Rate of Convergence of some Asymptotic Expansions for Distribution Approximations via an Esseen Type Estimate

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This work is dedicated to Dinis D. F. Pestana,
as a token of recognition for a lifelong
enthusiasm for Statistics and for his
never failing support of young mathematicians.

Abstract

Some asymptotic expansions non necessarily related to the central limit theorem are studied. We first observe that the smoothing inequality of Esseen implies the proximity, in the Kolmogorov distance sense, of the distributions of the random variables of two random sequences satisfying a sort of general asymptotic relation. We then present several instances of this observation. A first example, partially motivated by the the statistical theory of high precision measurements, is given by a uniform asymptotic approximation to $(g(X + \mu_n))_{n\in\mathbb{N}}$, where g is some smooth function, X is a random variable and $(\mu_n)_{n\in\mathbb{N}}$ is a sequence going to infinity; a multivariate version is also stated and proved. We finally present a second class of examples given by a randomization of the interesting parameter in some classical asymptotic formulas; namely, a generic Laplace's type integral, randomized by the sequence $(\mu_n X)_{n\in\mathbb{N}}$, X being a Gamma distributed random variable.

1 Introduction

In this work we explore some asymptotic uniform approximations, for distribution functions, that do not require a central limit theorem as a starting point. The matter under investigation is well illustrated by the following example. Let X, Y be standard independent Gaussian variables, μ and ν parameters and $\alpha, \beta \geq 3$; the (exact) distribution of the random variable $(\mu + X)^\alpha \times (\nu + Y)^\beta$ is not easily described but, for instance, performing a simulation experiment with varying and growing $\mu = \nu$ parameter, we get

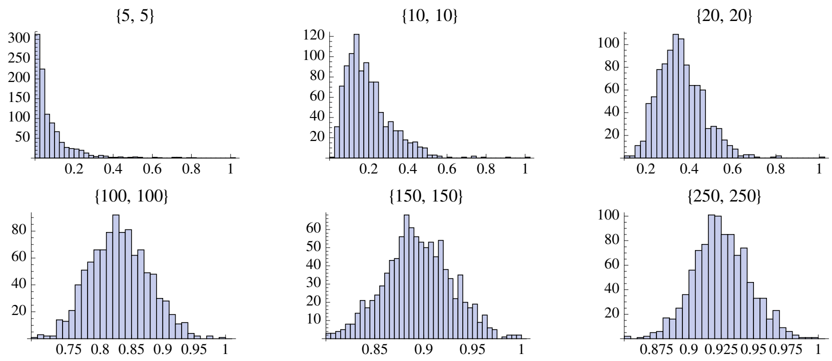

Figure 1.1: Simulated $(\mu + X)^{\alpha} \times (\nu + Y)^{\beta}$ with increasing $\mu = \nu$ ($\alpha = 3$, $\beta = 5$).

the results in the next figure. A natural intuition is that as $\mu = \nu$ increase the empirical distribution of $(\mu + X)^{\alpha} \times (\nu + Y)^{\beta}$ gets closer and closer to a Gaussian distribution. In section 3.3 we will show that, as μ and ν increase independently, the distribution of $(\mu + X)^{\alpha} \times (\nu + Y)^{\beta}$ is well approximated, in the Kolmogorov distance sense, by the distribution of $\mu^{\alpha}\nu^{\beta} + \left(\alpha\mu^{\alpha-1}\nu^{\beta}X + \beta\nu^{\beta-1}\mu^{\alpha}Y\right)$ which is Gaussian.

Asymptotic expansions related to limit theorems, both in the central and non-central case, are the main subject of asymptotic analysis for random variables and their distributions. Complementarily, asymptotic relations for random variables appear in the probability and statistics literature in a variety of aspects, mostly in asymptotic expansions for normalized sums of random variables like Gram-Charlier and Edgeworth expansions, the so-called *delta method*, the theory of quadratic differentiability in the mean of Le Cam (see [Le Cam 86]) and the stochastic differentiability of Hoffmann-Jørgensen ([Hoffmann-Jørgensen 92]), among others. The use of asymptotic expansions for the detailed study of a distribution, which otherwise could be impossible to perform, is an ancient and established subject in probability theory, at least, since Edgeworth ([Edgeworth 04]) and the first edition of ([Cramer 99]) in 1946. In the classic and masterful exposition of [Feller 71], the subject is treated as being mainly related to the central limit theorem. A more recent and encyclopedic presentation, is given in [Bhattacharya et al. 76]. Although somehow outdated, [Wallace 58] is a synthetic presentation of various aspects of asymptotic expansions depending on a parameter commonly linked to the sample size. An analytical expository presentation in the context of astronomy and astrophysics is given in [Blinnikov et al. 98] confirming the contemporary interest of the theme for practical applications.

We must refer next other aspects of the theme of asymptotic expansions which are, somehow, related to our approach. Asymptotic expansions for normalized sums of an

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i.i.d. sequence of random variables, assuming that this sum has a stable limit distribution, are discussed in [Inzhevitov 83], [Kristof 82]. Nonuniform bounds to the remainder term of asymptotic expansions for normalized sums of i.i.d. sequences of random variables with common distribution function F, assuming that F belongs to the domain of attraction of a stable distribution function, are presented in [Christoph 91]. In [Barbe et al. 05], the sum of independent heavy-tailed random variables satisfying an asymptotic smoothing condition is considered. The asymptotic expansion and the correspondent error bounds for a random variable that may be written as the product of two independent random variables, one of which may be a multivariate normal distribution among others with a smooth density function, is studied in [Fujikoshi et al. 89], [Fujikoshi et al 90], [Fujikoshi et al. 89b] and [Fujikoshi et al. 05]. An asymptotic formula for $(1/\lambda)\int_0^{\lambda} f(X_t)dt$ is given in [Khaniyev et al. 04], when λ goes to infinity, that quantity being a path mean functional for bounded measurable functions f and a semi-Markov random walk X. The important subject of asymptotic expansions for quadratic forms, for example, of the type $\sum a_{jk}X_jX_k$ with the random variables X_i forming an i.i.d. sequence and $\{a_{jk}\}$ a symmetric matrix, is studied in [Götze et al. 05] and complemented in a long series of papers there referenced.

The approximate normality discussed ahead as an application of section 3.1 was first observed in [Ramos 07] and was highlighted in [Areia et al. 08] by means of a simulation study. In [Mexia et al. 10], under the hypothesis of asymptotic linearity detailed in section 3.2, results similar to those in section 3 are obtained; the convergence is in probability, instead of convergence in the Kolmogorov distance sense obtained here, and there is no usable information on the rate of convergence. By observing that uniform integrability of the relevant family of functions follows from the asymptotic linearity hypothesis, the extension of these results to moment convergence is proved in [Fonseca et al. 10]. In [Esquivel et al. 09] there is a partial announcement of some of the results here presented and also a simulation study which clarifies the statistical quality of the asymptotic approximation given in one of the examples of section 4.

2 Asymptotics for random variables

Following an idea of [Pestana 07] apropos the study in [Ramos 07] and [Ramos et al. 07], we start by considering a generic question in the asymptotic theory for random variables. Given that $X_n = Y_n + Z_n$ what are the appropriate conditions to impose on the relation $Z_n \ll Y_n$ in order to get

$$X_n \overset{\mathrm{d}}{\approx} Y_n \text{ as } n \gg 1 \, .$$

That is, under what conditions may we approximate the distribution of X_n, which may be hard to compute, by the distribution of Y_n (in a sense to be determined, preferably, in the Kolmogorov distance sense), for large n? We observe that writing for $n \gg 1$ that $X_n = Y_n + Z_n$ with $Z_n \ll Y_n$ is a sort of asymptotic equality between the sequences of random variables $(X_n)_{n\geq 1}$ and $(Y_n)_{n\geq 1}$. We refer to [De Bruijn 81] or [Zorich 04 II, p. 595–641] for an exposition of traditional asymptotic analysis and section 4 for some examples of asymptotic relations for random variables. It is to be noticed that, to the best of our knowledge, a comprehensive theory of asymptotic relations for random variables is not yet available although some authors tackled this topic in particular points. See, for instance, [Hoffmann-Jørgensen 94, p. 443] with the definition of a stochastic remainder term.

Remark 1. The answer for the question above is not, in general, an usual convergence result issue (e.g. a central limit theorem) due to the fact that, possibly, the limiting values X_∞, Y_∞ and Z_∞ are not defined.

Always with $X_n = Y_n + Z_n$, one possible sense to be given to the expression $Z_n \ll Y_n$ is as follows.

$$\forall \epsilon > 0 \lim_{n \to +\infty} \mathbb{P}[|Z_n| > \epsilon \, |Y_n|] = 0 \,. \tag{2.1}$$

For context's sake we quote next a general result giving a first answer to the question above, under the conditions given by formula (2.1), for non negative random variables. It is a nice easy result but with limited practical usefulness as, in general, we don't control the rate convergence of the error term $\delta_n(\epsilon)$.

