Asymptotic comparison at optimal levels of minimum-variance reduced-bias tail index estimators

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Abstract In this paper we are interested in the asymptotic comparison of a set of semi-parametric minimum variance reduced-bias (MVRB) tail index estimators, at optimal levels and for a wide class of models. Again, as in the classical case, there is not any estimator that can always dominate the alternatives, but interesting clear-cut patterns are found. Consequently, and in practice, a suitable choice of a set of tail index estimators, will jointly enable us to better estimate the tail index γ, the primary parameter of extreme events.

1 The estimators under study and scope of the paper

Let us consider the common set-up of independent, identically distributed (i.i.d.) random variables (r.v.'s) $X_1, X_2, \cdots, X_n$, with a common distribution function (d.f.) F. Let us denote the associated ascending order statistics (o.s.) by $X_{1:n} \le X_{2:n} \le \cdots \le X_{n:n}$ and let us assume that there exist sequences of real constants $\{a_n > 0\}$ and $\{b_n \in R\}$ such that the maximum, linearly normalized, i.e., $(X_{n:n} - b_n)/a_n$, converges in distribution towards a non-degenerate r.v. Then F belongs to the max-domain of attraction of an *extreme value* (EV) d.f.,

$$EV_\gamma(x) = exp(-(1+\gamma x)^{-1/\gamma}), \quad 1+\gamma x > 0, \quad \gamma \in \mathrm{R}. \tag{1}$$

and we write $F \in \mathscr{D}_{\mathscr{M}}\left(EV_\gamma\right)$. The parameter γ is the *extreme value index*, the primary parameter of extreme events, with a low frequency, but a high impact. This index measures the heaviness of the right *tail function* $\overline{F} := 1 - F$, and the heavier the tail, the larger γ is. In this paper we shall work with heavy-tailed models, i.e.,

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Pareto-type underlying d.f.'s, with a strict positive extreme value index, often called *tail index*.

The *second order parameter*, ρ (≤ 0), rules the rate of convergence in the first order condition, and is the non-positive parameter appearing in the limiting relation

$$\lim_{t\to\infty} \frac{\ln U(tx) - \ln U(t) - \gamma \ln x}{A(t)} = \frac{x^\rho - 1}{\rho}, \tag{2}$$

which is assumed to hold for every $x > 0$, and where $|A|$ must then be of regular variation with index ρ (Geluk and de Haan [6]). To obtain information on the order of the asymptotic bias of second-order reduced-bias extreme value index estimators, it is necessary to further assume a third order condition, ruling the rate of convergence in (2), and which guarantees that, for all $x > 0$,

$$\lim_{t\to\infty} \frac{\frac{\ln U(tx) - \ln U(t) - \gamma \ln x}{A(t)} - \frac{x^\rho - 1}{\rho}}{B(t)} = \frac{x^{\rho+\rho'} - 1}{\rho+\rho'}, \tag{3}$$

where $|B(t)|$ must then be of regular variation with index $\rho' \le 0$.

In this paper, similarly to what has been done in Gomes *et al.* [10], we consider a Pareto-type class of models with a tail function

$$1 - F(x) = Cx^{-1/\gamma}\big(1 + D_1 x^{\rho/\gamma} + D_2 x^{2\rho/\gamma} + o\big(x^{2\rho/\gamma}\big)\big), \quad \text{as} \quad x \to \infty, \tag{4}$$

with $C > 0, D_1,\ D_2 \ne 0, \rho < 0$. Note that to assume (4) is equivalent to say that (3) holds with $\rho = \rho' < 0$ and that we may there choose

$$A(t) = \alpha\, t^\rho =: \gamma\, \beta\, t^\rho, \qquad B(t) = \beta'\, t^\rho = \beta' A(t)/(\beta\gamma), \qquad \beta,\ \beta' \ne 0, \tag{5}$$

with β and β' "scale" second and third order parameters, respectively.

