburger *et al.* [15], at $(\bar u, \bar y)$. Our result can then be seen as a qualified version of the optimality conditions already established by Abergel and Casas in [1] for a slightly different functional.

Corollary 5.2 *Assume that the extra-stress tensor has the form* $\tau(\eta) = 2\nu\eta$. *Let* $(\bar u, \bar y)$ *be a solution of* (P_α) *with* $\bar u$ *satisfying* (3.11). *There then exists a unique* $\bar p \in V_2$ *such that the following conditions hold*

$$\begin{cases} -\nu\,\Delta\bar y + (\bar y\cdot\nabla)\,\bar y + \nabla\bar\pi = \bar u & \textit{in } \Omega, \\ \nabla\cdot\bar y = 0 & \textit{in } \Omega, \\ \bar y = 0 & \textit{on } \Gamma, \end{cases}$$

$$\begin{cases} -\nu\,\Delta\bar p - (\bar y\cdot\nabla)\,\bar p + (\nabla\bar y)^T\bar p + \nabla\widetilde\pi = \bar y - y_d & \textit{in } \Omega, \\ \nabla\cdot\bar p = 0 & \textit{in } \Omega, \\ \bar p = 0 & \textit{on } \Gamma, \end{cases}$$

$$(\bar p + \lambda\bar u, v - \bar u) \ge 0 \qquad \textit{for all } v \in U_{ad}.$$

6 Approximate optimal control problem

When deriving the first order optimality conditions, we have to manage several combined difficulties related with the local Lipschitz continuity (and thus with the Gâteaux differentiability) in adequate functional spaces of the control-to-state mapping $u \longmapsto y_u$.

To clarify the ideas, let us first assume that we are dealing with generalized Stokes systems (no convective term). In the case $\alpha \ge 2$, we can follow the ideas developed in [4] and [6] to study optimal control problems governed by quasi-linear elliptic equations and prove that the sequence $(z_\rho)_{\rho>0}$ defined by

$$z_\rho = \tfrac{y_{u+\rho(v-u)} - y_u}{\rho}, \qquad u, v \in U_{ad}, \quad \rho \in]0,1[,$$

converges weakly in the weighted Sobolev space $H_\alpha^{y_u}$ and strongly in V_2. To prove that the limit belongs to $H_\alpha^{y_u}$, it is essential that $(z_\rho)_{\rho>0}$ be uniformly bounded in $V_\alpha \subset H_\alpha^{y_u}$. If $\alpha < 2$, we can argue similarly and show that $(z_\rho)_{\rho>0}$ is uniformly bounded in V_α. Nevertheless, in this case $H_\alpha^{y_u} \subset V_\alpha$ and we can only prove that there exist subsequences converging to elements wich are solutions of a linearized system in the distributional sense and belonging to $V_\alpha^{y_u}$. Differentiability of the control-to-state mapping is equivalent to the equality of all these limit points and this problem of uniqueness leads us to the problem of density of the function space $\mathcal{V}$ in $V_\alpha^{y_u}$. To overcome this difficulty, we can consider a family of approximate problems (P_α^ε) governed by the following regularized equation

$$-\varepsilon\,\Delta y - \nabla\cdot(\tau(Dy)) + \nabla\pi = u \quad \text{in } \Omega, \qquad \nabla\cdot y = 0 \quad \text{in } \Omega, \qquad y = 0 \quad \text{on } \Gamma$$

and falling into the case $\alpha = 2$. Adequate estimates for $\big(z_\rho^\varepsilon = \frac{y^\varepsilon_{u+\rho(v-u)} - y^\varepsilon_u}{\rho}\big)_{\rho>0}$ can be established in V_2 (here y_u^ε denotes the solution of the previous equation corresponding to u), allowing to derive the approximate optimality conditions and to obtain optimality conditions for (P_α) by passing to the limit.

The case of problems governed by Navier-Stokes equations and generalized Navier-Stokes equations is more delicate since a direct adaptation of these arguments, managing the convective term, may restrain not only u but also v (and by extension, all the admissible controls) to satisfy condition (3.11) (see for example [10], [26] and [27]). This difficulty is overcome in the case of shear-thickening and Navier-Stokes fluids by observing that, in order to establish uniform estimates of $(z_\rho)_{\rho>0}$ (consequence of the local Lipchitz continuity of the state with respect to the control) in V_α, the terms we need to restrain are related to the convective term and only depend on u. This fact is particularly important and enables us, when deriving the necessary optimality conditions, to impose a constraint only on the optimal control (see [3]). In the case of shear-thinning flows, the problems are even more difficult to handle because of the combined effect of the convective term and the nonlinear stress tensor. Unlike the case of generalized Stokes systems, obtaining uniform estimates for $(z^\varepsilon_\rho)_{\rho>0}$ in V_2 without restraining both u and v is not an easy issue. In this section, by carrying out a careful analysis, we prove that uniform estimates for $(z^\varepsilon_\rho)_{\rho>0}$ can be established under a condition involving ρ and the regularization parameter ε and by imposing restriction (3.7) only on u. The approximate optimality conditions are then derived.

6.1 Setting and approximate optimality conditions

For $\varepsilon > 0$ and u in $L^2(\Omega)$, consider the following problem

$$\begin{cases} -\varepsilon\,\Delta y - \nabla\cdot(\tau(Dy)) + y\cdot\nabla y + \nabla\pi = u & \text{in } \Omega, \\ \nabla\cdot y = 0 & \text{in } \Omega, \\ y = 0 & \text{on } \Gamma. \end{cases} \tag{6.1}$$

Let $(\bar u, \bar y)$ be a fixed solution of (P_α) and assume that $\bar u$ satisfies condition (3.7). Introduce the cost functional

$$I(u,y) = J(u,y) + \tfrac{1}{2}\int_\Omega |u-\bar u|^2\,dx$$

and the control problem

$$(P^\varepsilon_\alpha)\quad \begin{cases} \text{minimize} \quad I(u,y^\varepsilon) \\ \text{subject to} \quad (u,y^\varepsilon)\in U_{ad}\times H^1_0(\Omega) \text{ satisfies (6.1) for some } \pi_\varepsilon\in L^2(\Omega). \end{cases}$$

The main result of this section deals with the necessary optimality conditions for the approximate problem (P^ε_α).

Theorem 6.1 *Assume that* $\mathbf{A}_1$-$\mathbf{A}_2$ *are fulfilled with* $\frac{3n}{n+2}\le\alpha\le 2$. *For each* $\varepsilon>0$, *there exists at least one solution* $(\bar u^\varepsilon, \bar y^\varepsilon)$ *of* (P^ε_α). *Moreover, if* $\bar u^\varepsilon$ *satisfies* (3.7), *then there exists* $\bar p^\varepsilon\in V_2$ *such that*

$$\begin{cases} -\varepsilon\Delta\bar y^\varepsilon - \nabla\cdot(\tau(D\bar y^\varepsilon)) + \bar y^\varepsilon\cdot\nabla\bar y^\varepsilon + \nabla\pi^\varepsilon = \bar u^\varepsilon & \textit{in } \Omega, \\ \nabla\cdot\bar y^\varepsilon = 0 & \textit{in } \Omega, \\ \bar y^\varepsilon = 0 & \textit{on } \Gamma, \end{cases}$$

$$\begin{cases} -\varepsilon\Delta\bar p^\varepsilon - \nabla\cdot(\tau'(D\bar y^\varepsilon):D\bar p^\varepsilon) + (\nabla\bar y^\varepsilon)^T\bar p^\varepsilon - \bar y^\varepsilon\cdot\nabla\bar p^\varepsilon + \nabla\tilde\pi^\varepsilon = \bar y^\varepsilon - y_d & \textit{in } \Omega, \\ \nabla\cdot\bar p^\varepsilon = 0 & \textit{in } \Omega, \\ \bar p^\varepsilon = 0 & \textit{on } \Gamma, \end{cases} \tag{6.2}$$

$$(\bar p^\varepsilon + (\lambda+1)\bar u^\varepsilon - \bar u, v - \bar u^\varepsilon) \ge 0 \qquad \textit{for all } v\in U_{ad}. \tag{6.3}$$

6.2 Approximate state equation

In the following proposition we state an existence and uniqueness result for the approximate state equation (6.1) and related a priori estimates.

Proposition 6.2 *Assume that* $\mathbf{A}_1$-$\mathbf{A}_2$ *are fulfilled with* $\frac{3n}{n+2} \le \alpha \le 2$. *Then for* $u \in L^2(\Omega)$, *equation* (6.1) *admits at least a weak solution* $y_u^\varepsilon \in V_2$ *and the following estimates hold*

$$\|Dy_u^\varepsilon\|_\alpha \le \kappa_2 \left(1 + \frac{\|u\|_2}{\nu}\right)^{\frac{2-\alpha}{\alpha-1}} \frac{\|u\|_2}{\nu}, \tag{6.4}$$

$$\|Dy_u^\varepsilon\|_\alpha^\alpha \le \left(2^{\frac{2-\alpha}{2}} C_\alpha \frac{\|u\|_2}{\nu}\right)^{\frac{\alpha}{\alpha-1}} + |\Omega|, \tag{6.5}$$

$$2\varepsilon \left\|Dy_u^\varepsilon\right\|_2 \le C_2 \left\|u\right\|_2, \tag{6.6}$$

$$2\varepsilon \left\|Dy_u^\varepsilon\right\|_2^2 \le C_\alpha \|u\|_2 \left(\left(2^{\frac{2-\alpha}{2}} C_\alpha \frac{\|u\|_2}{\nu}\right)^{\frac{1}{\alpha-1}} + |\Omega|^{\frac{1}{\alpha}}\right). \tag{6.7}$$

Moreover, if u *satisfies condition* (3.7)*, then the solution is unique.*

Proof. The weak formulation associated with problem (6.1) reads as

$$(2\varepsilon Dy + \tau(Dy), D\varphi) + b\,(y, y, \varphi) = (u, \varphi) \qquad \text{for all } \varphi \in V_2.$$