Theorem 2.1 ([Ramos 07] p. 12)**.** *Suppose that $X_n, Y_n, Z_n \geq 0$ are as above, that is, satisfying (2.1). For all $0 < \epsilon < 1$ There exists a sequence $(\delta_n(\epsilon))_{n\in\mathbb{N}}$ such that $\lim_{n\to+\infty} \delta_n(\epsilon) = 0$ and:*

$$F_{Y_n}\left(\frac{x}{1+\epsilon}\right) - \delta_n(\epsilon) \leq F_{X_n}(x) \leq F_{Y_n}\left(\frac{x}{1-\epsilon}\right) + \delta_n(\epsilon)$$

Proof. See [Ramos et al. 07] or observe that with $\delta_n(\epsilon) := \mathbb{P}[|X_n - Y_n| > \epsilon \, |Y_n|]$:

$$F_{Y_n}\left(\frac{x}{1+\epsilon}\right) \leq \mathbb{P}\left[Y_n \leq \frac{x}{1+\epsilon}, X_n \leq (1+\epsilon)Y_n\right] + \delta_n(\epsilon) \leq F_{X_n} + \delta_n(\epsilon)$$

and that

$$F_{X_n} \leq \mathbb{P}\left[X_n \leq x, Y_n \leq \frac{1}{1-\epsilon} X_n\right] + \delta_n(\epsilon) \leq F_{Y_n}\left(\frac{x}{1-\epsilon}\right) + \delta_n(\epsilon) \,.$$

□

Remark 2. With $A_n(\epsilon) := \bigcap_{m=n}^{+\infty} \{|X_m - Y_m| < \epsilon \, |Y_m|\}$ the result above may also be proved with $\delta_n(\epsilon) := 1 - \mathbb{P}[A_n(\epsilon)]$. In any case, the practical usefulness of the result is limited to the cases where a detailed estimate of the asymptotic behavior of $X_n - Y_n$ is available.

The next results are obvious consequences of theorem 2.1 that allow a complete answer to the generic question above, whenever the sequence $(Y_n)_{n\in\mathbb{N}}$ converges in distribution to some random variable.

Theorem 2.2. *If $X_n, Y_n, Z_n \geq 0$ are as above, that is, satisfying* (2.1)*. then for all $0 < \varepsilon < 1$:*

$$\underline{\lim}_{n\to+\infty} F_{|Y_n|}\left(\frac{x}{1+\varepsilon}\right) \leq \underline{\lim}_{n\to+\infty} F_{|X_n|}(x) \leq \overline{\lim}_{n\to+\infty} F_{|X_n|}(x) \leq \overline{\lim}_{n\to+\infty} F_{|Y_n|}\left(\frac{x}{1-\varepsilon}\right) .$$

Proof. Just apply theorem 2.1 to the sequences $(|X_n|)_{n\in\mathbb{N}}$, and $(|Y_n|)_{n\in\mathbb{N}}$. □

The next result is a well known Slutsky's type result which follows immediately from theorem 2.2 .

Corolary 1. *Let $(X_n)_{n\in\mathbb{N}}$, $(Y_n)_{n\in\mathbb{N}}$, $(Z_n)_{n\in\mathbb{N}}$ be sequences of random variables such that $X_n = Y_n + Z_n$ and for all $\epsilon > 0$ we have that $\lim_{n\to+\infty} \mathbb{P}[|Z_n| > \epsilon |Y_n|] = 0$. If $Y_n \xrightarrow{\mathrm{d}} Y$ then $X_n \xrightarrow{\mathrm{d}} Y$.*

From here on we do not suppose that condition (2.1) is satisfied. The next important technical lemma will give us a way of proving the asymptotic approximation of the relevant distributions, in the Kolmogorov distance sense, as soon as we can control the asymptotic behavior of the product of a central moment of the remainder term by the maximum of the density of the first term of the asymptotic formula.

Lemma 1 (Esseen's type estimate)**.** *Let $(X_n)_{n\in\mathbb{N}}$, $(Y_n)_{n\in\mathbb{N}}$ and $(Z_n)_{n\in\mathbb{N}}$ be sequences of random variables such that for each $n \geq 1$ we have that $X_n = Y_n + Z_n$, Y_n admits a density F'_{Y_n} and that for some $\delta \in]0,1]$ we have $\mathbb{E}[|Z_n|^\delta] < +\infty$. Then:*

$$\sup_x |F_{X_n}(x) - F_{Y_n}(x)| \leq C_\delta \cdot \left(\mathbb{E}\left[|Z_n|^\delta\right]^{\frac{1}{\delta}} \cdot \sup_x F'_{Y_n}(x)\right)^{\frac{\delta}{1+\delta}} . \tag{2.2}$$

with $C_\delta = (1/\pi)2^{\frac{2-\delta}{1+\delta}} 24^{\frac{\delta}{1+\delta}}(1 + 1/\delta)$.

Proof. With $X_n = Y_n + Z_n$, we apply the estimate $\left|e^{ix} - 1\right| = 2\left|\sin(x/2)\right| \leq 2^{1-\delta} |x|^\delta$ valid for all $0 \leq \delta \leq 1$ to write:

$$|\phi_{X_n}(t) - \phi_{Y_n}(t)| = \left|\mathbb{E}\left[e^{itY_n}\left(e^{itZ_n} - 1\right)\right]\right| \leq 2^{1-\delta} |t|^\delta \, \mathbb{E}\left[|Z_n|^\delta\right] ,$$

so that for $0 \leq \delta \leq 1$:

$$\int_0^T \left|\frac{\phi_{X_n}(t) - \phi_{Y_n}(t)}{t}\right| dt \leq \frac{2^{1-\delta}T^\delta}{\delta} \mathbb{E}[|Z_n|^\delta] .$$

Now, by Esseen's smoothing inequality (see [Esseen 45] or as quoted in [Feller 71, p. 538] or [Shiryaev 96, p. 296]) for all $T > 0$:

$$\sup_x |F_{X_n}(x) - F_{Y_n}(x)| \leq \frac{2}{\pi}\int_0^T \left|\frac{\phi_{X_n}(t) - \phi_{Y_n}(t)}{t}\right| dt + \frac{24}{\pi T} \sup_x \left|F'_{Y_n}(x)\right| ,$$

giving an estimation which depends mainly on $\mathbb{E}\left[|Z_n|^\delta\right]$ and on $\sup_x \left|F'_{Y_n}(x)\right|$, that is

$$\sup_x |F_{X_n}(x) - F_{Y_n}(x)| \leq \frac{2^{2-\delta}T^\delta}{\delta\pi}\mathbb{E}[|Z_n|^\delta] + \frac{24}{\pi T}\sup_x \left|F'_{Y_n}(x)\right| \ . \tag{2.3}$$

Let us define $m_n = \mathbb{E}\left[|Z_n|^\delta\right]$ and $M_n = \sup_x F'_{Y_n}(x)$. Then the right hand side of (2.3) takes the form $A_n T^\delta + B_n T^{-1}$ with $A_n = \frac{2^{2-\delta}}{\delta\pi}m_n$ and $B_n = (24/\pi)M_n$. This expression attains its minimum for $T = \left(\frac{B_n}{\delta A_n}\right)^{\frac{1}{\delta+1}}$. Choosing this minimizer we will have formula (2.2) as wanted. □

As a consequence, we have the following immediate corollary, a first general result which we next show to have several particular relevant applications to specific situations.

Theorem 2.3. *Let $(X_n)_{n\in\mathbb{N}}$, $(Y_n)_{n\in\mathbb{N}}$ and $(Z_n)_{n\in\mathbb{N}}$ be sequences of random variables such that for each $n \geq 1$ we have that $X_n = Y_n + Z_n$, Y_n admits a density F'_{Y_n} and that for some $\delta \in]0,1]$ we have $\mathbb{E}[|Z_n|^\delta] < +\infty$. Then, if*

$$\lim_{n\to+\infty} \mathbb{E}\left[|Z_n|^\delta\right]^{\frac{1}{\delta}} \cdot \sup_x \left|F'_{Y_n}(x)\right| = 0 \ ,$$

we have that:

$$\lim_{n\to+\infty} \sup_x |F_{X_n}(x) - F_{Y_n}(x)| = 0 \ , \tag{2.4}$$

that is, we have the uniform approximation, for large values of n, of the distribution function of X_n by the distribution function of Y_n.