For heavy-tailed models, the classical tail index estimator is Hill's estimator (Hill [13]), the average of the scaled log-spacings U_i or of the log-excesses V_{ik}:

$$H_n(k) \equiv H(k) := \frac{1}{k}\sum_{i=1}^{k} U_i = \frac{1}{k}\sum_{i=1}^{k} V_{ik}, \tag{6}$$

where

$$U_i := i\left\{\ln \frac{X_{n-i+1:n}}{X_{n-i:n}}\right\}, \quad \text{and} \quad V_{ik} := \ln \frac{X_{n-i+1:n}}{X_{n-k:n}}, \quad 1 \le i \le k < n. \tag{7}$$

For intermediate k, i.e., a sequence of integers $k = k_n$, $1 \le k < n$, such that

$$k = k_n \to \infty \quad \text{and} \quad k_n = o(n), \quad \text{as} \quad n \to \infty, \tag{8}$$

the Hill estimator in (6) is consistent for $\gamma > 0$ whenever $F \in \mathscr{D}_{\mathscr{M}}(EV_\gamma)_{\gamma>0}$ holds.

The adequate accommodation of the bias of Hill's estimator has been extensively addressed in recent years in the literature. Recently, several authors [2, 11, 10, 9]

consider, in different ways and under the second order framework in (2), minimum-variance reduced-bias (MVRB) tail index estimators based on the joint external estimation of both the "scale" and the "shape" parameters, β and ρ, respectively. These estimators are called MVRB due to the fact that, under adequate restrictions, are able to reduce the bias without increasing the asymptotic variance, which is shown to be kept at the value γ^2, the asymptotic variance of Hill's estimator, at least for values k such that $\sqrt{k}A(n/k) \to \lambda$, finite, as $n \to \infty$.
Gomes *et al.* [10] consider an extreme value index estimator based on a linear combination of the log-excesses V_{ik} in (7), and given by

$$\overline{WH}_{\hat{\beta},\hat{\rho}}(k) := \frac{1}{k}\sum_{i=1}^{k} e^{-\hat{\beta}\ (n/k)^{\hat{\rho}}\ \psi_{ik}(\hat{\rho})}\ V_{ik}, \quad \psi_{ik}(\rho) = \psi_{ik} = -\frac{(i/k)^{-\rho}-1}{\rho \ln(i/k)}, \tag{9}$$

WH standing here for *weighted Hill* estimator. Caeiro *et al.* [2] consider two estimators of this same type, here denoted,

$$CH_{\hat{\beta},\hat{\rho}}(k) := H(k)\Big(1 - \frac{\hat{\beta}}{1-\hat{\rho}}\left(\frac{n}{k}\right)^{\hat{\rho}}\Big), \tag{10}$$

$$\overline{CH}_{\hat{\beta},\hat{\rho}}(k) := H(k)\exp\Big(-\frac{\hat{\beta}}{1-\hat{\rho}}\left(\frac{n}{k}\right)^{\hat{\rho}}\Big), \tag{11}$$

where the dominant component of the bias of Hill's estimator in (6), given by $A(n/k)\ /\ (1-\rho)\ =\ \gamma\ \beta\ \ (n/k)^{\rho}\ /(1-\rho)$, is thus essentially estimated through $H(k)\ \hat{\beta}\,(n/k)^{\hat{\rho}}\ /(1-\hat{\rho})$, and directly removed from Hill's classical extreme value index estimator. The notation CH stands for *corrected Hill*. A third class has been introduced in Gomes *et al.* [10], and it has the functional form

$$ML_{\widehat{\beta},\widehat{\rho}}(k) := H(k) - \widehat{\beta}\ \left(\frac{n}{k}\right)^{\widehat{\rho}}\ \Big(\frac{1}{k}\sum_{i=1}^{k}\left(\frac{i}{k}\right)^{-\widehat{\rho}} U_i\Big), \tag{12}$$

with U_i given in (7). These authors consider also the estimator

$$\overline{ML}_{\widehat{\beta},\widehat{\rho}}(k) := \frac{1}{k}\sum_{i=1}^{k} \exp\big(-\hat{\beta}(n/i)^{\hat{\rho}}\big)U_i, \tag{13}$$

the estimator directly derived from the likelihood equation for γ with β and ρ fixed and based upon the exponential approximation $U_i \approx \gamma \exp(\beta(n/i)^{\rho})E_i$, $1 \le i \le k$, being claimed a better performance of the ML estimator, comparatively to the $\overline{ML}$ estimator, for a large class of models. This is the reason why we shall also work with the bias-corrected Hill estimator

$$WH_{\hat{\beta},\hat{\rho}}(k) := H(k) - \hat{\beta}\ \left(\frac{n}{k}\right)^{\hat{\rho}}\Big(\frac{1}{k}\sum_{i=1}^{k}\psi_{ik}\ V_{ik}\Big), \tag{14}$$

with ψ_{ik} given in (9).