Since $2\varepsilon I + \tau$ satisfies assumptions $\mathbf{A}_1$-$\mathbf{A}_2$ with exponent 2, classical arguments show that the approximate system (6.1) admits at least a weak solution y_u^ε in V_2. Setting $\varphi = y_u^\varepsilon$ in the weak formulation of (6.1) yields

$$2\varepsilon \left\|Dy_u^\varepsilon\right\|_2^2 + \left(\tau\left(Dy_u^\varepsilon\right), Dy_u^\varepsilon\right) + b\left(y_u^\varepsilon, y_u^\varepsilon, y_u^\varepsilon\right) = \left(u, y_u^\varepsilon\right).$$

Estimates (6.4) and (6.5) can then be obtained with arguments similar to those used in the proof of Theorem 3.2. On the other hand, by taking into account the monotonicity condition (2.2), Lemma 2.6 and Lemma 2.4, we obtain

$$2\varepsilon \left\|Dy_u^\varepsilon\right\|_2^2 \le 2\varepsilon \left\|Dy_u^\varepsilon\right\|_2^2 + \left(\tau\left(Dy_u^\varepsilon\right), Dy_u^\varepsilon\right) = \left(u, y_u^\varepsilon\right) \le C_2 \left\|u\right\|_2 \left\|Dy_u^\varepsilon\right\|_2$$

which gives estimate (6.6). Similarly, by taking into account Lemma 2.4 and estimate (6.5) we have

$$2\varepsilon \left\|Dy_u^\varepsilon\right\|_2^2 \le \left(u, y_u^\varepsilon\right) \le C_\alpha \left\|u\right\|_2 \left\|Dy_u^\varepsilon\right\|_\alpha$$

and estimate (6.7) is proven. The uniqueness result can be obtained with arguments similar to those used in the proof of Theorem 3.3. ☐

Remark 6.3 *Notice that the existence result as well as the estimates stated in Proposition* 6.2 *are valid for* $\alpha \in]1, 2]$. *To prove the uniqueness result, we need to restrain the values of* α *in order to guarantee the embedding of* $W_0^{1,\alpha}(\Omega)$ *into* $L^{\frac{2\alpha}{\alpha-1}}(\Omega)$ (*cf. Lemma* 2.5).

6.3 Linearized equation

We next investigate the following linearized equation

$$\begin{cases} -\varepsilon\Delta z - \nabla \cdot \left(\tau'\left(Dy_u^\varepsilon\right) : Dz\right) + z \cdot \nabla y_u^\varepsilon + y_u^\varepsilon \cdot \nabla z + \nabla\pi = w & \text{in } \Omega, \\ \nabla \cdot z = 0 & \text{in } \Omega, \\ z = 0 & \text{on } \Gamma, \end{cases} \tag{6.8}$$

where $u \in L^2(\Omega)$, $y_u^\varepsilon \in V_2$ a corresponding solution of (6.1) and $w \in L^2(\Omega)$.

Proposition 6.4 *Assume that* $\mathbf{A}_1$-$\mathbf{A}_2$ *are fulfilled with* $\frac{3n}{n+2} \leq \alpha \leq 2$. *Let* $u \in L^2(\Omega)$ *satisfying* (3.7) *and let* $y_u^\varepsilon \in V_2$ *be the corresponding solution of* (6.1). *For* $w \in L^2(\Omega)$, *problem* (6.8) *admits a unique solution* z_{uw}^ε *in* V_2. *Moreover, the following estimates hold*

$$\begin{aligned} &\|Dz_{uw}^\varepsilon\|_\alpha \leq C_\alpha \, \mathcal{L}\left(\tfrac{\|u\|_2}{\nu}\right) \|w\|_2 \,, \\ &2\varepsilon \, \|Dz_{uw}^\varepsilon\|_2 \leq C_2 \, \|w\|_2 \\ &2\varepsilon \, \|Dz_{uw}^\varepsilon\|_2^2 \leq C_\alpha^2 \mathcal{L}\left(\tfrac{\|u\|_2}{\nu}\right) \|w\|_2^2 \,, \end{aligned}$$

where $\mathcal{L}(t) = \frac{\kappa_3 (1+t)^{\frac{2-\alpha}{\alpha-1}}}{\nu - \bar{\kappa} t (1+t)^{\frac{2(2-\alpha)}{\alpha-1}}}$ *with* $\bar{\kappa} = \kappa_1 \kappa_2 \kappa_3$.

Proof. Let us first recall that a function z is a weak solution of (6.8) if

$$2\varepsilon \, (Dz, D\varphi) + (\tau' \,(Dy_u^\varepsilon) : Dz, D\varphi) + b\,(z, y_u^\varepsilon, \varphi) + b\,(y_u^\varepsilon, z, \varphi) = (w, \varphi) \qquad \text{for all } \varphi \in V_2.$$

Consider then the bilinear form defined by

$$B(z_1, z_2) = 2\varepsilon \, (Dz_1, Dz_2) + (\tau' \,(Dy_u^\varepsilon) : Dz_1, Dz_2) + b\,(z_1, y_u^\varepsilon, z_2) + b\,(y_u^\varepsilon, z_1, z_2) \,.$$

Taking into account Lemma 2.6, we get

$$\begin{aligned} B(z, z) \;&= 2\varepsilon \, \|Dz\|_2^2 + (\tau' \,(Dy_u^\varepsilon) : Dz, Dz) + b\,(z, y_u^\varepsilon, z) + b\,(y_u^\varepsilon, z, z) \\ &= 2\varepsilon \, \|Dz\|_2^2 + (\tau' \,(Dy_u^\varepsilon) : Dz, Dz) + b\,(z, y_u^\varepsilon, z) \end{aligned}$$

for every $z \in V_2$. On the other hand, by using assumption $\mathbf{A}_2$, Lemma 2.9, estimate (6.5) and arguing as in (3.10), we deduce that

$$\begin{aligned} (\tau' \,(Dy_u^\varepsilon) : Dz, Dz) \;&\geq \nu \int_\Omega \left(1 + |Dy_u^\varepsilon|^2\right)^{\frac{\alpha}{2}-1} |Dz|^2 \, dx \\ &\geq \frac{\nu \|Dz\|_\alpha^2}{\left(|\Omega| + \|Dy_u^\varepsilon\|_\alpha^\alpha\right)^{\frac{2-\alpha}{\alpha}}} \geq \frac{\nu \|Dz\|_\alpha^2}{\kappa_3 \left(1 + \frac{\|u\|_2}{\nu}\right)^{\frac{2-\alpha}{\alpha-1}}} . \end{aligned}$$

Moreover, due to Lemma 2.5 and estimate (6.4), we have

$$|b\,(z, y_u^\varepsilon, z)| \leq \kappa_1 \, \|Dy_u^\varepsilon\|_\alpha \, \|Dz\|_\alpha^2 \leq \kappa_1 \kappa_2 \left(1 + \tfrac{\|u\|_2}{\nu}\right)^{\frac{2-\alpha}{\alpha-1}} \tfrac{\|u\|_2}{\nu} \, \|Dz\|_\alpha^2 \,.$$

Therefore, we obtain

$$B(z, z) \geq 2\varepsilon \, \|Dz\|_2^2 + \left(\frac{\nu}{\kappa_3 \left(1 + \frac{\|u\|_2}{\nu}\right)^{\frac{2-\alpha}{\alpha-1}}} - \kappa_1 \kappa_2 \left(1 + \tfrac{\|u\|_2}{\nu}\right)^{\frac{2-\alpha}{\alpha-1}} \tfrac{\|u\|_2}{\nu} \right) \|Dz\|_\alpha^2 \tag{6.9}$$

which shows that B is coercive on V_2 since u satisfies (3.7). Let us now prove that B is continuous. Similarly, due to $\mathbf{A}_1$, (6.4) and the fact that u satisfies condition (3.7), we have

$$\begin{aligned} \tfrac{1}{\gamma} \, |(\tau' \,(Dy_u^\varepsilon) : Dz_1, Dz_2)| \;&\leq \int_\Omega \left(1 + |Dy_u^\varepsilon|^2\right)^{\frac{\alpha-2}{\alpha}} |Dz_1||Dz_2| \, dx \\ &\leq \int_\Omega |Dz_1||Dz_2| \, dx \leq \|Dz_1\|_2 \, \|Dz_2\|_2 \end{aligned}$$

and

$$
\begin{aligned}
|b(z_1, y_u^\varepsilon, z_2) + b(y_u^\varepsilon, z_1, z_2)| &\le 2\kappa_1 \|Dy_u^\varepsilon\|_\alpha \|Dz_1\|_\alpha \|Dz_2\|_\alpha \\
&\le 2\kappa_1 |\Omega|^{\frac{2-\alpha}{\alpha}} \|Dy_u^\varepsilon\|_\alpha \|Dz_1\|_2 \|Dz_2\|_2 \\
&\le 2\kappa_1\kappa_2 |\Omega|^{\frac{2-\alpha}{\alpha}} \left(1 + \tfrac{\|u\|_2}{\nu}\right)^{\frac{2-\alpha}{\alpha-1}} \tfrac{\|u\|_2}{\nu} \|Dz_1\|_2 \|Dz_2\|_2 \\
&\le \frac{2|\Omega|^{\frac{2-\alpha}{\alpha}}\nu}{\kappa_3\left(1+\frac{\|u\|_2}{\nu}\right)^{\frac{2-\alpha}{\alpha-1}}} \|Dz_1\|_2 \|Dz_2\|_2 \\
&\le 2\nu \|Dz_1\|_2 \|Dz_2\|_2
\end{aligned}
$$

for every $z_1, z_2 \in V_2$. Therefore,

$$
B(z_1, z_2) \le (2\varepsilon + \gamma + 2\nu) \|Dz_1\|_2 \|Dz_2\|_2 .
$$

The bilinear form B is then continuous and coercive on V_2. Applying the Lax-Milgram theorem, we deduce that problem (6.8) admits a unique solution z_{uw}^ε in V_2. Taking into account (6.9), we obtain

$$
\begin{aligned}
\left(\frac{\nu}{\kappa_3\left(1+\frac{\|u\|_2}{\nu}\right)^{\frac{2-\alpha}{\alpha-1}}} - \kappa_1\kappa_2\left(1 + \tfrac{\|u\|_2}{\nu}\right)^{\frac{2-\alpha}{\alpha-1}} \tfrac{\|u\|_2}{\nu}\right) \|Dz_{uw}^\varepsilon\|_\alpha^2 \quad &\le B(z_{uw}^\varepsilon, z_{uw}^\varepsilon) = (w, z_{uw}^\varepsilon) \\
&\le C_\alpha \|w\|_2 \|Dz_{uw}^\varepsilon\|_\alpha .
\end{aligned}
$$

This gives the first estimate which combined with (6.9) imply

$$
2\varepsilon \|Dz_{uw}^\varepsilon\|_2^2 \le B(z_{uw}^\varepsilon, z_{uw}^\varepsilon) \le \begin{cases} C_2 \|w\|_2 \|Dz_{uw}^\varepsilon\|_2 \\ C_\alpha \|w\|_2 \|Dz_{uw}^\varepsilon\|_\alpha \le C_\alpha^2 \mathcal{L}\left(\frac{\|u\|_2}{\nu}\right) \|w\|_2^2 \end{cases}
$$

and we derive the second and third estimate. □

6.4 Analysis of the control-to-state mapping

In order to study the local Lipschitz continuity of the approximate state with respect to the control, we first establish some useful estimates.