3 Linear transform approximation results

In this section we study some approximation results, instances of theorem 2.3, considering cases where the first two terms in the asymptotic expansion define an affine transformation of the initial random variable. The driving tool, both in the univariate and multivariate cases, is to consider asymptotic Taylor type expansions. A first idea to deal with the problem studied in this question would be to apply some form of the delta method (see [Oehlert 92]). Nevertheless, as already pointed out in remark 1 the delta method relies on the central limit theorem which, in general, is not applicable to the situation under scrutiny.

Our interest in a statistical perspective of high precision measurements, in section 3.3 drove the discussion of the examples in section 3.1.

3.1 The univariate case

We recall briefly the notion of regularly varying function which will be used in the following (see [Galambos et al. 73] or [Bingham et al. 78, p. 12]). A positive measurable function R is regularly varying (at $+\infty$) if and only if for some $B > 0$ and some finite ρ it admits the representation

$$\forall x \geq B \;\; R(x) = x^{\rho} \exp\left(\eta(x) + \int_B^x \frac{\epsilon(t)}{t} dt\right) ,$$

with η and ϵ bounded measurable functions such that $\lim_{x\to+\infty} \eta(x) = c \in \mathbb{R}$ and $\lim_{t\to+\infty} \epsilon(t) = 0$. It is natural to say that R is regularly varying at $-\infty$ if $R(-x)$ is is regularly varying at $+\infty$. This will be the sense of the hypothesis made in the following theorem.

Theorem 3.1. *Let $g\colon \mathbb{R} \longrightarrow \mathbb{R}$ be a $C^2(\mathbb{R})$ function and X a real valued random variable such that:*

1. *With $g'(x) \neq 0$ for x large enough, we have that $\lim_{n\to+\infty} g''(x)/g'(x) = 0$;*
2. *$|g''|$ is a regularly varying function at $+\infty$ and at $-\infty$ admitting an integral representation given by:*

$$\forall\, |x| \geq B \quad \left|g''(x)\right| = |x|^{\rho} \exp\left(\eta(|x|) + \int_B^{|x|} \frac{\epsilon(t)}{t} dt\right) \tag{3.1}$$

   with $B > 0$, $\rho > 1$ some constants, η and ϵ measurable and bounded functions such that $\lim_{t\to+\infty} \epsilon(t) = 0$;

3. *X has a density bounded by a constant $D_1 > 0$;*
4. *X has an absolute moment of order 2ρ, that is, $\mathbb{E}[|X|^{2\rho}] < +\infty$.*

Then, for any non-random real sequence $(\mu_n)_{n\in\mathbb{N}}$ verifying $\lim_{n\to+\infty} \mu_n = +\infty$ we have that for some $n \in \mathbb{N}$ large enough on, and for some constant D_2 bounding:

$$\mathbb{E}\left[\left|\int_0^1 (1-t) \frac{g''(\mu_n + tX)}{|g''(\mu_n)|} dt\right|\right] ,$$

that

$$\sup_{x\in\mathbb{R}} \left|F_{g(X+\mu_n)}(x) - F_{g(\mu_n)+g'(\mu_n)X}(x)\right| \leq C_{\frac{\rho}{\rho+1}} \left(D_1 D_2\right)^{\frac{\rho}{2\rho+1}} \mathbb{E}[|X|^{2\rho}]^{\frac{1}{2\rho+1}} \left|\frac{g''(\mu_n)}{g'(\mu_n)}\right|^{\frac{\rho}{2\rho+1}} ,$$

thus showing that for large n, the law of $g(X + \mu_n)$ may be approximated, in the Kolmogorov distance sense, by the law of $g(\mu_n) + g'(\mu_n)X$ with a rate of convergence given by $|g''(\mu_n)/g'(\mu_n)|^{\rho/2\rho+1}$.

Proof. The theorem being a consequence of lemma 1, we expose the proof in three steps.

- First step: The density of $g(\mu_n) + g'(\mu_n)X$ is bounded by $D_1/|g'(\mu_n)|$.

We have that

$$F_{g(\mu_n)+g'(\mu_n)X}(x) = \begin{cases} F_X\left(\dfrac{x-g(\mu_n)}{g'(\mu_n)}\right) & \text{if} \quad g'(\mu_n) > 0 \\ \\ 1 - F_X\left(\dfrac{x-g(\mu_n)}{g'(\mu_n)}\right) & \text{if} \quad g'(\mu_n) < 0 \end{cases}$$

thus ensuring that

$$\sup_{x\in\mathbb{R}}\left(F'_{g(\mu_n)+g'(\mu_n)X}(x)\right) = \frac{1}{|g'(\mu_n)|}\cdot\sup_{x\in\mathbb{R}} F'_X\left(\frac{x-g(\mu_n)}{g'(\mu_n)}\right) \leq \frac{D_1}{|g'(\mu_n)|}$$

with D_1 independent of n (note that, necessarily, $\lim_{N\to+\infty}|g'(\mu_n)| = +\infty$).

- Second step: Defining Z_n as the integral remainder term of the Taylor expansion of $g(X+\mu_n)$ we show that with $\delta = \rho/(\rho+1) < 1$ we have:

$$\mathbb{E}\left[|Z_n|^\delta\right]^{\frac{1}{\delta}} \leq \mathbb{E}\left[|X|^{\frac{2\delta}{1-\delta}}\right]^{\frac{1-\delta}{\delta}}\mathbb{E}\left[\left|\int_0^1(1-t)g''(\mu_n+tX)dt\right|\right]. \tag{3.2}$$

The Taylor expansion of g with integral remainder allows us to write:

$$g(\mu_n+X) = g(\mu_n) + g'(\mu_n)X + X^2\int_0^1(1-t)g''(\mu_n+tX)\,dt.$$

For the record, with the notations of previous sections we have $X_n := g(\mu_n+X)$, $Y_n := g(\mu_n) + g'(\mu_n)X$ and

$$Z_n := X^2\int_0^1(1-t)g''(\mu_n+tX)\,dt$$

which is a random variable with a not easily described distribution. Considering $\delta = \rho/(\rho+1)$ observing that $2\delta/(1-\delta) = 2\rho$ and, with $\frac{1}{\delta^{-1}} + \frac{1}{(1-\delta)^{-1}} = 1$, applying Hölder inequality in order to estimate $\mathbb{E}[|Z_n|^\delta]$ we get

$$\begin{aligned}\mathbb{E}[|Z_n|^\delta] = \mathbb{E}\left[X^{2\delta}\cdot\left(\int_0^1(1-t)g''(\mu_n+tX)\,dt\right)^\delta\right] \leq \\ \leq \mathbb{E}\left[|X|^{\frac{2\delta}{1-\delta}}\right]^{1-\delta}\cdot\mathbb{E}\left[\left|\int_0^1(1-t)g''(\mu_n+tX)dt\right|\right]^\delta.\end{aligned}$$

thus obtaining formula (3.2). Now aiming at applying lemma (1) we have as a consequence of formula (3.2) that

$$\mathbb{E}\left[|Z_n|^\delta\right]^{\frac{1}{\delta}}\cdot\sup_{x\in\mathbb{R}}\left|F'_{Y_n}(x)\right| \leq \mathbb{E}\left[|X|^{2\rho}\right]^{\frac{1}{\rho}} D_1\left|\frac{g''(\mu_n)}{g'(\mu_n)}\right|\mathbb{E}\left[\left|\int_0^1(1-t)\frac{g''(\mu_n+tX)}{|g''(\mu_n)|}dt\right|\right].$$

- Third step: The announced result will be proved if we show that

$$\mathbb{E}\left[\left|\int_0^1 (1-t)\frac{g''(\mu_n+tX)}{|g''(\mu_n)|}dt\right|\right]$$

is bounded by a constant D_2. As $\lim_{n\to+\infty} |g''(\mu_n+tX)/g''(\mu_n)| = 1$, by the continuity of g'', this is a consequence of Lebesgue dominated convergence theorem if we show that the integrand is bounded uniformly in n by an integrable function with respect to $\mathbb{P}\otimes\lambda$.