Remark 1. In all the above MVRB tail index estimators, $\hat{\beta}$ and $\hat{\rho}$ need to be adequate consistent estimators of the second order parameters β and ρ, respectively. For more details related with the estimation of these parameters see, for instance, [5, 8]

In this paper, we compare asymptotically, at optimal levels, the above mentioned MVRB statistics, denoted generically $UH_{\beta,\rho}(k)$ (assuming thus that β and ρ are known or adequately estimated). In Section 2, we shall state for the class of models in (4), the asymptotic properties of $UH_{\beta,\rho}(k)$, and in Section 3, we provide a full asymptotic comparison, at optimal levels, of $UH_{\beta,\rho}(k)$ for $UH = CH$, ML and WH.

2 The asymptotic behaviour of the MVRB tail index estimators

Let $\{E_i\}$ denote a sequence of i.i.d. standard exponential r.v.'s, and define

$$Z_k := \frac{1}{k}\sum_{i=1}^{k} E_i \quad \text{and} \quad \overline{Z}_k := \sqrt{k}\big(Z_k - 1\big). \tag{15}$$

Assuming the third order framework in (4), we state the following result, a particular case, with a few additions related with the $\overline{UH}$ statistics, of Theorem 3.1 in [4].

Theorem 1. *Under the third order framework in* (4)*, with $A(t)$ given in* (5)*, $\overline{Z}_k$ given in* (15)*, and for intermediate k, i.e., if* (8) *holds, we can write*

$$UH_{\beta,\rho}(k) \overset{d}{=} \gamma + \frac{\gamma \overline{Z}_k}{\sqrt{k}} + \Big(b_{UH}A^2(n/k) + O_p\Big(\frac{A(n/k)}{\sqrt{k}}\Big)\Big)(1+o_p(1)), \tag{16}$$

where, with $\xi = \beta'/\beta$,

$$a_2(\rho) := -\frac{1}{\rho^2}\left(\ln(1-2\rho) - 2\ln(1-\rho)\right),$$

$$b_{CH} = \frac{1}{\gamma}\Big(\frac{\xi}{1-2\rho} - \frac{1}{(1-\rho)^2}\Big), \qquad b_{\overline{CH}} = \frac{1}{\gamma}\Big(\frac{\xi}{1-2\rho} - \frac{1}{2(1-\rho)^2}\Big),$$

$$b_{ML} = \frac{1}{\gamma(1-2\rho)}(\xi-1), \qquad b_{\overline{ML}} = \frac{1}{2\gamma(1-2\rho)}(\xi-1),$$

$$b_{WH} = \frac{1}{\gamma}\Big(\frac{\xi}{1-2\rho} - a_2(\rho)\Big), \qquad b_{\overline{WH}} = \frac{1}{\gamma}\Big(\frac{\xi}{1-2\rho} - \frac{a_2(\rho)}{2}\Big).$$

Consequently, even if $\sqrt{k}\,A(n/k) \to \infty$, with $\sqrt{k}\,A^2(n/k) \to \lambda_A$, finite,

$$\sqrt{k}\ \big(UH_{\beta,\rho}(k) - \gamma\big) \xrightarrow[n\to\infty]{d} Normal\big(b_{UH}, \sigma^2_{UH} = \gamma^2\big).$$

Remark 2. If $\sqrt{k}\,A^2(n/k) \to \infty$, $\big(UH_{\beta,\rho}(k) - \gamma\big)$ is $O_p(A^2(n/k))$.

Remark 3. Note that $b_{_{ML}} = b_{_{\overline{ML}}} = 0$ whenever $\xi = \beta'/\beta = 1$. This happens for important models like the Burr and the *GP*, and it is a point in favour of the *ML*-statistic.