Lemma 6.5 *Assume that* $\mathbf{A}_1$-$\mathbf{A}_2$ *are fulfilled with* $\frac{3n}{n+2} \le \alpha \le 2$. *Let* u_1, u_2 *be in* $L^2(\Omega)$ *and let* $y_{u_1}^\varepsilon$ *and* $y_{u_2}^\varepsilon$ *be two corresponding solutions of* (6.1)*. Then the following estimate holds*

$$
\begin{aligned}
&2\varepsilon \left\|D\left(y_{u_1}^\varepsilon - y_{u_2}^\varepsilon\right)\right\|_2^2 + \frac{\nu}{\kappa_3\left(1+\frac{\|u_1\|_2}{2\nu}+\frac{\|u_2\|_2}{2\nu}\right)^{\frac{2-\alpha}{\alpha-1}}} \left\|D\left(y_{u_1}^\varepsilon - y_{u_2}^\varepsilon\right)\right\|_\alpha^2 \\
&\le \kappa_1\kappa_2\left(1 + \tfrac{\|u_2\|_2}{\nu}\right)^{\frac{2-\alpha}{\alpha-1}} \tfrac{\|u_2\|_2}{\nu} \left\|D\left(y_{u_1}^\varepsilon - y_{u_2}^\varepsilon\right)\right\|_\alpha^2 + C_2 \|u_1 - u_2\|_2 \left\|D\left(y_{u_1}^\varepsilon - y_{u_2}^\varepsilon\right)\right\|_2 .
\end{aligned}
$$

Proof. The arguments are very similar to those used in the proof of Theorem 3.3. For the confort of the reader, we will give the principal ideas. Setting $\varphi = y_{u_1}^\varepsilon - y_{u_2}^\varepsilon$ in the corresponding weak formulation and taking into account Lemma 2.8 and Lemma 2.6, we obtain

$$
2\varepsilon \left\|D\left(y_{u_1}^\varepsilon - y_{u_2}^\varepsilon\right)\right\|_2^2 + \frac{\nu\left\|D(y_{u_1}^\varepsilon - y_{u_2}^\varepsilon)\right\|_\alpha^2}{\left(|\Omega| + \left\|Dy_{u_1}^\varepsilon\right\|_\alpha^\alpha + \left\|Dy_{u_2}^\varepsilon\right\|_\alpha^\alpha\right)^{\frac{2-\alpha}{\alpha}}}
$$

$$\leq 2\varepsilon \left\|D\left(y^\varepsilon_{u_1}-y^\varepsilon_{u_2}\right)\right\|_2^2 + \left(\tau(Dy^\varepsilon_{u_1})-\tau(Dy^\varepsilon_{u_2}), D\left(y^\varepsilon_{u_1}-y^\varepsilon_{u_2}\right)\right)$$
$$= -b\left(y^\varepsilon_{u_1}-y^\varepsilon_{u_2}, y^\varepsilon_{u_2}, y^\varepsilon_{u_1}-y^\varepsilon_{u_2}\right) + \left(u_1-u_2, y^\varepsilon_{u_1}-y^\varepsilon_{u_2}\right). \tag{6.10}$$

Lemma 2.5 and estimate (3.1) then yield

$$\begin{aligned}\left|b\left(y^\varepsilon_{u_1}-y^\varepsilon_{u_2}, y^\varepsilon_{u_2}, y^\varepsilon_{u_1}-y^\varepsilon_{u_2}\right)\right| &\leq \kappa_1 \left\|D\left(y^\varepsilon_{u_1}-y^\varepsilon_{u_2}\right)\right\|_\alpha^2 \left\|Dy^\varepsilon_{u_2}\right\|_\alpha \\ &\leq \kappa_1\kappa_2\left(1+\tfrac{\|u_2\|_2}{\nu}\right)^{\frac{2-\alpha}{\alpha-1}} \tfrac{\|u_2\|_2}{\nu} \left\|D\left(y^\varepsilon_{u_1}-y^\varepsilon_{u_2}\right)\right\|_\alpha^2. \end{aligned} \tag{6.11}$$

On the other hand, by taking into account estimate (3.2), we obtain

$$\begin{aligned}\left(|\Omega| + \left\|Dy^\varepsilon_{u_1}\right\|_\alpha^\alpha + \left\|Dy^\varepsilon_{u_2}\right\|_\alpha^\alpha\right)^{\frac{2-\alpha}{\alpha}} &\leq \left(3|\Omega| + \left(2^{\frac{2-\alpha}{2}} C_\alpha \tfrac{\|u_1\|_2}{\nu}\right)^{\frac{\alpha}{\alpha-1}} + \left(2^{\frac{2-\alpha}{2}} C_\alpha \tfrac{\|u_2\|_2}{\nu}\right)^{\frac{\alpha}{\alpha-1}}\right)^{\frac{2-\alpha}{\alpha}} \\ &\leq \kappa_3\left(1+\tfrac{\|u_1\|_2}{2\nu}+\tfrac{\|u_2\|_2}{2\nu}\right)^{\frac{2-\alpha}{\alpha-1}}. \end{aligned} \tag{6.12}$$

The conclusion follows by combining (6.10), (6.11), (6.12) and by taking into account Lemma 2.4. □

A direct consequence of the previous lemma is that the control-to-state mapping $G^\varepsilon : u \longmapsto y^\varepsilon_u$ is locally Lipschitz continuous in V_2 if the following condition holds

$$\bar{\kappa}\left(1+\tfrac{\|u_1\|_2}{2\nu}+\tfrac{\|u_2\|_2}{2\nu}\right)^{\frac{2-\alpha}{\alpha-1}}\left(1+\tfrac{\|u_2\|_2}{\nu}\right)^{\frac{2-\alpha}{\alpha-1}} \tfrac{\|u_2\|_2}{\nu^2} < 1$$

with $\bar{\kappa} = \kappa_1\kappa_2\kappa_3$. Unlike the Navier-Stokes case or the shear-thickening case, where the corresponding conditions only involve u_2 (see [3]), the previous sufficient constraint is quite restrictive since it requires both controls u_1 and u_2 to be sufficiently "small". It holds if both controls u_1 and u_2 satisfies (3.7) and guarantees the Lipchitz continuity only for restricted admissible controls.

To overcome this difficulty, we impose condition (3.7) only on u_2 and refine the analysis of the result obtained in Lemma 6.5.

Lemma 6.6 *Assume that $\mathbf{A}_1$-$\mathbf{A}_2$ are fulfilled with $\frac{3n}{n+2} \leq \alpha \leq 2$. Let u_1, u_2 be in $L^2(\Omega)$ and let $y^\varepsilon_{u_1}$ and $y^\varepsilon_{u_2}$ be two corresponding solutions of* (6.1)*. If u_2 satisfies condition* (3.7)*, then the following estimate holds*

$$2\varepsilon\left\|D\left(y^\varepsilon_{u_1}-y^\varepsilon_{u_2}\right)\right\|_2 \leq \nu s_\alpha \left(\tfrac{\max(\|u_1\|_2-\|u_2\|_2,0)}{2\nu}\right)^{\frac{2-\alpha}{\alpha-1}} \left\|D\left(y^\varepsilon_{u_1}-y^\varepsilon_{u_2}\right)\right\|_2 + C_2\left\|u_1-u_2\right\|_2 \tag{6.13}$$

with $s_\alpha = 0$ if $\alpha = 2$ and $s_\alpha = 1$ if $\alpha < 2$.