In the representation given in formula (3.1), let us consider bounds $M_\eta > 0$ and $M_\epsilon > 0$ for the functions η and ϵ, respectively. Also, as $\lim_{t\to+\infty}\epsilon(t) = 0$, let $x_\rho > 0$ be such that for $|x| \geq x_\rho$ we have $|\epsilon(x)| \leq \rho - 1$. Let us first observe that:

$$\mu_n \geq \max(B, x_\rho) \Rightarrow \left|g''(\mu_n)\right| \geq e^{-M_\eta}B^\rho \ . \tag{3.3}$$

This results from having for $|x| \geq \max(B, x_\rho)$:

$$\eta(|x|) + \int_B^{|x|}\frac{\epsilon(t)+\rho}{t}dt \geq -M_\eta + \log(\frac{|x|}{B}) \Rightarrow \eta(|x|) + \int_B^{|x|}\frac{\epsilon(t)}{t}dt \geq -M_\eta + \log(\frac{|x|}{B})^{(1-\rho)} \ .$$

As a second observation we also have that for $|x| \geq B$:

$$\exp\left(\eta(|x|) + \int_B^{|x|}\frac{\epsilon(t)}{t}dt\right) \leq e^{M_\eta}\left(\frac{|x|}{B}\right)^{M_\epsilon} \ , \tag{3.4}$$

this resulting from having:

$$\eta(|x|) + \int_B^{|x|}\frac{\epsilon(t)}{t}dt \leq \left|\eta(|x|) + \int_B^{|x|}\frac{\epsilon(t)}{t}dt\right| \leq |\eta(|x|)| + \int_B^{|x|}\frac{|\epsilon(t)|}{t}dt \leq M_\eta + \log(\frac{|x|}{B})^{M_\epsilon} \ .$$

Now, for $|\mu_n + tX| \leq \max(B, x_\rho)$ we have, by the continuity of g'',

$$\left|\frac{g''(\mu_n+tX)}{g''(\mu_n)}\right| \leq \sup_{|x|\leq\max(B,x_\rho)}\left|g''(x)\right|\cdot\frac{1}{e^{-M_\eta}B^\rho} < +\infty \ .$$

Also for the complementary case, that is, for $|\mu_n + tX| > \max(B, x_\rho)$ and $\mu_n \geq \max(B, x_\rho)$:

$$\begin{aligned}\left|\frac{g''(\mu_n+tX)}{g''(\mu_n)}\right| &= \frac{|\mu_n+tX|^\rho\exp\left(\eta(|\mu_n+tX|) + \int_B^{|\mu_n+tX|}\frac{\epsilon(t)}{t}dt\right)}{|\mu_n|^\rho\exp\left(\eta(\mu_n) + \int_B^{\mu_n}\frac{\epsilon(t)}{t}dt\right)} \\ &\leq \left|1+\frac{tX}{\mu_n}\right|^\rho e^{2M_\eta}\exp\left(\int_B^{|\mu_n+tX|}\frac{\epsilon(t)}{t}dt - \int_B^{\mu_n}\frac{\epsilon(t)}{t}dt\right)\end{aligned} \tag{3.5}$$

We will now split in two sub-cases the estimation of the exponential right-hand term of formula (3.5) of this second case. Let us suppose first that $|\mu_n + tX| \geq \mu_n \geq \max(B, x_\rho)$. Then as

$$\int_B^{|\mu_n+tX|} \frac{\epsilon(t)}{t} dt - \int_B^{\mu_n} \frac{\epsilon(t)}{t} dt = \int_{\mu_n}^{|\mu_n+tX|} \frac{\epsilon(t)}{t} dt \leq \left| \int_{\mu_n}^{|\mu_n+tX|} \frac{\epsilon(t)}{t} dt \right| \leq$$
$$\leq \int_{\mu_n}^{|\mu_n+tX|} \frac{|\epsilon(t)|}{t} dt \leq \log \left(\left| 1 + \frac{tX}{\mu_n} \right|^\rho \right)$$

we have that

$$\exp \left(\int_B^{|\mu_n+tX|} \frac{\epsilon(t)}{t} dt - \int_B^{\mu_n} \frac{\epsilon(t)}{t} dt \right) \leq \left| 1 + \frac{tX}{\mu_n} \right|^\rho . \tag{3.6}$$

In the complementary sub-case, that is, whenever $\max(B, x_\rho) < |\mu_n + tX| < \mu_n$, as:

$$\int_B^{|\mu_n+tX|} \frac{\epsilon(t)}{t} dt - \int_B^{\mu_n} \frac{\epsilon(t)}{t} dt = - \int_{|\mu_n+tX|}^{\mu_n} \frac{\epsilon(t)}{t} dt \leq \left| \int_{|\mu_n+tX|}^{\mu_n} \frac{-\epsilon(t)}{t} dt \right| \leq$$
$$\leq \int_{|\mu_n+tX|}^{\mu_n} \frac{|\epsilon(t)|}{t} dt \leq \log \left(\left| 1 + \frac{tX}{\mu_n} \right|^{-\rho} \right)$$

also, we have that:

$$\exp \left(\int_B^{|\mu_n+tX|} \frac{\epsilon(t)}{t} dt - \int_B^{\mu_n} \frac{\epsilon(t)}{t} dt \right) \leq \left| 1 + \frac{tX}{\mu_n} \right|^{-\rho} . \tag{3.7}$$

Finally, gathering the results in formulas (3.6) and (3.7) we have that:

$$\left| \frac{g''(\mu_n + tX)}{g''(\mu_n)} \right| \leq \left| 1 + \frac{tX}{\mu_n} \right|^\rho e^{2M_\eta} \cdot \max \left(\left| 1 + \frac{tX}{\mu_n} \right|^\rho , \left| 1 + \frac{tX}{\mu_n} \right|^{-\rho} \right) =$$
$$= e^{2M_\eta} \max \left(\left| 1 + \frac{tX}{\mu_n} \right|^{2\rho} , 1 \right) \leq e^{2M_\eta} 2^{2\rho} \left(1 + \frac{|X|^{2\rho}}{\max(B, x_\rho)^{2\rho}} \right) , \tag{3.8}$$

which allow us to conclude as $\mathbb{E}[|X|^{2\rho}] < +\infty$. □

Remark. A characterization of the largest class of functions g for which an approximation similar to the one obtained in theorem 3.1 is an interesting problem. Some restriction on the growth of g is necessary; for instance, for $r > 0$, the function $g(x) = \exp(x^r)$ will satisfy the hypothesis 1 of the theorem if and only if $r < 1$. In this case, using the fact that for $0 < s < 1$:

$$\forall x \geq 0 \quad (1 + x)^s - 1 \leq sx \sup_{y \in [0,x]} \frac{1}{(1 + y)^{1-s}} = sx ,$$

it is easy to see that a theorem similar to theorem 3.1 holds for a non negative random variable X under the condition that for some $c > 0$ we have $\mathbb{E}[\exp(cX)] < +\infty$, that is, for X having a exponential decaying tail. Also, if a function g with $g' > 0$ and $g'' \geq 0$ satisfies the hypothesis 1 of the theorem then, as a consequence of Gronwall lemma, it will be bounded at infinity by an exponential. In fact, if for any given $\epsilon > 0$ and t_ϵ such that for $t \geq t_\epsilon$ we have that $g''(t) \leq \epsilon g'(t)$ and so for $t \geq t_\epsilon$:

$$g''(t) \leq \epsilon g'(t_\epsilon) + \epsilon(g'(t) - g'(t_\epsilon)) = \epsilon g'(t_\epsilon) + \epsilon \int_{t_\epsilon}^{t_\epsilon} g''(s)ds ,$$

by Gronwall lemma it follows that for $t \geq t_\epsilon$, we will have $g''(t) \leq \epsilon g'(t_\epsilon)e^{\epsilon t}$.

3.2 The multidimensional case

We will use next the following notations. For an integer r, $\mathbf{x} = (x_1, \dots x_r)$ or $\mathbf{x_0} = (x_{0,1}, \dots x_{0,r})$ will denote generic points of $\mathbb{R}^r$. By $\mathbf{X} = (X_1, \dots X_r)$ we will denote a vector of random variables, $F_{\mathbf{X}}(\mathbf{x})$ being its distribution function, $f_{\mathbf{X}}(\mathbf{x})$ the corresponding density and $f_{X_i}(x_i)$ for $i \in \{1, \dots r\}$ a marginal density. Let $\boldsymbol{\mu}_n = (\mu_{n,1}, \dots \mu_{n,r})$ be a vector such that for every $i \in \{1, \dots r\}$, $\lim_{n \to +\infty} \mu_{n,i} = +\infty$. Let $\mathbf{g} : \mathbb{R}^r \mapsto \mathbb{R}$ be a $C^2(\mathbb{R}^r)$ map. Recall that $D\mathbf{g}(\mathbf{x_0})$ the differential of $\mathbf{g}$ at a generic point $\mathbf{x_0}$ is a linear form over $\mathbb{R}^p$ which whenever applied to a point $\mathbf{x} - \mathbf{x_0} \in \mathbb{R}^r$ admits the following representation:

$$D\mathbf{g}(\mathbf{x_0})(\mathbf{x} - \mathbf{x_0}) = \sum_{i=1}^{r} \frac{\partial \mathbf{g}}{\partial x_i}(\mathbf{x_0})(x_i - x_{0,i}) .$$