Remark 4. We also add that the results for $\overline{UH}$ follow straightforwardly from the results for UH. Indeed, as $n \to \infty$, $\overline{WH}_{\beta,\rho} - WH_{\beta,\rho} \overset{p}{\sim} a_2(\rho)A^2(n/k)/(2\gamma)$, $\overline{CH}_{\beta,\rho} - CH_{\beta,\rho} \overset{p}{\sim} A^2(n/k)/(2\gamma(1-\rho)^2)$ and $\overline{ML}_{\beta,\rho} - ML_{\beta,\rho} \overset{p}{\sim} A^2(n/k)/(2\gamma(1-2\rho))$.

Remark 5. Note that, as already mentioned in Caeiro *et al.* [4], since $\lambda_{_A} \geq 0$ and $1/(2a_2(\rho)) > (1-\rho)^2 > 1/a_2(\rho) > 1-2\rho$ for any $\rho < 0$, $b_{_{\overline{WH}}} \geq b_{_{CH}} \geq b_{_{WH}} \geq b_{_{ML}}$. All depends then on the sign of the bias.

3 Asymptotic comparison of the MVRB tail index estimators

We shall next proceed to the comparison of the MVRB estimators under study at their optimal levels. This is again done in a way similar to the one used in de Haan and Peng [12], Gomes and Martins [7], for the classical tail index estimators. Let us assume that $\widehat{\gamma}^{\bullet}_{n,k}$ denotes any arbitrary reduced-bias semi-parametric estimator of the extreme value index γ, for which we have, for any intermediate $k = k_n$,

$$\widehat{\gamma}^{\bullet}_{n,k} = \gamma + \frac{\sigma_\bullet}{\sqrt{k}} Z^{\bullet}_n + b_\bullet\, A^2(n/k) + o_p(A(n/k)), \tag{17}$$

with $Z^{\bullet}_n$ an asymptotically standard normal r.v. Then, $\sqrt{k}\left[\widehat{\gamma}^{\bullet}_{n,k} - \gamma\right] \longrightarrow^d N(\lambda b_\bullet, \sigma^2_\bullet)$ provided that k is such that $\sqrt{k}\, A^2(n/k) \to \lambda_{_A}$, finite, as $n \to \infty$. We then write $Bias_\infty\left[\widehat{\gamma}^{\bullet}_{n,k}\right] := b_\bullet\, A^2(n/k)$ and $Var_\infty\left[\widehat{\gamma}^{\bullet}_{n,k}\right] := \sigma^2_\bullet/k$. The so-called Asymptotic Mean Square Error ($AMSE$) is then given by

$$AMSE\left[\widehat{\gamma}^{\bullet}_{n,k}\right] := \frac{\sigma^2_\bullet}{k} + b^2_\bullet\, A^4(n/k).$$

Regular variation theory (Bingham *et al.* [1]), enables us to show that, whenever $b_\bullet \neq 0$, there exists a function $\varphi(n) = \varphi(n, \gamma, \rho)$, such that

$$\lim_{n\to\infty} \varphi(n)\, AMSE\left[\widehat{\gamma}^{\bullet}_{n0}\right] = \left(\sigma^2_\bullet\right)^{-\frac{4\rho}{1-4\rho}} \left(b^2_\bullet\right)^{\frac{1}{1-4\rho}} =: LMSE\left[\widehat{\gamma}^{\bullet}_{n0}\right],$$

where $\widehat{\gamma}^{\bullet}_{n0} := \widehat{\gamma}^{\bullet}_{n,k^{\bullet}_0(n)}$ and $k^{\bullet}_0(n) := \arg\inf_k AMSE\left[\widehat{\gamma}^{\bullet}_{n,k}\right]$. It is then sensible to consider the following:

Definition 1. Given two biased estimators $\widehat{\gamma}^{(1)}_{n,k}$ and $\widehat{\gamma}^{(2)}_{n,k}$, for which a distributional representation of the type of the one in (17) holds, with constants (σ_1, b_1) and (σ_2, b_2), $b_1, b_2 \neq 0$, respectively, both computed at their optimal levels, the *Asymptotic Root Efficiency* ($AREFF$) of $\widehat{\gamma}^{(1)}_{n0}$ relatively to $\widehat{\gamma}^{(2)}_{n0}$ is

$$AREFF_{1|2} \equiv AREFF_{\widehat{\gamma}_{n0}^{(1)}|\widehat{\gamma}_{n0}^{(2)}} := \sqrt{\frac{LMSE\left[\widehat{\gamma}_{n0}^{(2)}\right]}{LMSE\left[\widehat{\gamma}_{n0}^{(1)}\right]}} = \left(\left(\frac{\sigma_2}{\sigma_1}\right)^{-4\rho}\left|\frac{b_2}{b_1}\right|\right)^{\frac{1}{1-4\rho}}. \quad (18)$$

Remark 6. Note that this $AREFF$ indicator has been conceived so that the highest the $AREFF$ indicator is, the better is the first estimator.