Proof. Let us first notice that if $\|u_1\|_2 \leq \|u_2\|_2$, then

$$\left(1+\tfrac{\|u_1\|_2}{2\nu}+\tfrac{\|u_2\|_2}{2\nu}\right)^{\frac{2-\alpha}{\alpha-1}} \leq \left(1+\tfrac{\|u_2\|_2}{\nu}\right)^{\frac{2-\alpha}{\alpha-1}}.$$

Due to Lemma 6.5, it follows that

$$2\varepsilon\left\|D\left(y^\varepsilon_{u_1}-y^\varepsilon_{u_2}\right)\right\|_2^2 + \left(\frac{\nu}{\kappa_3\left(1+\frac{\|u_2\|_2}{\nu}\right)^{\frac{2-\alpha}{\alpha-1}}} - \kappa_1\kappa_2\left(1+\tfrac{\|u_2\|_2}{\nu}\right)^{\frac{2-\alpha}{\alpha-1}}\tfrac{\|u_2\|_2}{\nu}\right)\left\|D\left(y^\varepsilon_{u_1}-y^\varepsilon_{u_2}\right)\right\|_\alpha^2$$

$$\leq C_2 \|u_1 - u_2\|_2 \left\|D\left(y^\varepsilon_{u_1} - y^\varepsilon_{u_2}\right)\right\|_2$$

which by taking into account the fact that u_2 satisfies (3.7) gives (6.13). Similarly, observing that $\frac{2-\alpha}{\alpha-1} \in [0,1]$ and that for $x > y \geq 0$ we have

$$(1+x)^{\frac{2-\alpha}{\alpha-1}} - (1+y)^{\frac{2-\alpha}{\alpha-1}} \leq s_\alpha (x-y)^{\frac{2-\alpha}{\alpha-1}}$$

we deduce from Lemma 6.5 that if $\|u_1\|_2 > \|u_2\|_2$ then

$$2\varepsilon \left\|D\left(y^\varepsilon_{u_1} - y^\varepsilon_{u_2}\right)\right\|_2^2 + \left(\frac{\nu}{\kappa_3\left(1+\frac{\|u_2\|_2}{\nu}\right)^{\frac{2-\alpha}{\alpha-1}}} - \kappa_1\kappa_2\left(1+\frac{\|u_2\|_2}{\nu}\right)^{\frac{2-\alpha}{\alpha-1}} \frac{\|u_2\|_2}{\nu}\right) \left\|D\left(y^\varepsilon_{u_1} - y^\varepsilon_{u_2}\right)\right\|_\alpha^2$$

$$\leq \frac{\nu\left\|D\left(y^\varepsilon_{u_1} - y^\varepsilon_{u_2}\right)\right\|_\alpha^2}{\kappa_3\left(1+\frac{\|u_2\|_2}{\nu}\right)^{\frac{2-\alpha}{\alpha-1}}} - \frac{\nu\left\|D\left(y^\varepsilon_{u_1} - y^\varepsilon_{u_2}\right)\right\|_\alpha^2}{\kappa_3\left(1+\frac{\|u_1\|_2}{2\nu}+\frac{\|u_2\|_2}{2\nu}\right)^{\frac{2-\alpha}{\alpha-1}}} + C_2 \|u_1 - u_2\|_2 \left\|D\left(y^\varepsilon_{u_1} - y^\varepsilon_{u_2}\right)\right\|_2$$

$$\leq \frac{\nu s_\alpha}{\kappa_3}\left(\frac{\|u_1\|_2 - \|u_2\|_2}{2\nu}\right)^{\frac{2-\alpha}{\alpha-1}} \left\|D\left(y^\varepsilon_{u_1} - y^\varepsilon_{u_2}\right)\right\|_\alpha^2 + C_2 \|u_1 - u_2\|_2 \left\|D\left(y^\varepsilon_{u_1} - y^\varepsilon_{u_2}\right)\right\|_2$$

$$\leq \frac{\nu s_\alpha |\Omega|^{\frac{2-\alpha}{\alpha}}}{\kappa_3}\left(\frac{\|u_1\|_2 - \|u_2\|_2}{2\nu}\right)^{\frac{2-\alpha}{\alpha-1}} \left\|D\left(y^\varepsilon_{u_1} - y^\varepsilon_{u_2}\right)\right\|_2^2 + C_2 \|u_1 - u_2\|_2 \left\|D\left(y^\varepsilon_{u_1} - y^\varepsilon_{u_2}\right)\right\|_2$$

$$\leq \nu s_\alpha \left(\frac{\|u_1\|_2 - \|u_2\|_2}{2\nu}\right)^{\frac{2-\alpha}{\alpha-1}} \left\|D\left(y^\varepsilon_{u_1} - y^\varepsilon_{u_2}\right)\right\|_2^2 + C_2 \|u_1 - u_2\|_2 \left\|D\left(y^\varepsilon_{u_1} - y^\varepsilon_{u_2}\right)\right\|_2$$

which gives the claimed result. □

Remark 6.7 *From Lemma* 6.6*, we deduce that the Lipschitz continuity of G^ε in V_2 holds if u_1 and u_2 satisfy*

$$\nu s_\alpha \left(\frac{\max(\|u_1\|_2 - \|u_2\|_2, 0)}{2\nu}\right)^{\frac{2-\alpha}{\alpha-1}} < 2\varepsilon$$

with u_2 satisfying condition (3.7)*. The previous inequality is obviously valid if $\alpha = 2$ or if $\|u_1\|_2 \leq \|u_1\|_2$, but also if the difference $\|u_1\|_2 - \|u_2\|_2$ can be "controlled".*

Let u, v be in U_{ad} with u satisfying condition (3.7) and let ρ in $]0,1[$. Set $u_\rho = u + \rho(v-u)$, and let $y^\varepsilon_{u_\rho}$ be a solution of (6.1) corresponding to u_ρ and y^ε_u be the solution of (6.1) corresponding to u. In the remaining part of this section, and in order to simplify the notation, we set y^ε and y^ε_ρ instead of y^ε_u and $y^\varepsilon_{u_\rho}$ and $z^\varepsilon_\rho = \frac{y^\varepsilon_\rho - y^\varepsilon}{\rho}$.

Lemma 6.8 *For every ρ such that*

$$\rho < \frac{\nu}{U}\left(\frac{\varepsilon}{\nu s_\alpha}\right)^{\frac{\alpha-1}{2-\alpha}}, \tag{6.14}$$

the following estimate holds

$$\varepsilon \left\|Dz^\varepsilon_\rho\right\|_2 \leq C_2 \|u-v\|_2 .$$

Proof. First recall that since u e v belong to U_{ad}, then $\|u\|_2 \leq U$ and $\|v\|_2 \leq U$. Due to Lemma 6.6, we have

$$\begin{aligned} 2\varepsilon \left\|Dz^\varepsilon_\rho\right\|_2 &\leq \nu s_\alpha \left(\frac{\max(\|u_\rho\|_2 - \|u\|_2, 0)}{2\nu}\right)^{\frac{2-\alpha}{\alpha-1}} \left\|Dz^\varepsilon_\rho\right\|_2 + C_2\|v-u\|_2 \\ &\leq \nu s_\alpha \left(\frac{\|u_\rho - u\|_2}{2\nu}\right)^{\frac{2-\alpha}{\alpha-1}} \left\|Dz^\varepsilon_\rho\right\|_2 + C_2\|v-u\|_2 \\ &\leq \nu s_\alpha \left(\frac{\|v-u\|_2}{2\nu}\right)^{\frac{2-\alpha}{\alpha-1}} \rho^{\frac{2-\alpha}{\alpha-1}} \left\|Dz^\varepsilon_\rho\right\|_2 + C_2\|v-u\|_2 \\ &\leq \nu s_\alpha \left(\frac{U}{\nu}\right)^{\frac{2-\alpha}{\alpha-1}} \rho^{\frac{2-\alpha}{\alpha-1}} \left\|Dz^\varepsilon_\rho\right\|_2 + C_2\|v-u\|_2. \end{aligned}$$

Choosing ρ such that
$$\nu s_\alpha \left(\frac{U}{\nu}\right)^{\frac{2-\alpha}{\alpha-1}} \rho^{\frac{2-\alpha}{\alpha-1}} < \varepsilon$$
we obtain the claimed result. $\square$

Remark 6.9 *Notice that in the case $\alpha = 2$, condition (6.14) holds for every $\rho \in]0,1[$. Moreover, a direct consequence of Lemma 6.8 is that $(y^\varepsilon_\rho)_{\rho>0}$ strongly converges to y^ε in $H^1_0(\Omega)$.*

Lemma 6.10 *If $\left(z^\varepsilon_{\rho_k}\right)_k$ weakly converges to z^ε in V_2 for some sequence $(\rho_k)_k$ converging to zero, then*
$$\lim_{k\to+\infty} \tfrac{1}{\rho_k}\left(b\left(y^\varepsilon_{\rho_k}, y^\varepsilon_{\rho_k}, \varphi\right) - b\left(y^\varepsilon, y^\varepsilon, \varphi\right)\right) = b\left(z^\varepsilon, y^\varepsilon, \varphi\right) + b\left(y^\varepsilon, z^\varepsilon, \varphi\right)$$
$$\lim_{k\to+\infty} \tfrac{1}{\rho_k}\left(\tau(Dy^\varepsilon_{\rho_k}) - \tau(Dy^\varepsilon), D\varphi\right) = \left(\tau'(Dy^\varepsilon) : Dz^\varepsilon, D\varphi\right)$$
for all $\varphi \in \mathcal{V}$.

Proof. Since $(\rho_k)_k$ converges to zero, we can assume without loss of generality that ρ_k satisfies condition (6.14). The rest of the proof is split into two steps.

Step 1. Notice that
$$\begin{aligned}
&\left|\tfrac{1}{\rho_k}\left(b\left(y^\varepsilon_{\rho_k}, y^\varepsilon_{\rho_k}, \varphi\right) - b\left(y^\varepsilon, y^\varepsilon, \varphi\right)\right) - \left(b\left(z^\varepsilon, y^\varepsilon, \varphi\right) + b\left(y^\varepsilon, z^\varepsilon, \varphi\right)\right)\right| \\
&= \left|b\left(z^\varepsilon_{\rho_k}, y^\varepsilon_{\rho_k}, \varphi\right) + b\left(y^\varepsilon, z^\varepsilon_{\rho_k}, \varphi\right) - \left(b\left(z^\varepsilon, y^\varepsilon, \varphi\right) + b\left(y^\varepsilon, z^\varepsilon, \varphi\right)\right)\right| \\
&\le \left|b\left(z^\varepsilon_{\rho_k}, y^\varepsilon_{\rho_k}, \varphi\right) - b\left(z^\varepsilon, y^\varepsilon, \varphi\right)\right| + \left|b\left(y^\varepsilon, z^\varepsilon_{\rho_k}, \varphi\right) - b\left(y^\varepsilon, z^\varepsilon, \varphi\right)\right| \\
&\le \left|b\left(z^\varepsilon_{\rho_k} - z^\varepsilon, y^\varepsilon_{\rho_k}, \varphi\right)\right| + \left|b\left(z^\varepsilon, y^\varepsilon_{\rho_k} - y^\varepsilon, \varphi\right)\right| + \left|b\left(y^\varepsilon, z^\varepsilon_{\rho_k} - z^\varepsilon, \varphi\right)\right| \\
&\le \left\|z^\varepsilon_{\rho_k} - z^\varepsilon\right\|_4 \left\|\nabla y^\varepsilon_{\rho_k}\right\|_2 \|\varphi\|_4 + \|z^\varepsilon\|_4 \left\|\nabla\left(y^\varepsilon_{\rho_k} - y^\varepsilon\right)\right\|_2 \|\varphi\|_4 + \left|b\left(y^\varepsilon, z^\varepsilon_{\rho_k} - z^\varepsilon, \varphi\right)\right|
\end{aligned}$$
The result is then a consequence of the strong convergence of $\left(y^\varepsilon_{\rho_k}\right)_k$ to y^ε in $H^1_0(\Omega)$, the weak convergence of $\left(z^\varepsilon_{\rho_k}\right)_k$ to z^ε in $H^1_0(\Omega)$ and its strong convergence in $L^4(\Omega)$.