Similarly, the second differential $D^2\mathbf{g}(\mathbf{x_0})$ of $\mathbf{g}$ at a generic point $\mathbf{x_0}$ is a bilinear form over $\mathbb{R}^p \times \mathbb{R}^p$ which whenever applied to a point $(\mathbf{x} - \mathbf{x_0})^2 \in \mathbb{R}^r \times \mathbb{R}^r$ admits the following representation:

$$D^2\mathbf{g}(\mathbf{x_0})(\mathbf{x} - \mathbf{x_0})^{(2)} = \sum_{i=1}^{r} \sum_{j=1}^{r} \frac{\partial^2 \mathbf{g}}{\partial x_i \partial x_j}(\mathbf{x_0})(x_i - x_{0,i})(x_j - x_{0,j})$$

We will also use the notations $D\mathbf{g}(\mathbf{x})$ for the gradient of $\mathbf{g}$ taken at a point $\mathbf{x}$, that is, for the vector $(\frac{\partial \mathbf{g}}{\partial x_1}(\mathbf{x}), \dots, \frac{\partial \mathbf{g}}{\partial x_r}(\mathbf{x}))$, and $D^2\mathbf{g}(\mathbf{x})$ for the Hessian matrix of $\mathbf{g}$ a point $\mathbf{x}$, that is, for the matrix $(\partial^2 \mathbf{g}/\partial x_i \partial x_j(\mathbf{x}))_{i,j,\in\{1,\dots,r\}}$ and also $\|D\mathbf{g}(\mathbf{x})\|$ and $|||D^2\mathbf{g}(\mathbf{x})|||$, respectively, for the corresponding norms.

In the following we will need some results on random vectors in Euclidean spaces and their densities, via Hausdorff measures. A very complete reference we will follow is [Hoffmann-Jørgensen 94, chapter 8]. The classical reference for Hausdorff measure and some of its applications is [Federer 69]. Friendly and very readable texts on the subject are [Evans et al. 92] and [Mattila 95]. For reference purposes we state very briefly some definitions and important results used next. For $A \subseteq \mathbb{R}^r$ let $|A|$ denote the diameter of A, that is, $|A| = \sup\{|x - y| : x, y \in A\}$. For $\delta > 0$ and $s \in]0, r]$ let

$$\lambda_{r,s}^{\delta}(A) := \inf_{A \subseteq \bigcup_{i=1}^{+\infty} A_i \,,\, |A_i| \leq \delta} \sum_{i=1}^{+\infty} \frac{\pi^{\frac{s}{2}}}{2^s \Gamma(\frac{s}{2} + 1)} \times |A_i|^s \quad ,$$

and $\lambda_{r,s}(A) := \lim_{\delta \to 0} \lambda_{r,s}^{\delta}(A) = \sup_{\delta > 0} \lambda_{r,s}^{\delta}(A)$. Then $\lambda_{r,s}$ is the s-dimensional Hausdorff outer measure in $\mathbb{R}^r$. A first important result is that the $\lambda_{r,s}$ measurable sets, in the sense of Carathéodory, form a σ-algebra containing the Borel sets of $\mathbb{R}^r$. Other remarkable properties of Hausdorff measuresare the following: $\lambda_{r,s}$ is a regular Borel measure and for every affine isometry $M : \mathbb{R}^r \mapsto \mathbb{R}^r$ and every $A \subseteq \mathbb{R}^r$ we have $\lambda_{r,s}(M(A)) = \lambda_{r,s}(A)$. Furthermore, as a consequence of the isodiametric inequality we have that $\lambda_{r,r}$ coincides with the Lebesgue measure over $\mathbb{R}^r$. Finally we have the following consequence of the change of variables theorem (see [Hoffmann-Jørgensen 94, p. 11, formula 8.8.6]). For $M : \mathbb{R}^{r-1} \mapsto \mathbb{R}^r$ smooth Borel function, $\mathbf{h} : \mathbb{R}^{r-1} \mapsto \mathbb{R}$ Borel function and $B \subset \mathbb{R}^{r-1}$ such that M is injective on B and with $J_M(\mathbf{y})$ being the Jacobiant of M evaluated at $\mathbf{y}$:

$$\int_{M(B)} \mathbf{h}(\mathbf{x}) \, d\lambda_{r,r-1}(\mathbf{x}) = \int_B \mathbf{h}(M(\mathbf{y})) \, J_M(\mathbf{y}) \, d\mathbf{y} \tag{3.9}$$

The following result is the multivariate analog of theorem 3.1.

Theorem 3.2. *Let $\mathbf{g} : \mathbb{R}^r \mapsto \mathbb{R}$ be a $C^2(\mathbb{R}^r)$ function and $\mathbf{X} = (X_1, \ldots, X_r)$ a $\mathbb{R}^r$ valued random variable such that:*

1. *With $D\mathbf{g}(\mathbf{x}) \neq \mathbf{0}$ for $\mathbf{x}$ large enough, we have $\lim_{n \to +\infty} |||D^2\mathbf{g}(\mathbf{x})||| \, / \, \|D\mathbf{g}(\mathbf{x})\| = 0$;*

2. *$|||D^2\mathbf{g}(\mathbf{x})|||$ is a multivariate almost radial regularly varying function admitting an integral representation given by:*

$$\forall \mathbf{x}, \;\; \|\mathbf{x}\| \geq B \quad |||D^2\mathbf{g}(\mathbf{x})||| = \|\mathbf{x}\|^{\rho} \exp\left(\boldsymbol{\eta}(\mathbf{x}) + \int_B^{\|\mathbf{x}\|} \frac{\epsilon(t)}{t} dt\right) \tag{3.10}$$

   with $B > 0$ and $\rho > 1$ some constants, $\boldsymbol{\eta} : \mathbb{R}^r \mapsto \mathbb{R}$ and $\epsilon : \mathbb{R} \mapsto \mathbb{R}$ measurable and bounded functions such that $\lim_{t \to +\infty} \epsilon(t) = 0$;

3. *One of the marginals of $\mathbf{X}$ say, from now on and with no generality loss, X_r has a density f_{X_r} bounded by a constant $D_1 > 0$;*

4. *With f_{X_r} bounded by a constant, the density of $\mathbf{X}$ satisfies for some constant D_1:*

$$\forall \mathbf{x} = (x_1, \ldots x_r) \in \mathbb{R}^r \;\; f_{\mathbf{X}}(\mathbf{x}) \leq D_1 f_{(X_1, \ldots, X_{r-1})}(x_1, \ldots x_{r-1}) \; ;$$

5. *With f_{X_r} bounded by a constant,*

$$\forall \mathbf{x}, \;\; \|\mathbf{x}\| \geq B \;\; \frac{\partial \mathbf{g}}{\partial x_r}(\mathbf{x}) \neq 0 \; ;$$

6. *$\mathbf{X}$ has an absolute moment of order 2ρ, that is, $\mathbb{E}[\|\mathbf{X}\|^{2\rho}] < +\infty$;*

Then, for any vector $\boldsymbol{\mu}_n = (\mu_{n,1}, \dots \mu_{n,r})$ *such that for every* $i \in \{1, \dots r\}$ *we have* $\lim_{n\to+\infty} \mu_{n,i} = +\infty$ *we have that for some* $n \in \mathbb{N}$ *large enough on, and for some constant* D_2 *bounding:*

$$\mathbb{E}\left[\int_0^1 (1-t)\frac{|||D^2\mathbf{g}(\boldsymbol{\mu}_n + t\mathbf{X})|||}{|||D^2\mathbf{g}(\boldsymbol{\mu}_n)|||}dt\right] ,$$

that

$$\sup_{\mathbf{x}\in\mathbb{R}^r} \left|F_{\mathbf{g}(\mathbf{X}+\boldsymbol{\mu}_n)}(\mathbf{x}) - F_{\mathbf{g}(\boldsymbol{\mu}_n)+D\mathbf{g}(\boldsymbol{\mu}_n)\cdot\mathbf{X}}(\mathbf{x})\right| \leq$$

$$\leq C_{\frac{\rho}{\rho+1}} \, (D_1D_2)^{\frac{\rho}{2\rho+1}} \, \mathbb{E}[\|\mathbf{X}\|^{2\rho}]^{\frac{1}{2\rho+1}} \left(\frac{|||D^2\mathbf{g}(\boldsymbol{\mu}_n)|||}{\|D\mathbf{g}(\boldsymbol{\mu}_n)\|}\right)^{\frac{\rho}{2\rho+1}} ,$$

thus showing that for large n*, the law of* $\mathbf{g}(\mathbf{X}+\boldsymbol{\mu}_n)$ *may be approximated, in the Kolmogorov distance sense, by the law of* $\mathbf{g}(\boldsymbol{\mu}_n) + D\mathbf{g}(\boldsymbol{\mu}_n)\cdot\mathbf{X}$ *with a rate of convergence given by the sequence* $\left(|||D^2\mathbf{g}(\boldsymbol{\mu}_n)||| \,/\, \|D\mathbf{g}(\boldsymbol{\mu}_n)\|\right)^{\rho/(2\rho+1)}$.