We first present in Figures 1, 2 and 3, the measure $AREFF_{UH|\overline{UH}}$ for $UH = CH, ML, WH$, respectively, in the (ξ, ρ)-plane. Figure 4 show us the MVRB tail index estimator with minimum LMSE in the (ξ, ρ)-plane.

From these figures it is possible to see that there is practically no difference between the relative behaviour between UH and $\overline{UH}$ for $UH = CH, ML$ and CH. Figure 4 also shows us that, at optimal levels, none of these estimators outperform the others in the all the (ξ, ρ)-plane, but their simultaneous use will for sure enable us to better estimate γ, the primary parameter of extreme events.

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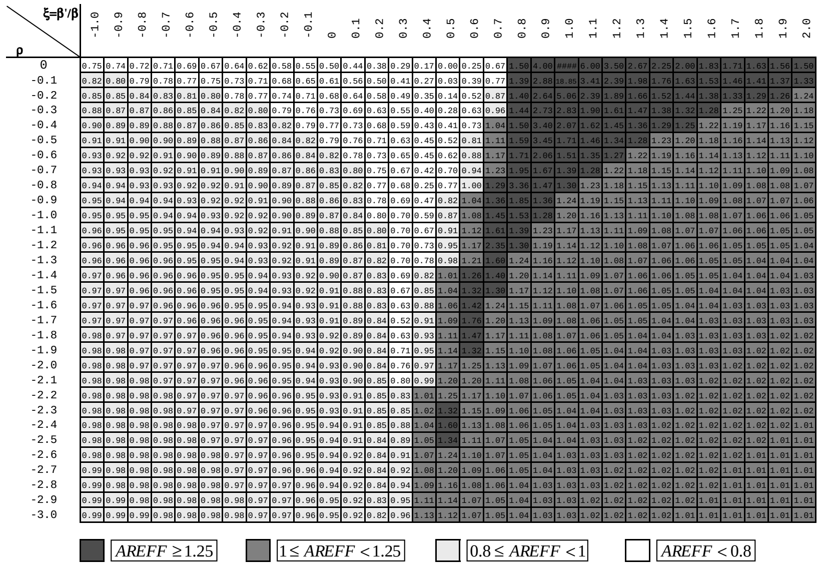

Fig. 1 $AREFF_{CH|\overline{CH}}$, in the (ξ,ρ)-plane.

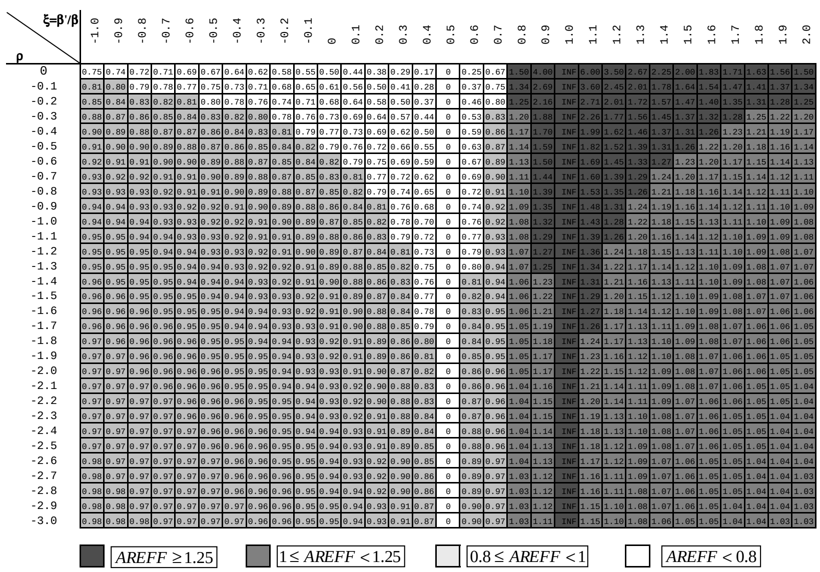

Fig. 2 $AREFF_{ML|\overline{ML}}$, in the (ξ,ρ)-plane.