Step 2. To prove the second result, fix $\varphi \in \mathcal{V}$ and use the mean value theorem to get
$$\begin{aligned}
\tfrac{1}{\rho_k}\left(\tau(Dy^\varepsilon_{\rho_k}) - \tau(Dy^\varepsilon), D\varphi\right) &= \int_\Omega\int_0^1 \tau'(Dy^\varepsilon(x) + \theta D(y^\varepsilon_{\rho_k} - y^\varepsilon)(x)) : Dz^\varepsilon_{\rho_k}(x) : D\varphi(x)\, d\theta\, dx \\
&= \left(\tau'\left(\sigma^{\varphi,\varepsilon}_{\rho_k}\right) : Dz^\varepsilon_{\rho_k}, D\varphi\right) = \left(D\varphi : \tau'\left(\sigma^{\varphi,\varepsilon}_{\rho_k}\right), Dz^\varepsilon_{\rho_k}\right),
\end{aligned} \tag{6.15}$$
where $\sigma^{\varphi,\varepsilon}_{\rho_k}(x) = \theta^{\varphi,\varepsilon}_{\rho_k}(x)\left(Dy^\varepsilon_{\rho_k}(x) - Dy^\varepsilon(x)\right) + Dy^\varepsilon(x)$ with $0 < \theta^{\varphi,\varepsilon}_{\rho_k}(x) < 1$ being a number (depending on $\varphi(x)$) arising when applying the mean values theorem to the integral in the interval $[0,1]$. Convergence of $(\sigma^\varepsilon_{\rho_k})_k$ to Dy^ε in $L^2(\Omega)$ and continuity of τ' imply that
$$D\varphi : \tau'(\sigma^\varepsilon_{\rho_k}) \longrightarrow D\varphi : \tau'(Dy^\varepsilon) \qquad \text{a.e. in } \Omega.$$
On the other hand, due to $\mathbf{A}_1$, for all $x \in \Omega$ and $i, j, m, \ell = 1, \cdots, n$
$$\left|\left(\tau'(\sigma^\varepsilon_{\rho_k})(x)\right)_{ijm\ell}\right| \le \gamma\left(1 + |\sigma^\varepsilon_{\rho_k}(x)|^2\right)^{\frac{\alpha-2}{2}} \le \gamma$$
and thus
$$\left|\left(D\varphi(x) : \tau'(\sigma^\varepsilon_{\rho_k})(x)\right)_{m\ell}\right| \le \gamma\sum_{i,j}|D_{ij}\varphi(x)| \le n\gamma|D\varphi|.$$
Due to the dominated convergence theorem, we deduce that $\left(D\varphi : \tau'(\sigma^\varepsilon_{\rho_k})\right)_k$ strongly converges to $D\varphi : \tau'(Dy^\varepsilon)$ in $L^2(\Omega)$, which together with the weak convergence of $(Dz^\varepsilon_{\rho_k})_k$ to Dz^ε in $L^2(\Omega)$ prove the claimed result. $\square$

Proposition 6.11 *If $\left(z^\varepsilon_{\rho_k}\right)_k$ weakly converges to z^ε in V_2 for some sequence $(\rho_k)_k$ converging to zero, then z^ε is the unique solution of the linearized problem* (6.8) *corresponding to $(y^\varepsilon, v-u)$. Moreover, $\left(z^\varepsilon_{\rho_k}\right)_k$ converges strongly to z^ε in V_2.*

Proof. Since $(\rho_k)_k$ converges to zero, we can assume without loss of generality that ρ_k satisfies condition (6.14). The first assertion of the proposition is a direct consequence of Lemma 6.10 together with the density of $\mathcal{V}$ in V_2. To prove the strong convergence, let us set

$$M^\varepsilon(x) = 2\varepsilon I + \tau'(Dy^\varepsilon(x)), \qquad M^\varepsilon_\rho(x) = 2\varepsilon I + \tau'(\sigma^\varepsilon_\rho(x)),$$

where I denotes the identity matrix $n\times n$ and where σ^ε_ρ is defined as in (6.15) with φ substituted by $y^\varepsilon_\rho - y^\varepsilon$. Due to $\mathbf{A}_2$, the matrices

$$M^{\varepsilon,S}(x) = \tfrac{M^\varepsilon(x)+(M^\varepsilon(x))^T}{2}, \qquad M^{\varepsilon,S}_\rho(x) = \tfrac{M^\varepsilon_\rho(x)+(M^\varepsilon_\rho(x))^T}{2},$$

are symmetric and positive definite. Applying the Cholesky method to $M^{\varepsilon,S}(x)$ and $M^{\varepsilon,S}_\rho(x)$, we deduce the existence of lower triangular matrices $L^\varepsilon(x)$ and $L^\varepsilon_\rho(x)$ such that

$$M^{\varepsilon,S}(x) = L^\varepsilon(x)(L^\varepsilon(x))^T \qquad \text{and} \qquad M^{\varepsilon,S}_\rho(x) = L^\varepsilon_\rho(x)(L^\varepsilon_\rho(x))^T.$$

Substituing in the weak formulation of (6.1), we obtain

$$2\varepsilon\left(D(y^\varepsilon_\rho - y^\varepsilon), D\varphi\right) + \left(\tau(Dy^\varepsilon_\rho) - \tau(Dy^\varepsilon), D\varphi\right) = b\left(y^\varepsilon, y^\varepsilon, \varphi\right) - b\left(y^\varepsilon_\rho, y^\varepsilon_\rho, \varphi\right) + \rho(v-u,\varphi)$$

for all $\varphi \in V_2$. Therefore, taking into account (6.15), Lemma 2.6, Lemma 2.5, Lemma 2.4, Lemma 6.8 and estimate (6.6), we have

$$\begin{aligned}
\left\|(L^\varepsilon_{\rho_k})^T Dz^\varepsilon_{\rho_k}\right\|_2^2 &= \left(M^\varepsilon_{\rho_k} : Dz^\varepsilon_{\rho_k}, Dz^\varepsilon_{\rho_k}\right)\\
&= -b\left(y^\varepsilon, z^\varepsilon_{\rho_k}, z^\varepsilon_{\rho_k}\right) - b\left(z^\varepsilon_{\rho_k}, y^\varepsilon_{\rho_k}, z^\varepsilon_{\rho_k}\right) + \left(v-u, z^\varepsilon_{\rho_k}\right)\\
&= -b\left(z^\varepsilon_{\rho_k}, y^\varepsilon_{\rho_k}, z^\varepsilon_{\rho_k}\right) + \left(v-u, z^\varepsilon_{\rho_k}\right)\\
&\le \kappa_1 \left\|Dy^\varepsilon_{\rho_k}\right\|_2 \left\|Dz^\varepsilon_{\rho_k}\right\|_2^2 + C_2\left\|v-u\right\|_2 \left\|Dz^\varepsilon_{\rho_k}\right\|_2\\
&\le \tfrac{C_2^2}{\varepsilon}\left(\tfrac{\kappa_1 C_2}{2\varepsilon^2}\|u_{\rho_k}\|_2 + 1\right)\|v-u\|_2^2 \le \tfrac{C_2^2}{\varepsilon}\left(\tfrac{U\kappa_1 C_2}{2\varepsilon^2} + 1\right)\|v-u\|_2^2 \qquad (6.16)
\end{aligned}$$

and the sequence $((L^\varepsilon_{\rho_k})^T Dz^\varepsilon_{\rho_k})_k$ is then bounded in $L^2(\Omega)$. On the other hand, due to $\mathbf{A}_1$ we have

$$\left|L^\varepsilon_{\rho_k}(x)\right|^2 = \left|M^\varepsilon_{\rho_k}(x)\right| \le 2\sqrt{n}\varepsilon + C(\gamma,n) \qquad \text{for all } x\in\Omega.$$

Taking into account the convergence of $\left(Dy^\varepsilon_{\rho_k}\right)_k$ to Dy^ε into $L^2(\Omega)$ and the continuity of τ', we deduce that $M^\varepsilon_{\rho_k}(x)$ converges to $M^\varepsilon(x)$ and, consequently, $L^\varepsilon_{\rho_k}(x)$ converges to $L^\varepsilon(x)$ for a.e. $x\in\Omega$. The dominated convergence theorem then implies

$$L^\varepsilon_{\rho_k} \longrightarrow L^\varepsilon \qquad \text{strongly in } L^2(\Omega) \tag{6.17}$$

which together with the weak convergence of $(z^\varepsilon_{\rho_k})_k$ to z^ε in V_2, guarantees that $(L^\varepsilon_{\rho_k})^T Dz^\varepsilon_{\rho_k}$ weakly converges to $(L^\varepsilon)^T Dz^\varepsilon$ in $L^2(\Omega)$. Moreover, taking into account (6.16), we deduce that

$$\begin{aligned}
\left\|(L^\varepsilon)^T Dz^\varepsilon\right\|_2^2 &\le \liminf_k \left\|(L^\varepsilon_{\rho_k})^T Dz^\varepsilon_{\rho_k}\right\|_2^2\\
&\le \limsup_k \left\|(L^\varepsilon_{\rho_k})^T Dz^\varepsilon_{\rho_k}\right\|_2^2 = \limsup_k \left(M^\varepsilon_{\rho_k} : Dz^\varepsilon_{\rho_k}, Dz^\varepsilon_{\rho_k}\right)\\
&= \limsup_k \left(-b\left(z^\varepsilon_{\rho_k}, y^\varepsilon_{\rho_k}, z^\varepsilon_{\rho_k}\right) + \left(v-u, z^\varepsilon_{\rho_k}\right)\right)\\
&= -b\left(z^\varepsilon, y^\varepsilon, z^\varepsilon\right) + \left(v-u, z^\varepsilon\right) = -b\left(y^\varepsilon, z^\varepsilon, z^\varepsilon\right) - b\left(z^\varepsilon, y^\varepsilon, z^\varepsilon\right) + \left(v-u, z^\varepsilon\right)\\
&= \left(M^\varepsilon : Dz^\varepsilon, Dz^\varepsilon\right) = \left\|(L^\varepsilon)^T Dz^\varepsilon\right\|_2^2.
\end{aligned}$$