Proof. The proof follows the same steps as the proof of the theorem for the univariate version. The only step requiring some different ideas being the first one.

- First step: The density of $\mathbf{g}(\boldsymbol{\mu}_n) + D\mathbf{g}(\boldsymbol{\mu}_n)\cdot\mathbf{X}$ is bounded by $D_1/\|D\mathbf{g}(\boldsymbol{\mu}_n)\|$.

We will need some notations for what follows. Let $F_n : \mathbb{R}^r \mapsto \mathbb{R}$ be the linear form defined by:

$$\forall \mathbf{x} = (x_1, \dots, x_r) \quad F_n(\mathbf{x}) = D\mathbf{g}(\boldsymbol{\mu}_n)\cdot\mathbf{x} = \sum_{n=1}^{r} \frac{\partial\mathbf{g}}{\partial x_i}(\boldsymbol{\mu}_n)\cdot x_i$$

and for each $y \in \mathbb{R}$ let $T_n(\mathbf{x}) = \mathbf{g}(\boldsymbol{\mu}_n) + F_n(\mathbf{x})$ and so:

$$T_n^{-1}(y) := \left\{\mathbf{x} \in \mathbb{R}^r : \mathbf{g}(\boldsymbol{\mu}_n) + \sum_{n=1}^{r} \frac{\partial\mathbf{g}}{\partial x_i}(\boldsymbol{\mu}_n)\cdot x_i = y\right\} = F_n^{-1}(y - \mathbf{g}(\boldsymbol{\mu}_n)) .$$

Considering $\mathbf{x}_0^{y-\mathbf{g}(\boldsymbol{\mu}_n)} \in F_n^{-1}(y - \mathbf{g}(\boldsymbol{\mu}_n))$ and $K_n := F_n^{-1}(\{0\})$ we have that $T_n^{-1}(y) = \mathbf{x}_0^{y-\mathbf{g}(\boldsymbol{\mu}_n)} + K_n$. With the hypothesis assuring that $(\partial\mathbf{g}/\partial x_r)(\boldsymbol{\mu}_n) \neq 0$, we have that:

$$\mathbf{x} = (x_1, \dots, x_r) \in K_n \Rightarrow \mathbf{x} = \left(x_1, \dots, x_{r-1}, -\frac{\sum_{n=1}^{r-1} \frac{\partial\mathbf{g}}{\partial x_i}(\boldsymbol{\mu}_n)\cdot x_i}{\frac{\partial\mathbf{g}}{\partial x_r}(\boldsymbol{\mu}_n)}\right) , \tag{3.11}$$

and, as a consequence, if we define the matrix M by

$$M := \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ . & . & . & . & \dots & . & 0 \\ . & . & . & . & \dots & 1 & 0 \\ -\frac{\frac{\partial\mathbf{g}}{\partial x_1}(\boldsymbol{\mu}_n)}{\frac{\partial\mathbf{g}}{\partial x_r}(\boldsymbol{\mu}_n)} & -\frac{\frac{\partial\mathbf{g}}{\partial x_2}(\boldsymbol{\mu}_n)}{\frac{\partial\mathbf{g}}{\partial x_r}(\boldsymbol{\mu}_n)} & -\frac{\frac{\partial\mathbf{g}}{\partial x_3}(\boldsymbol{\mu}_n)}{\frac{\partial\mathbf{g}}{\partial x_r}(\boldsymbol{\mu}_n)} & -\frac{\frac{\partial\mathbf{g}}{\partial x_4}(\boldsymbol{\mu}_n)}{\frac{\partial\mathbf{g}}{\partial x_r}(\boldsymbol{\mu}_n)} & \dots & -\frac{\frac{\partial\mathbf{g}}{\partial x_{r-1}}(\boldsymbol{\mu}_n)}{\frac{\partial\mathbf{g}}{\partial x_r}(\boldsymbol{\mu}_n)} & 1 \end{bmatrix}$$

we will have that $M : \mathbb{R}^r \mapsto K_n$ has determinant equal to 1 and so defines a invertible linear transformation. It now immediate to verify that

$$\mathbf{x} = (x_1, \ldots, x_r) \in M^{-1}(K_n) \Leftrightarrow x_r = 0 \ . \tag{3.12}$$

We now proceed to show the conclusion stated in the first step. For that purpose, we observe that as a consequence of the co-area formula (see the second transformation formula in [Hoffmann-Jørgensen 94, p. 11] or proposition 3 [Evans et al. 92, p. 119]) as T_n is a smooth regular Borel function then $T_n(\mathbf{X}) = \mathbf{g}(\boldsymbol{\mu}_n) + D\mathbf{g}(\boldsymbol{\mu}_n) \cdot \mathbf{X}$ admits a density with respect to $\lambda_{1,1}$ given, for every $y \in T_n(\mathbb{R}^r)$ by:

$$f_{T_n(\mathbf{X})}(y) = \int_{T_n^{-1}(y)} \frac{f_{\mathbf{X}}(\mathbf{x})}{J_{T_n(\mathbf{x})}} \, d\lambda_{r,r-1}(\mathbf{x}) \ ,$$

where $J_{T_n(\mathbf{x})}$ is the absolute Jacobiant of T_n, and such that $f_{T_n(\mathbf{X})}(y) = 0$ for $y \notin T_n(\mathbb{R}^r)$. Observing that $J_{T_n(\mathbf{x})} = \|DT_n(\mathbf{x})\| = \|D\mathbf{g}(\boldsymbol{\mu}_n)\|$ is constant, we only have to estimate the integral of the density of $\mathbf{X}$ with respect to $\lambda_{r,r-1}$ For that, we first observe that by a trivial change of variables and by the fact that Hausdorff measure $\lambda_{r,r-1}$ is invariant by translations that:

$$\begin{aligned} I := \int_{T_n^{-1}(y)} f_{\mathbf{X}}(\mathbf{x}) \, d\lambda_{r,r-1}(\mathbf{x}) &= \int_{\mathbf{x}_0^{y-\mathbf{g}(\boldsymbol{\mu}_n)} + K_n} f_{\mathbf{X}}(\mathbf{x}) \, d\lambda_{r,r-1}(\mathbf{x}) = \\ &= \int_{K_n} f_{\mathbf{X}}(\mathbf{z} + \mathbf{x}_0^{y-\mathbf{g}(\boldsymbol{\mu}_n)}) \, d\lambda_{r,r-1}(\mathbf{z}) \ . \end{aligned}$$

Now consider $\mathbf{h} = \mathbf{h}_{\mathbf{x}_0^{y-\mathbf{g}(\boldsymbol{\mu}_n)}}$ such that $\mathbf{h} : \mathbb{R}^r \mapsto \mathbb{R}$ is defined for $\mathbf{z} \in \mathbb{R}^r$ by $\mathbf{h}(\mathbf{z}) = f_{\mathbf{X}}(\mathbf{z} + \mathbf{x}_0^{y-\mathbf{g}(\boldsymbol{\mu}_n)})$. As by (3.12) we have $M^{-1}(K_n) = \{x_r = 0\}$ in $\mathbb{R}^r$, and also as the transformation M is linear and so its Jacobiant is $J_M = \det(M^t \cdot M) = \det(M \cdot M^t) = 1$, applying formula (3.9) we have that:

$$\begin{aligned} I &= \int_{K_n} f_{\mathbf{X}}(\mathbf{z} + \mathbf{x}_0^{y-\mathbf{g}(\boldsymbol{\mu}_n)}) \, d\lambda_{r,r-1}(\mathbf{z}) = \int_{M(M^{-1}(K_n))} \mathbf{h}(\mathbf{z}) \, d\lambda_{r,r-1}(\mathbf{z}) = \\ &= \int_{M^{-1}(K_n)} \mathbf{h}(M(\mathbf{x})) J_M \, dx_1 \ldots dx_{r-1} = \\ &= \int_{\{x_r=0\}} \mathbf{h}\left(x_1, \ldots, x_{r-1}, -\frac{\sum_{i=1}^{r-1} \frac{\partial \mathbf{g}}{\partial x_i}(\boldsymbol{\mu}_n) \cdot x_i}{\frac{\partial \mathbf{g}}{\partial x_r}(\boldsymbol{\mu}_n)} \right) dx_1 \ldots dx_{r-1} = \\ &= \int_{\{x_r=0\}} f_{\mathbf{X}}\left(x_1 + [\mathbf{x}_0^{y-\mathbf{g}(\boldsymbol{\mu}_n)}]_1, \ldots, -\frac{\sum_{i=1}^{r-1} \frac{\partial \mathbf{g}}{\partial x_i}(\boldsymbol{\mu}_n) \cdot x_i}{\frac{\partial \mathbf{g}}{\partial x_r}(\boldsymbol{\mu}_n)} + [\mathbf{x}_0^{y-\mathbf{g}(\boldsymbol{\mu}_n)}]_r \right) dx_1 \ldots dx_{r-1} = \\ &= \int_{\{x_r=0\}} f_{\mathbf{X}}\left(x_1, \ldots, x_{r-1}, -\frac{\sum_{i=1}^{r-1} \frac{\partial \mathbf{g}}{\partial x_i}(\boldsymbol{\mu}_n) \cdot x_i}{\frac{\partial \mathbf{g}}{\partial x_r}(\boldsymbol{\mu}_n)} + [\mathbf{x}_0^{y-\mathbf{g}(\boldsymbol{\mu}_n)}]_r \right) dx_1 \ldots dx_{r-1} \leq \\ &\leq D_1 \int_{\mathbb{R}^{r-1}} f_{\mathbf{X}}\left(x_1, \ldots, x_{r-1}\right) \, dx_1 \ldots dx_{r-1} \leq D_1 \ . \end{aligned}$$

where in the two last lines we have made a trivial change of variable by translation and next used the hypothesis on the density of $\mathbf{X}$.

- Second step: Defining $\mathbf{Z}_n$ as the integral remainder term of the Taylor expansion of $\mathbf{g}(X + \mu_n)$

$$\mathbf{g}(\boldsymbol{\mu}_n + \boldsymbol{X}) = \mathbf{g}(\boldsymbol{\mu}_n) + D\mathbf{g}(\boldsymbol{\mu}_n) \cdot \boldsymbol{X} + \int_0^1 (1-t) D^2 \mathbf{g}(\boldsymbol{\mu}_n + t\boldsymbol{X}) \cdot \boldsymbol{X}^2 \, dt \,,$$

  we show that with $\delta = \rho/(\rho+1) < 1$ we have:

$$\mathbb{E}\left[|\mathbf{Z}_n|^\delta \right]^{\frac{1}{\delta}} \leq \mathbb{E}\left[\|\mathbf{X}\|^{\frac{2\delta}{1-\delta}} \right]^{\frac{1-\delta}{\delta}} \mathbb{E}\left[\int_0^1 (1-t) \left|\left|\left| D^2 \mathbf{g}(\boldsymbol{\mu}_n + t\boldsymbol{X}) \right|\right|\right| dt \right] . \tag{3.13}$$

This proof follow exactly the same proof as presented for the univariate case.

- Third step: The announced result will be proved if we show that

$$\mathbb{E}\left[\int_0^1 (1-t) \frac{\left|\left|\left| D^2 \mathbf{g}(\boldsymbol{\mu}_n + t\mathbf{X}) \right|\right|\right|}{|||D^2 \mathbf{g}(\boldsymbol{\mu}_n)|||} dt \right] ,$$

  is bounded by a constant D_2. As $\lim_{n \to +\infty} \left|\left|\left| D^2 \mathbf{g}(\boldsymbol{\mu}_n + t\mathbf{X}) \right|\right|\right| / \left|\left|\left| D^2 \mathbf{g}(\boldsymbol{\mu}_n) \right|\right|\right| = 1$, by the continuity of $D^2\mathbf{g}$, this is a consequence of Lebesgue dominated convergence theorem if we show that the integrand is bounded uniformly in n by an integrable function with respect to $\mathbb{P} \otimes \lambda$.

In the representation given in formula (3.1), let us consider bounds $M_{\boldsymbol{\eta}} > 0$ and $M_\epsilon > 0$ for the functions $\boldsymbol{\eta}$ and ϵ, respectively. Also, as $\lim_{t \to +\infty} \epsilon(t) = 0$, let $x_\rho > 0$ be such that for $\|\mathbf{x}\| \geq x_\rho$ we have $|\epsilon(x)| \leq \rho - 1$. Let us first observe that:

$$\boldsymbol{\mu}_n \geq \max(B, x_\rho) \Rightarrow \left\| D^2 \mathbf{g}(\boldsymbol{\mu}_n) \right\| \geq e^{-M_{\boldsymbol{\eta}}} B^\rho \,. \tag{3.14}$$

This results from having for $\|\mathbf{x}\| \geq \max(B, x_\rho)$:

$$\boldsymbol{\eta}(\mathbf{x}) + \int_B^{\|\mathbf{x}\|} \frac{\epsilon(t) + \rho}{t} dt \geq -M_{\boldsymbol{\eta}} + \log(\frac{\|\mathbf{x}\|}{B}) \Rightarrow \boldsymbol{\eta}(\mathbf{x}) + \int_B^{\|\mathbf{x}\|} \frac{\epsilon(t)}{t} dt \geq$$
$$\geq -M_{\boldsymbol{\eta}} + \log(\frac{\|\mathbf{x}\|}{B})^{(1-\rho)} \,.$$

As a second observation we also have that for $\|\mathbf{x}\| \geq B$:

$$\exp\left(\boldsymbol{\eta}(\mathbf{x}) + \int_B^{\|\mathbf{x}\|} \frac{\epsilon(t)}{t} dt \right) \leq e^{M_{\boldsymbol{\eta}}} \left(\frac{\|\mathbf{x}\|}{B} \right)^{M_\epsilon} \,, \tag{3.15}$$

this resulting from having:

$$\boldsymbol{\eta}(\mathbf{x}) + \int_B^{\|\mathbf{x}\|} \frac{\epsilon(t)}{t} dt \leq \left| \boldsymbol{\eta}(\mathbf{x}) + \int_B^{\|\mathbf{x}\|} \frac{\epsilon(t)}{t} dt \right| \leq |\boldsymbol{\eta}(\mathbf{x})| + \int_B^{\|\mathbf{x}\|} \frac{|\epsilon(t)|}{t} dt \leq$$
$$\leq M_{\boldsymbol{\eta}} + \log(\frac{\|\mathbf{x}\|}{B})^{M_\epsilon} \,.$$

Now, for $\|\boldsymbol{\mu}_n + t\mathbf{X}\| \leq \max(B, x_\rho)$ we have by the continuity of $D^2\mathbf{g}$:

$$\left\| \frac{D^2\mathbf{g}(\boldsymbol{\mu}_n + t\mathbf{X})}{D^2\mathbf{g}(\boldsymbol{\mu}_n)} \right\| \leq \sup_{\|\mathbf{x}\| \leq \max(B, x_\rho)} \left\| D^2\mathbf{g}(\mathbf{x}) \right\| \cdot \frac{1}{e^{-M_{\boldsymbol{\eta}}} B^\rho} < +\infty \, .$$

Also for the complementary case, that is, for $\|\boldsymbol{\mu}_n + t\mathbf{X}\| > \max(B, x_\rho)$ and $\boldsymbol{\mu}_n \geq \max(B, x_\rho)$:

$$\begin{aligned} \frac{\left\| D^2\mathbf{g}(\boldsymbol{\mu}_n + t\mathbf{X}) \right\|}{\left\| D^2\mathbf{g}(\boldsymbol{\mu}_n) \right\|} &= \frac{\|\boldsymbol{\mu}_n + t\mathbf{X}\|^\rho \exp\left(\boldsymbol{\eta}(\|\boldsymbol{\mu}_n + t\mathbf{X}\|) + \int_B^{\|\boldsymbol{\mu}_n + t\mathbf{X}\|} \frac{\epsilon(t)}{t} dt\right)}{\|\boldsymbol{\mu}_n\|^\rho \exp\left(\boldsymbol{\eta}(\boldsymbol{\mu}_n) + \int_B^{\|\boldsymbol{\mu}_n\|} \frac{\epsilon(t)}{t} dt\right)} \\ &\leq \left(\frac{\|\boldsymbol{\mu}_n + t\mathbf{X}\|}{\|\boldsymbol{\mu}_n\|} \right)^\rho e^{2M_{\boldsymbol{\eta}}} \exp\left(\int_B^{\|\boldsymbol{\mu}_n + t\mathbf{X}\|} \frac{\epsilon(t)}{t} dt - \int_B^{\|\boldsymbol{\mu}_n\|} \frac{\epsilon(t)}{t} dt \right) \end{aligned} \tag{3.16}$$