ρ \ ξ=β′/β	-1.0	-0.9	-0.8	-0.7	-0.6	-0.5	-0.4	-0.3	-0.2	-0.1	0	0.1	0.2	0.3	0.4	0.5
0	####	####	####	####	####	####	####	####	####	####	####	####	####	####	####	####
-0.1	0.81	0.80	0.79	0.78	0.77	0.75	0.73	0.71	0.68	0.65	0.61	0.56	0.50	0.41	0.28	0.02
-0.2	0.85	0.85	0.84	0.83	0.81	0.80	0.78	0.76	0.74	0.71	0.68	0.64	0.58	0.49	0.36	0.09
-0.3	0.88	0.87	0.86	0.86	0.85	0.83	0.82	0.80	0.78	0.76	0.73	0.69	0.64	0.56	0.42	0.20
-0.4	0.90	0.89	0.88	0.88	0.87	0.86	0.85	0.83	0.81	0.79	0.77	0.73	0.68	0.61	0.47	0.31
-0.5	0.91	0.91	0.90	0.89	0.89	0.88	0.87	0.85	0.84	0.82	0.79	0.76	0.71	0.64	0.51	0.40
-0.6	0.92	0.92	0.91	0.91	0.90	0.89	0.88	0.87	0.86	0.84	0.82	0.79	0.74	0.67	0.54	0.49
-0.7	0.93	0.93	0.92	0.92	0.91	0.90	0.89	0.88	0.87	0.85	0.83	0.80	0.76	0.70	0.56	0.56
-0.8	0.94	0.93	0.93	0.92	0.92	0.91	0.90	0.89	0.88	0.87	0.85	0.82	0.78	0.72	0.57	0.62
-0.9	0.94	0.94	0.94	0.93	0.93	0.92	0.91	0.90	0.89	0.88	0.86	0.83	0.80	0.73	0.58	0.67
-1.0	0.95	0.95	0.94	0.94	0.93	0.93	0.92	0.91	0.90	0.89	0.87	0.85	0.81	0.75	0.58	0.72
-1.1	0.95	0.95	0.95	0.94	0.94	0.93	0.93	0.92	0.91	0.90	0.88	0.86	0.82	0.76	0.58	0.75
-1.2	0.96	0.95	0.95	0.95	0.94	0.94	0.93	0.92	0.91	0.90	0.89	0.87	0.83	0.77	0.57	0.79
-1.3	0.96	0.96	0.95	0.95	0.95	0.94	0.94	0.93	0.92	0.91	0.89	0.87	0.84	0.78	0.55	0.82
-1.4	0.96	0.96	0.96	0.95	0.95	0.95	0.94	0.93	0.93	0.91	0.90	0.88	0.85	0.79	0.48	0.84
-1.5	0.96	0.96	0.96	0.96	0.95	0.95	0.94	0.94	0.93	0.92	0.91	0.89	0.85	0.79	0.51	0.86
-1.6	0.97	0.96	0.96	0.96	0.96	0.95	0.95	0.94	0.93	0.92	0.91	0.89	0.86	0.80	0.61	0.88
-1.7	0.97	0.97	0.96	0.96	0.96	0.95	0.95	0.94	0.94	0.93	0.91	0.90	0.87	0.80	0.67	0.90
-1.8	0.97	0.97	0.97	0.96	0.96	0.96	0.95	0.95	0.94	0.93	0.92	0.90	0.87	0.81	0.71	0.91
-1.9	0.97	0.97	0.97	0.97	0.96	0.96	0.96	0.95	0.94	0.93	0.92	0.90	0.87	0.81	0.74	0.93
-2.0	0.97	0.97	0.97	0.97	0.96	0.96	0.96	0.95	0.95	0.94	0.93	0.91	0.88	0.81	0.77	0.94
-2.1	0.98	0.97	0.97	0.97	0.97	0.96	0.96	0.95	0.95	0.94	0.93	0.91	0.88	0.82	0.80	0.95
-2.2	0.98	0.97	0.97	0.97	0.97	0.97	0.96	0.96	0.95	0.94	0.93	0.91	0.89	0.82	0.82	0.96
-2.3	0.98	0.98	0.97	0.97	0.97	0.97	0.96	0.96	0.95	0.95	0.93	0.92	0.89	0.82	0.83	0.97
-2.4	0.98	0.98	0.98	0.97	0.97	0.97	0.96	0.96	0.95	0.95	0.94	0.92	0.89	0.82	0.85	0.98
-2.5	0.98	0.98	0.98	0.97	0.97	0.97	0.97	0.96	0.96	0.95	0.94	0.92	0.89	0.82	0.86	0.99
-2.6	0.98	0.98	0.98	0.98	0.97	0.97	0.97	0.96	0.96	0.95	0.94	0.93	0.90	0.82	0.88	0.99
-2.7	0.98	0.98	0.98	0.98	0.97	0.97	0.97	0.96	0.96	0.95	0.94	0.93	0.90	0.82	0.89	1.00
-2.8	0.98	0.98	0.98	0.98	0.97	0.97	0.97	0.96	0.96	0.95	0.94	0.93	0.90	0.81	0.90	1.01
-2.9	0.98	0.98	0.98	0.98	0.98	0.97	0.97	0.97	0.96	0.96	0.95	0.93	0.90	0.81	0.91	1.01
-3.0	0.98	0.98	0.98	0.98	0.98	0.97	0.97	0.97	0.96	0.96	0.95	0.93	0.91	0.81	0.92	1.02