Weak convergence together with norm convergence implies strong convergence of $((L^\varepsilon_{\rho_k})^T Dz^\varepsilon_{\rho_k})_k$ to $(L^\varepsilon)^T Dz^\varepsilon$ in $L^2(\Omega)$. There then exist a subsequence, still indexed by ρ_k, and a function $h^\varepsilon \in L^2(\Omega)$ such that

$$\left|(L^\varepsilon_{\rho_k}(x))^T Dz^\varepsilon_{\rho_k}(x)\right| \le h^\varepsilon(x) \qquad \text{for a.e. } x \in \Omega \text{ and } k > k_0,$$

$$(L^\varepsilon_{\rho_k}(x))^T Dz^\varepsilon_{\rho_k}(x) \longrightarrow (L^\varepsilon(x))^T Dz^\varepsilon(x) \qquad \text{for a.e. } x \in \Omega.$$

Therefore, taking into account $\mathbf{A}_2$, we obtain

$$\begin{aligned} 2\varepsilon \left|Dz^\varepsilon_{\rho_k}(x)\right|^2 &\le M^\varepsilon_{\rho_k}(x) : Dz^\varepsilon_{\rho_k}(x) : Dz^\varepsilon_{\rho_k}(x) = (Dz^\varepsilon_{\rho_k}(x))^T M^\varepsilon_{\rho_k}(x) Dz^\varepsilon_{\rho_k}(x) \\ &= \left|(L^\varepsilon_{\rho_k})^T(x) Dz^\varepsilon_{\rho_k}(x)\right|^2 \le (h^\varepsilon(x))^2 \quad \text{for a.e. } x \in \Omega \text{ and } k > k_0. \end{aligned}$$

Since (6.17) implies that $\left((L^\varepsilon_{\rho_k}(x))^T\right)^{-1}$ converges to $\left((L^\varepsilon(x))^T\right)^{-1}$ for a.e. $x \in \Omega$, we deduce that

$$\begin{aligned} Dz^\varepsilon_{\rho_k}(x) &= \left((L^\varepsilon_{\rho_k}(x))^T\right)^{-1} \left((L^\varepsilon_{\rho_k}(x))^T\right) Dz^\varepsilon_{\rho_k}(x) \\ &\longrightarrow \left((L^\varepsilon(x))^T\right)^{-1} \left((L^\varepsilon(x))^T\right) Dz^\varepsilon(x) = Dz^\varepsilon(x) \end{aligned}$$

for a.e. $x \in \Omega$. The conclusion follows by applying the dominated convergence theorem. □

6.5 Proof of the approximate optimality conditions

Let us now prove Theorem 6.1. Existence of an optimal solution $(\bar u^\varepsilon, \bar y^\varepsilon)$ for problem (P^ε_α) can be established arguing as in the proof of Theorem 4.1. Assume in the rest of the proof that $\bar u^\varepsilon$ satisfies condition (3.7). For $\rho \in]0,1[$ satisfying (6.14) and $v \in U_{ad}$, let $u^\varepsilon_\rho = \bar u^\varepsilon + \rho(v - \bar u^\varepsilon)$, $y^\varepsilon_\rho = y_{u^\varepsilon_\rho}$ and $z^\varepsilon_\rho = \frac{y^\varepsilon_\rho - \bar y^\varepsilon}{\rho}$. Due to Lemma 6.8, we deduce that $\left(z^\varepsilon_\rho\right)_\rho$ is bounded in V_2. There then exist a subsequence $\left(z^\varepsilon_{\rho_k}\right)_k$ and $z^\varepsilon \in V_2$ such that $\left(z^\varepsilon_{\rho_k}\right)_k$ weakly converges to z^ε in V_2. Due to Proposition 6.11, z^ε is the unique solution of

$$\begin{cases} -\varepsilon\Delta z - \nabla\cdot(\tau'(D\bar y^\varepsilon) : Dz) + z\cdot\nabla\bar y^\varepsilon + \bar y^\varepsilon\cdot\nabla z + \nabla\pi = v - \bar u^\varepsilon & \text{in } \Omega, \\ \nabla\cdot z = 0 & \text{in } \Omega, \\ z = 0 & \text{on } \Gamma, \end{cases}$$

and $\left(z^\varepsilon_{\rho_k}\right)_k$ strongly converges to z^ε in V_2. Therefore

$$\lim_{k\to+\infty} \frac{I(u^\varepsilon_{\rho_k}, y^\varepsilon_{\rho_k}) - I(\bar u^\varepsilon, \bar y^\varepsilon)}{\rho_k} = (z^\varepsilon, \bar y^\varepsilon - y_d) + ((\lambda+1)\bar u^\varepsilon - \bar u, v - \bar u^\varepsilon)\,. \tag{6.18}$$

On the other hand, since $(u^\varepsilon_{\rho_k}, y^\varepsilon_{\rho_k})$ is admissible for (P^ε_α) and $(\bar u^\varepsilon, \bar y^\varepsilon)$ is an optimal solution, we deduce that

$$\lim_{k\to+\infty} \frac{I(u^\varepsilon_{\rho_k}, y^\varepsilon_{\rho_k}) - I(\bar u^\varepsilon, \bar y^\varepsilon)}{\rho_k} \ge 0 \qquad \text{for all } v \in U_{ad}. \tag{6.19}$$

Let $\bar p^\varepsilon \in V_2$ be the unique solution of (6.2) (existence and uniqueness of a solution can be obtained with arguments similar to those used in the proof of Proposition 6.4). Setting $\phi = z^\varepsilon$ and taking into account the weak formulation of problem (6.8), we obtain

$$\begin{aligned} (\bar y^\varepsilon - y_d, z^\varepsilon) &= (2\varepsilon D\bar p^\varepsilon, Dz^\varepsilon) + (\tau'(D\bar y^\varepsilon) : Dz^\varepsilon, D\bar p^\varepsilon) + \left((\nabla\bar y^\varepsilon)^T\bar p^\varepsilon - \bar y^\varepsilon\cdot\nabla\bar p^\varepsilon, z^\varepsilon\right) \\ &= (2\varepsilon Dz^\varepsilon, D\bar p^\varepsilon) + (\tau'(D\bar y^\varepsilon) : Dz^\varepsilon, D\bar p^\varepsilon) + (z^\varepsilon\cdot\nabla\bar y^\varepsilon, \bar p^\varepsilon) + (\bar y^\varepsilon\cdot\nabla z^\varepsilon, \bar p^\varepsilon) \\ &= (v - \bar u^\varepsilon, \bar p^\varepsilon)\,. \end{aligned} \tag{6.20}$$

The result follows by combining (6.18), (6.19) and (6.20). □

7 Proof of the necessary optimality conditions

We begin this section by establishing a useful convergence result.

Proposition 7.1 *Assume that* $\mathbf{A}_1$-$\mathbf{A}_2$ *are fulfilled with* $\frac{3n}{n+2} \leq \alpha \leq 2$. *Let* $\left(v_\varepsilon, y^\varepsilon_{v_\varepsilon}\right)$ *be an admissible pair for* (P^ε_α). *There exists a subsequence* $(\varepsilon_k)_k$ *converging to zero and* (u, y)*, an admissible pair for* (P_α)*, such that*

$$v_{\varepsilon_k} \longrightarrow u \quad \textit{weakly in } L^2(\Omega) \qquad \textit{and} \qquad y^{\varepsilon_k}_{v_{\varepsilon_k}} \longrightarrow y \quad \textit{strongly in } W^{1,\alpha}_0(\Omega).$$

Proof. To simplify the redaction, let us set $y_\varepsilon = y^\varepsilon_{v_\varepsilon}$. Since $(v_\varepsilon)_\varepsilon$ is uniformly bounded in $L^2(\Omega)$, estimate (6.5) and arguments similar to those used in the proof of Theorem 4.1 show that $(y_\varepsilon)_\varepsilon$ and $(\tau(Dy_\varepsilon))_\varepsilon$ are uniformly bounded in V_α and $L^{\frac{\alpha}{\alpha-1}}(\Omega)$, respectively. There then exists a subsequence $(\varepsilon_k)_k$ converging to zero and $u \in L^2(\Omega)$, $y \in V_\alpha$ and $\widetilde{\tau} \in L^{\frac{\alpha}{\alpha-1}}(\Omega)$ such that $(v_{\varepsilon_k})_k$ weakly converges to u in $L^2(\Omega)$, $(y_{\varepsilon_k})_k$ weakly converges to y in V_α and $(\tau(Dy_{\varepsilon_k}))_k$ weakly converges to $\widetilde{\tau}$ in $L^{\frac{\alpha}{\alpha-1}}(\Omega)$. On the other hand, estimate (6.7) implies that $\left(\sqrt{\varepsilon_k}\, Dy_{\varepsilon_k}\right)_k$ is bounded in $L^2(\Omega)$ and that for $\varphi \in V_2$, we have

$$|\varepsilon_k\, (Dy_{\varepsilon_k}, D\varphi)| \leq \sqrt{\varepsilon_k}\, \|\sqrt{\varepsilon_k}\, Dy_{\varepsilon_k}\|_2\, \|D\varphi\|_2 \longrightarrow 0 \qquad \text{when } \varepsilon_k \to 0. \tag{7.1}$$

Taking into account (7.1) and passing to the limit in the weak formulation corresponding to y_{ε_k}, we deduce that

$$(\widetilde{\tau}, D\varphi) + b\,(y, y, \varphi) = (u, \varphi) \qquad \text{for all } \varphi \in \mathcal{V}$$

and thus for all $\varphi \in V_\alpha$. On the other hand, (2.2) gives

$$(\tau\,(Dy_{\varepsilon_k}) - \tau\,(D\varphi)\,, Dy_{\varepsilon_k} - D\varphi) \geq 0 \qquad \text{for all } \varphi \in V_\alpha,$$

and since

$$(\tau\,(Dy_{\varepsilon_k})\,, Dy_{\varepsilon_k}) = (v_{\varepsilon_k}, y_{\varepsilon_k}) - 2\varepsilon_k\, \|Dy_{\varepsilon_k}\|_2^2\,,$$

we obtain

$$(v_{\varepsilon_k}, y_{\varepsilon_k}) - (\tau\,(Dy_{\varepsilon_k})\,, D\varphi) - (\tau\,(D\varphi)\,, Dy_{\varepsilon_k} - D\varphi) \geq 2\varepsilon_k\, \|Dy_{\varepsilon_k}\|_2^2 \geq 0$$

for all $\varphi \in V_\alpha$. By passing to the limit in the previous inequality, and arguing as in the proof of Theorem 4.1, we can prove that

$$(\widetilde{\tau}, D\varphi) = (\tau(Dy), D\varphi) \qquad \text{for all } \varphi \in V_\alpha$$

and thus y is a solution of (1.1) corresponding to u. To prove the strong convergence of $(y_{\varepsilon_k})_k$ to y in $W^{1,\alpha}_0(\Omega)$, notice that estimate (6.5) together with (6.12) yield

$$\begin{aligned}(\tau\,(Dy_{\varepsilon_k}) - \tau\,(Dy)\,, D\,(y_{\varepsilon_k} - y)) &\geq \frac{\nu \left\|D(y_{\varepsilon_k} - y)\right\|_\alpha^2}{\left(|\Omega| + \|Dy_{\varepsilon_k}\|_\alpha^\alpha + \|Dy\|_\alpha^\alpha\right)^{\frac{2-\alpha}{\alpha}}} \\ &\geq \frac{\nu \left\|D(y_{\varepsilon_k} - y)\right\|_\alpha^2}{\kappa_3 \left(1 + \frac{\|u\|_2}{2\nu} + \frac{\|v_{\varepsilon_k}\|_2}{2\nu}\right)^{\frac{2-\alpha}{\alpha-1}}} \geq \frac{\nu \left\|D(y_{\varepsilon_k} - y)\right\|_\alpha^2}{\kappa_3 \left(1 + \frac{U}{\nu}\right)^{\frac{2-\alpha}{\alpha-1}}}.\end{aligned}$$

Therefore, by taking into account the previous convergence results, we deduce that

$$\frac{\nu}{\kappa_3 \left(1 + \frac{U}{\nu}\right)^{\frac{2-\alpha}{\alpha-1}}} \limsup_k \|D(y_{\varepsilon_k} - y)\|_\alpha^2$$

$$\leq \limsup_{k} \left(\tau\left(Dy_{\varepsilon_k}\right) - \tau\left(Dy\right), D\left(y_{\varepsilon_k} - y\right)\right) \leq \limsup_{k} \left(\tau\left(Dy_{\varepsilon_k}\right), D\left(y_{\varepsilon_k} - y\right)\right)$$
$$\leq \limsup_{k} \left(\left(v_{\varepsilon_k}, y_{\varepsilon_k}\right) - \left(\tau\left(Dy_{\varepsilon_k}\right), Dy\right)\right) \leq (u, y) - (\widetilde{\tau}, Dy) = 0$$

and the claimed result is proven. □

Next, we prove that the solutions of problems (P_α^ε) form an approximating family for (P_α).

Proposition 7.2 *Assume that* $\mathbf{A}_1$-$\mathbf{A}_2$ *are fulfilled with* $\frac{3n}{n+2} \leq \alpha \leq 2$. *Let* $(\bar{u}^\varepsilon, \bar{y}^\varepsilon)$ *be a solution of* (P_α^ε). *There exists a subsequence* $(\varepsilon_k)_k$ *converging to zero such that*

$$\lim_{k\to+\infty} \|\bar{u}^{\varepsilon_k} - \bar{u}\|_2 = 0, \qquad \lim_{k\to+\infty} \|\bar{y}^{\varepsilon_k} - \bar{y}\|_{1,\alpha} = 0, \qquad \lim_{k\to+\infty} I(\bar{u}^{\varepsilon_k}, \bar{y}^{\varepsilon_k}) = J(\bar{u}, \bar{y}).$$

Proof. Setting $v_\varepsilon = \bar{u}$ for all $\varepsilon > 0$ and $v_\varepsilon = \bar{u}^\varepsilon$ and applying Proposition 7.1, we deduce that there exists a subsequence $(\varepsilon_k)_k$ such that $(y_{\bar{u}}^{\varepsilon_k})_k$ converges in $W_0^{1,\alpha}(\Omega)$ to $\bar{y}$ (the unique solution of (1.1) corresponding to $\bar{u}$), $(\bar{u}^{\varepsilon_k})_k$ weakly converges in $L^2(\Omega)$ to some u and $(\bar{y}^{\varepsilon_k})_k$ converges in $W_0^{1,\alpha}(\Omega)$ to y (a solution of (1.1) corresponding to u). Using the lower semicontinuity of I and the admissibility of $(\bar{u}, y_{\bar{u}}^{\varepsilon_k})$ for $(P_\alpha^{\varepsilon_k})$, we obtain

$$\begin{aligned}
\tfrac{1}{2}\left\|y - y_d\right\|_2^2 + \tfrac{\lambda}{2}\left\|u\right\|_2^2 + \tfrac{1}{2}\left\|u - \bar{u}\right\|_2^2 &\leq \liminf_{k} I(\bar{u}^{\varepsilon_k}, \bar{y}^{\varepsilon_k}) \\
&\leq \limsup_{k} I(\bar{u}^{\varepsilon_k}, \bar{y}^{\varepsilon_k}) \\
&\leq \lim_{k} I(\bar{u}, y_{\bar{u}}^{\varepsilon_k}) = \tfrac{1}{2}\left\|\bar{y} - y_d\right\|_2^2 + \tfrac{\lambda}{2}\left\|\bar{u}\right\|_2^2
\end{aligned}$$

and consequently

$$J(u, y) + \tfrac{1}{2}\left\|u - \bar{u}\right\|_2^2 \leq J(\bar{u}, \bar{y}).$$

Since $(\bar{u}, \bar{y})$ is solution of (P_α), we have $J(\bar{u}, \bar{y}) \leq J(u, y)$ and thus $u = \bar{u}$. Recalling that $\bar{u}$ satisfies condition (3.7), we deduce that $y = \bar{y}$ and thus

$$\lim_{k\to+\infty} I(\bar{u}^{\varepsilon_k}, \bar{y}^{\varepsilon_k}) = J(\bar{u}, \bar{y}).$$

Finally, observing that

$$\begin{aligned}
\tfrac{1}{2}\limsup_{k} \left\|\bar{u}^{\varepsilon_k} - \bar{u}\right\|_2^2 &= \limsup_{k} \left(I(\bar{u}^{\varepsilon_k}, \bar{y}^{\varepsilon_k}) - \tfrac{1}{2}\|\bar{y}^{\varepsilon_k} - y_d\|_2^2 - \tfrac{\lambda}{2}\left\|\bar{u}^{\varepsilon_k}\right\|_2^2\right) \\
&\leq J(\bar{u}, \bar{y}) - \tfrac{1}{2}\|\bar{y} - y_d\|_2^2 - \tfrac{\lambda}{2}\liminf_{k} \left\|\bar{u}^{\varepsilon_k}\right\|_2^2 \\
&= \tfrac{\lambda}{2}\left\|\bar{u}\right\|_2^2 - \tfrac{\lambda}{2}\liminf_{k} \left\|\bar{u}^{\varepsilon_k}\right\|_2^2 \leq 0
\end{aligned}$$

we conclude that $(\bar{u}^{\varepsilon_k})_k$ converges to $\bar{u}$ strongly in $L^2(\Omega)$. □

Proof of Theorem 5.1. Let $(\bar{u}^{\varepsilon_k}, \bar{y}^{\varepsilon_k})$ be the solution of $(P_\alpha^{\varepsilon_k})$ given in Proposition 7.2. Since $\bar{u}$ satisfies condition (3.7), we deduce that there exists $k_1 \in I\!N$ such that $\bar{u}^{\varepsilon_k}$ also satisfies condition (3.7) for every $k > k_1$. The rest of the proof is split into two steps.