ρ \ ξ=β′/β	0.6	0.7	0.8	0.9	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
0	####	####	####	####	####	####	####	####	####	####	####	####	####	####	####
-0.1	0.38	0.76	1.36	2.78	30.77	3.50	2.42	2.00	1.77	1.63	1.54	1.47	1.41	1.37	1.33
-0.2	0.49	0.84	1.32	2.37	7.34	2.53	1.94	1.69	1.55	1.45	1.39	1.34	1.30	1.27	1.25
-0.3	0.58	0.89	1.31	2.20	3.81	2.05	1.68	1.51	1.41	1.35	1.30	1.26	1.23	1.21	1.19
-0.4	0.66	0.94	1.31	2.14	2.63	1.76	1.52	1.40	1.33	1.28	1.24	1.21	1.19	1.17	1.16
-0.5	0.72	0.98	1.32	2.16	2.09	1.59	1.41	1.32	1.27	1.23	1.20	1.17	1.16	1.14	1.13
-0.6	0.77	1.01	1.34	2.29	1.79	1.47	1.34	1.27	1.22	1.19	1.17	1.15	1.13	1.12	1.11
-0.7	0.82	1.04	1.35	2.72	1.61	1.38	1.28	1.23	1.19	1.16	1.14	1.13	1.11	1.10	1.10
-0.8	0.86	1.07	1.37	2.80	1.48	1.32	1.24	1.19	1.16	1.14	1.12	1.11	1.10	1.09	1.08
-0.9	0.89	1.09	1.39	1.94	1.39	1.27	1.21	1.17	1.14	1.12	1.11	1.10	1.09	1.08	1.07
-1.0	0.91	1.10	1.42	1.66	1.33	1.23	1.18	1.15	1.13	1.11	1.10	1.09	1.08	1.07	1.07
-1.1	0.94	1.12	1.46	1.51	1.28	1.20	1.16	1.13	1.11	1.10	1.09	1.08	1.07	1.06	1.06
-1.2	0.96	1.14	1.52	1.41	1.24	1.18	1.14	1.12	1.10	1.09	1.08	1.07	1.06	1.06	1.05
-1.3	0.98	1.15	1.62	1.34	1.21	1.16	1.13	1.10	1.09	1.08	1.07	1.06	1.06	1.05	1.05
-1.4	0.99	1.17	1.87	1.29	1.19	1.14	1.11	1.09	1.08	1.07	1.06	1.06	1.05	1.05	1.05
-1.5	1.01	1.18	1.80	1.25	1.17	1.13	1.10	1.09	1.07	1.07	1.06	1.05	1.05	1.04	1.04
-1.6	1.02	1.20	1.51	1.22	1.15	1.11	1.09	1.08	1.07	1.06	1.05	1.05	1.04	1.04	1.04
-1.7	1.03	1.22	1.39	1.19	1.13	1.10	1.09	1.07	1.06	1.06	1.05	1.05	1.04	1.04	1.04
-1.8	1.04	1.23	1.32	1.17	1.12	1.09	1.08	1.07	1.06	1.05	1.05	1.04	1.04	1.04	1.03
-1.9	1.05	1.26	1.27	1.15	1.11	1.09	1.07	1.06	1.05	1.05	1.04	1.04	1.04	1.03	1.03
-2.0	1.06	1.29	1.23	1.14	1.10	1.08	1.07	1.06	1.05	1.04	1.04	1.04	1.03	1.03	1.03
-2.1	1.07	1.32	1.21	1.13	1.09	1.07	1.06	1.05	1.05	1.04	1.04	1.03	1.03	1.03	1.03
-2.2	1.08	1.39	1.18	1.12	1.09	1.07	1.06	1.05	1.04	1.04	1.04	1.03	1.03	1.03	1.03
-2.3	1.09	1.55	1.16	1.11	1.08	1.06	1.05	1.05	1.04	1.04	1.03	1.03	1.03	1.03	1.02
-2.4	1.10	1.48	1.15	1.10	1.07	1.06	1.05	1.04	1.04	1.03	1.03	1.03	1.03	1.02	1.02
-2.5	1.11	1.34	1.13	1.09	1.07	1.06	1.05	1.04	1.04	1.03	1.03	1.03	1.02	1.02	1.02
-2.6	1.11	1.27	1.12	1.08	1.06	1.05	1.04	1.04	1.03	1.03	1.03	1.03	1.02	1.02	1.02
-2.7	1.12	1.23	1.11	1.08	1.06	1.05	1.04	1.04	1.03	1.03	1.03	1.02	1.02	1.02	1.02
-2.8	1.13	1.20	1.10	1.07	1.06	1.05	1.04	1.04	1.03	1.03	1.03	1.02	1.02	1.02	1.02
-2.9	1.14	1.18	1.10	1.07	1.05	1.04	1.04	1.03	1.03	1.03	1.02	1.02	1.02	1.02	1.02
-3.0	1.15	1.16	1.09	1.06	1.05	1.04	1.04	1.03	1.03	1.02	1.02	1.02	1.02	1.02	1.02