Step 1. Let us first prove (5.1) and (5.2). Due Theorem 6.1, there exists $\bar{p}^{\varepsilon_k} \in V_2$ such that

$$\begin{cases}
-\varepsilon_k \Delta p - \nabla \cdot \left(\tau'\left(D\bar{y}^{\varepsilon_k}\right) : Dp\right) + (\nabla \bar{y}^{\varepsilon_k})^T p - \bar{y}^{\varepsilon_k} \cdot \nabla p + \nabla \tilde{\pi}^{\varepsilon_k} = \bar{y}^{\varepsilon_k} - y_d, \\
\nabla \cdot p = 0, \\
p_{|\Gamma} = 0,
\end{cases} \tag{7.2}$$

$$(\bar{p}^{\varepsilon_k} + (\lambda+1)\bar{u}^{\varepsilon_k} - \bar{u}, v - \bar{u}^{\varepsilon_k}) \geq 0 \qquad \text{for all } v \in U_{ad}. \tag{7.3}$$

With arguments similar to those used in the proof of Proposition 6.4, we obtain the following estimates

$$\|D\bar{p}^{\varepsilon_k}\|_\alpha \leq C_\alpha \mathcal{L}\left(\tfrac{\|\bar{u}^{\varepsilon_k}\|_2}{\nu}\right) \|\bar{y}^{\varepsilon_k} - y_d\|_2 \leq C_\alpha \mathcal{L}\left(\tfrac{U}{\nu}\right) \|\bar{y}^{\varepsilon_k} - y_d\|_2\,, \tag{7.4}$$

$$2\varepsilon_k \|D\bar{p}^{\varepsilon_k}\|_2^2 \leq C_\alpha^2 \mathcal{L}\left(\tfrac{\|\bar{u}^{\varepsilon_k}\|_2}{\nu}\right) \|\bar{y}^{\varepsilon_k} - y_d\|_2^2 \leq C_\alpha^2 \mathcal{L}\left(\tfrac{U}{\nu}\right) \|\bar{y}^{\varepsilon_k} - y_d\|_2^2\,.$$

Therefore, $(\bar{p}^{\varepsilon_k})_k$ is bounded in V_α and $(\sqrt{\varepsilon_k}\, D\bar{p}^{\varepsilon_k})_k$ is bounded in $L^2(\Omega)$. There then exist a subsequence, still indexed by k, and $\bar{p}$ such that $(\bar{p}^{\varepsilon_k})_k$ weakly converges to $\bar{p}$ in V_α and, by using compactness results on Sobolev spaces, $(\bar{p}^{\varepsilon_k})_k$ strongly converges to $\bar{p}$ in $L^q(\Omega)$ for every $q < \frac{n\alpha}{n-\alpha}$. Moreover, for $\varphi \in \mathcal{V}$, we have

$$|\varepsilon_k\,(D\bar{p}^{\varepsilon_k}, D\varphi)| \leq \sqrt{\varepsilon_k}\, \|\sqrt{\varepsilon_k}\, D\bar{p}^{\varepsilon_k}\|_2 \|D\varphi\|_2 \longrightarrow 0 \qquad \text{when } \varepsilon_k \to 0. \tag{7.5}$$

Similarly, we may prove that

$$\lim_{k\to+\infty} \left((\nabla\bar{y}^{\varepsilon_k})^T \bar{p}^{\varepsilon_k} - \bar{y}^{\varepsilon_k}\cdot\nabla\bar{p}^{\varepsilon_k}, \varphi\right) = \left((\nabla\bar{y})^T\bar{p} - \bar{y}\cdot\nabla\bar{p}, \varphi\right). \tag{7.6}$$

Taking into account the convergence of $(D\bar{y}^{\varepsilon_k})_k$ to $D\bar{y}$ in $L^\alpha(\Omega)$ and the continuity of τ', with arguments similar to those used in the proof of Lemma 6.10, we deduce that

$$\lim_{k\to+\infty} \|\tau'(D\bar{y}^{\varepsilon_k}) : D\varphi - \tau'(D\bar{y}) : D\varphi\|_{\frac{\alpha}{\alpha-1}} = 0.$$

This result together with the convergence of $(D\bar{p}^{\varepsilon})_k$ to $D\bar{p}$ in the weak topology of $L^\alpha(\Omega)$ imply

$$\lim_{k\to+\infty} (\tau'(D\bar{y}^{\varepsilon_k}) : D\varphi, D\bar{p}^{\varepsilon_k}) = (\tau'(D\bar{y}) : D\varphi, D\bar{p})\,. \tag{7.7}$$

Taking into account (7.5), (7.6) and (7.7), and passing to the limit in (7.2) and (7.3), we obtain

$$(\tau'(D\bar{y}) : D\varphi, D\bar{p}) + \left((\nabla\bar{y})^T\bar{p} - \bar{y}\cdot\nabla\bar{p}, \varphi\right) = (\bar{y} - y_d, \varphi) \quad \text{for all } \varphi \in \mathcal{V},$$

and

$$(\bar{p} + \lambda\bar{u}, v - \bar{u}) \geq 0 \qquad \text{for all } v \in U_{ad}$$

which gives the claim result.

Step 2. Let us now prove (5.3). Set

$$M(x) = \tau'(D\bar{y}(x)), \qquad M^\varepsilon = \tau'(D\bar{y}^\varepsilon(x)),$$

and

$$M^S(x) = \tfrac{M(x)+(M(x))^T}{2}, \qquad M^{\varepsilon,S}(x) = \tfrac{M^\varepsilon(x)+(M^\varepsilon(x))^T}{2}.$$

Due to $\mathbf{A}_2$, the matrices $M^S(x)$ and $M^{\varepsilon,S}(x)$ are symmetric and positive definite. Applying the Cholesky method, we deduce the existence of lower triangular matrices $L(x)$ and $L^\varepsilon(x)$ such that

$$M^S(x) = L(x)(L(x))^T \qquad \text{and} \qquad M^{\varepsilon,S}(x) = L^\varepsilon(x)(L^\varepsilon(x))^T.$$

Therefore, setting $\varphi = p^{\varepsilon_k}$ in the weak formulation of (7.2) and due to lemma 2.5, Lemma 2.4 and estimates (6.5) and (7.4) , we obtain

$$\begin{aligned}
\left\|(L^{\varepsilon_k})^T D\bar{p}^{\varepsilon_k}\right\|_2^2 &= (M^{\varepsilon_k} : D\bar{p}^{\varepsilon_k}, D\bar{p}^{\varepsilon_k}) \\
&= -2\varepsilon_k \|D\bar{p}^{\varepsilon_k}\|_2^2 - \left((\nabla\bar{y}^{\varepsilon_k})^T\bar{p}^{\varepsilon_k} - \bar{y}^{\varepsilon_k}\cdot\nabla\bar{p}^{\varepsilon_k}, \bar{p}^{\varepsilon_k}\right) + (\bar{y}^{\varepsilon_k} - y_d, \bar{p}^{\varepsilon_k}) \\
&= -2\varepsilon_k \|D\bar{p}^{\varepsilon_k}\|_2^2 - (\bar{p}^{\varepsilon_k}\cdot\nabla\bar{y}^{\varepsilon_k}, \bar{p}^{\varepsilon_k}) + (\bar{y}^{\varepsilon_k} - y_d, \bar{p}^{\varepsilon_k}) \\
&\le -(\bar{p}^{\varepsilon_k}\cdot\nabla\bar{y}^{\varepsilon_k}, \bar{p}^{\varepsilon_k}) + (\bar{y}^{\varepsilon_k} - y_d, \bar{p}^{\varepsilon_k}) \\
&\le \kappa_1 \|D\bar{p}^{\varepsilon_k}\|_\alpha^2 \|D\bar{y}^{\varepsilon_k}\|_\alpha + C_\alpha \|\bar{y}^{\varepsilon_k} - y_d\|_2 \|D\bar{p}^{\varepsilon_k}\|_\alpha \\
&\le C_\alpha^2 \left(\mathcal{L}\left(\tfrac{U}{\nu}\right) + \kappa_1\left(\left(2^{\frac{2-\alpha}{2}} C_\alpha\left(\tfrac{\|\bar{u}^{\varepsilon_k}\|_2}{\nu}\right)\right)^{\frac{1}{\alpha-1}} + |\Omega|^{\frac{1}{\alpha}}\right)\mathcal{L}^2\left(\tfrac{U}{\nu}\right)\right)\|\bar{y}^{\varepsilon_k} - y_d\|_2^2 \\
&\le C_\alpha^2 \left(\mathcal{L}\left(\tfrac{U}{\nu}\right) + \kappa_1\left(\left(2^{\frac{2-\alpha}{2}} C_\alpha\left(\tfrac{U}{\nu}\right)\right)^{\frac{1}{\alpha-1}} + |\Omega|^{\frac{1}{\alpha}}\right)\mathcal{L}^2\left(\tfrac{U}{\nu}\right)\right)\|\bar{y}^{\varepsilon_k} - y_d\|_2^2
\end{aligned}$$

and the sequence $((L^{\varepsilon_k})^T D\bar{p}^{\varepsilon_k})_k$ is bounded in $L^2(\Omega)$. On the other hand, due to $\mathbf{A}_1$ we have

$$|L^{\varepsilon_k}(x)|^2 = |M^{\varepsilon_k}(x)| \le C(\gamma, n) \qquad \text{for all } x \in \Omega.$$

Taking into account the convergence of $(D\bar{y}^{\varepsilon_k})_k$ to $D\bar{y}$ in $L^\alpha(\Omega)$ and the continuity of τ', we deduce that $(M^{\varepsilon_k}(x))_k$ converges to $M(x)$ and thus $(L^{\varepsilon_k}(x))_k$ converges to $L(x)$ for a.e. $x \in \Omega$. The dominated convergence theorem then implies the convergence of $(L^{\varepsilon_k})_k$ to L in $L^q(\Omega)$ for every $q > 1$ and in particular for $q = \frac{\alpha}{\alpha-1}$. Since $(D\bar{p}^{\varepsilon_k})_k$ weakly converges to $D\bar{p}$ in $L^\alpha(\Omega)$, we deduce that

$$(L^{\varepsilon_k})^T Dp^{\varepsilon_k} \longrightarrow L^T Dp \qquad \text{weakly in } L^2(\Omega).$$

Therefore,

$$\begin{aligned}
\left\|L^T D\bar{p}\right\|_2^2 &\le \liminf_k \left\|(L^{\varepsilon_k})^T D\bar{p}^{\varepsilon_k}\right\|_2^2 \\
&\le \limsup_k \left\|(L^{\varepsilon_k})^T D\bar{p}^{\varepsilon_k}\right\|_2^2 = \limsup_k (M^{\varepsilon_k} : D\bar{p}^{\varepsilon_k}, D\bar{p}^{\varepsilon_k}) \\
&\le \limsup_k -(\bar{p}^{\varepsilon_k}\cdot\nabla\bar{y}^{\varepsilon_k}, \bar{p}^{\varepsilon_k}) + (\bar{y}^{\varepsilon_k} - y_d, \bar{p}^{\varepsilon_k}) \\
&= -(\bar{p}\cdot\nabla\bar{y}, \bar{p}) + (\bar{y} - y_d, \bar{p}),
\end{aligned}$$

and the claimed result is proven. □

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