$AREFF \geq 1.25$ | $1 \leq AREFF < 1.25$ | $0.8 \leq AREFF < 1$ | $AREFF < 0.8$

Fig. 3 $AREFF_{WH|\overline{WH}}$, in the (ξ, ρ)-plane.

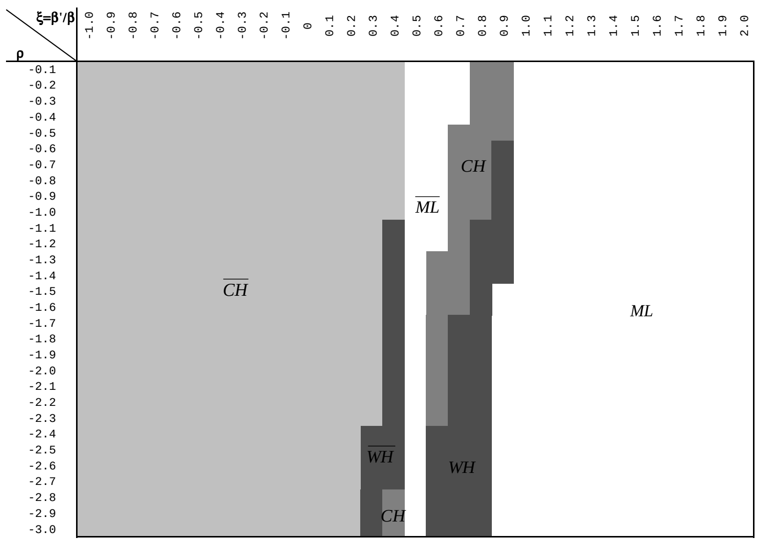

Fig. 4 Minimum LMSE among the CH, ML and WH statistics.

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