We will now split in two sub-cases the estimation of the exponential right-hand term of formula (3.15) of this second case. Let us suppose first that $\|\boldsymbol{\mu}_n + t\mathbf{X}\| \geq \boldsymbol{\mu}_n \geq \max(B, x_\rho)$. Then as

$$\begin{aligned} \int_B^{\|\boldsymbol{\mu}_n + t\mathbf{X}\|} \frac{\epsilon(t)}{t} dt - \int_B^{\|\boldsymbol{\mu}_n\|} \frac{\epsilon(t)}{t} dt &= \int_{\|\boldsymbol{\mu}_n\|}^{\|\boldsymbol{\mu}_n + t\mathbf{X}\|} \frac{\epsilon(t)}{t} dt \leq \left| \int_{\|\boldsymbol{\mu}_n\|}^{\|\boldsymbol{\mu}_n + t\mathbf{X}\|} \frac{\epsilon(t)}{t} dt \right| \leq \\ &\leq \int_{\|\boldsymbol{\mu}_n\|}^{\|\boldsymbol{\mu}_n + t\mathbf{X}\|} \frac{|\epsilon(t)|}{t} dt \leq \log \left(\frac{\|\boldsymbol{\mu}_n + t\mathbf{X}\|}{\|\boldsymbol{\mu}_n\|} \right)^\rho \end{aligned}$$

we have that

$$\exp\left(\int_B^{\|\boldsymbol{\mu}_n + t\mathbf{X}\|} \frac{\epsilon(t)}{t} dt - \int_B^{\|\boldsymbol{\mu}_n\|} \frac{\epsilon(t)}{t} dt \right) \leq \left(\frac{\|\boldsymbol{\mu}_n + t\mathbf{X}\|}{\|\boldsymbol{\mu}_n\|} \right)^\rho \, . \tag{3.17}$$

In the complementary sub-case, that is, whenever $\max(B, x_\rho) < \|\boldsymbol{\mu}_n + t\mathbf{X}\| < \boldsymbol{\mu}_n$, as:

$$\begin{aligned} \int_B^{\|\boldsymbol{\mu}_n + t\mathbf{X}\|} \frac{\epsilon(t)}{t} dt - \int_B^{\|\boldsymbol{\mu}_n\|} \frac{\epsilon(t)}{t} dt &= - \int_{\|\boldsymbol{\mu}_n + t\mathbf{X}\|}^{\|\boldsymbol{\mu}_n\|} \frac{\epsilon(t)}{t} dt \leq \left| \int_{\|\boldsymbol{\mu}_n + t\mathbf{X}\|}^{\|\boldsymbol{\mu}_n\|} \frac{-\epsilon(t)}{t} dt \right| \leq \\ &\leq \int_{\|\boldsymbol{\mu}_n + t\mathbf{X}\|}^{\|\boldsymbol{\mu}_n\|} \frac{|\epsilon(t)|}{t} dt \leq \log \left(\frac{\|\boldsymbol{\mu}_n + t\mathbf{X}\|}{\|\boldsymbol{\mu}_n\|} \right)^\rho \end{aligned}$$

also, we have that:

$$\exp\left(\int_B^{\|\boldsymbol{\mu}_n + t\mathbf{X}\|} \frac{\epsilon(t)}{t} dt - \int_B^{\boldsymbol{\mu}_n} \frac{\epsilon(t)}{t} dt \right) \leq \left(\frac{\|\boldsymbol{\mu}_n + t\mathbf{X}\|}{\|\boldsymbol{\mu}_n\|} \right)^{-\rho} \, . \tag{3.18}$$

Finally, gathering the results in formulas (3.17) and (3.18) we have that:

$$\left\|\frac{D^2\mathbf{g}(\boldsymbol{\mu}_n+t\mathbf{X})}{D^2\mathbf{g}(\boldsymbol{\mu}_n)}\right\| \leq \left(\frac{\|\boldsymbol{\mu}_n+t\mathbf{X}\|}{\|\boldsymbol{\mu}_n\|}\right)^{\rho} e^{2M_\eta}\cdot\max\left(\left(\frac{\|\boldsymbol{\mu}_n+t\mathbf{X}\|}{\|\boldsymbol{\mu}_n\|}\right)^{\rho},\left(\frac{\|\boldsymbol{\mu}_n+t\mathbf{X}\|}{\|\boldsymbol{\mu}_n\|}\right)^{-\rho}\right)=$$
$$= e^{2M_\eta}\max\left(\left(\frac{\|\boldsymbol{\mu}_n+t\mathbf{X}\|}{\|\boldsymbol{\mu}_n\|}\right)^{2\rho},1\right)\leq e^{2M_\eta}2^{2\rho}\left(1+\frac{\|\mathbf{X}\|^{2\rho}}{\max(B,x_\rho)^{2\rho}}\right), \tag{3.19}$$

which allow us to conclude as $\mathbb{E}[\|\mathbf{X}\|^{2\rho}]<+\infty$. □

Remark. In [Mexia et al. 10] asymptotic linearity for a a $C^2(\mathbb{R}^r)$ function $\mathbf{g}:\mathbb{R}^r\mapsto\mathbb{R}$ is defined by requiring that

$$\forall d>0\quad \lim_{u\to+\infty}\sup_{\|\mathbf{x}\|\geq u}\left(\frac{1}{\|D\mathbf{g}(\mathbf{x})\|}\sup_{\|\mathbf{x}-\mathbf{y}\|\leq d}\left|\left|\left|D^2\mathbf{g}(\mathbf{y})\right|\right|\right|\right)=0\,.$$

It is easy to see that if $\mathbf{g}$ is assimptotically linear then $\lim_{n\to+\infty}\left|\left|\left|D^2\mathbf{g}(\mathbf{x})\right|\right|\right|/\left\|D\mathbf{g}(\mathbf{x})\right\|=0$. On the other hand, for all fixed $d\geq 0$ and $x\geq 0$ with $x-d>0$ there exists $\lambda_x\in[0,1]$ such that, by the continuity of g'':

$$\frac{1}{|g'(x)|}\sup_{y\in[x-d,x+d]}\left|g''(y)\right|\leq\frac{|g''(\lambda_x(x-d)+(1-\lambda_x)(x+d))|}{|g'(x)|}=\frac{|g''(x+(1-2\lambda_x)d)|}{|g'(x)|}=$$
$$=\frac{|g''(x+(1-2\lambda_x)d)|}{|g''(x)|}\frac{|g''(x)|}{|g'(x)|}\,,$$

and so, as $1-2\lambda_x\in[-1,1]$, if $\lim_{x\to+\infty}|g''(x)|\,/\,|g'(x)|=0$, g will be assimptotically linear, thus showing the equivalence of the two hypothesis, at least in the univariate case.

3.3 An application to high precision measurements

Let us convey briefly an informal approach to a statistical perspective of high precision measurements. We are given a sequence of non degenerate square integrable random variables $V_1,\ldots V_n,\ldots$, the observations, with $m_n=\mathbb{E}[V_n]$, $\sigma_n=\sqrt{\mathbb{E}[V_n^2]-\mathbb{E}[V_n]^2}$ and $\mu_n=m_n/\sigma_n$. Suppose we are interested in determining the asymptotic behavior of the laws either of the random variables of the sequence $(g(V_n))_{n\in\mathbb{N}}$ or of the sequence of reduced observations $(g(V_n/\sigma_n))_{n\in\mathbb{N}}$. In either case, if the standardized observations $W_n=(V_n-m_n)/\sigma_n$ all have the same distribution say, the distribution of some determined random variable V, we may always write:

$$V_n=\sigma_n\left(\frac{V_n-m_n}{\sigma_n}+\frac{\mu_n}{\sigma_n}\right)\overset{\mathrm{d}}{\simeq}\sigma_n\left(V+\mu_n\right)\,. \tag{3.20}$$

Now, one main idea in high precision measurements is to consider that $\lim_{n\to+\infty}\sigma_n = 0$ and that the sequence $(m_n)_{n\in\mathbb{N}}$ is bounded, or even constant, so that $\lim_{n\to+\infty}\mu_n = +\infty$. This means that as the number of observations grows, the coefficient of variation (defined, in the literature, to be equal to $1/\mu_n$) gets smaller.

Formula 3.20 shows that the asymptotic behavior of the laws of the reduced observations is given by the asymptotic behavior of the laws of the random variables of the sequence $(g(V+\mu_n))_{n\in\mathbb{N}}$.

For the asymptotic behavior of the laws of the sequence $(g(V_n))_{n\in\mathbb{N}}$, suppose furthermore that for all x, y the function g verifies $g(xy) = g(x)g(y)$ and that $g(x) \neq 0$, for $x \neq 0$, in order to eliminate trivial solutions of this functional equation [1]. Thus, obtaining an asymptotic approximation in distribution for $g(V_n)$ amounts to obtaining an asymptotic approximation in distribution for $X_n := g(V+\mu_n)$. In fact, if for some Y_n we have $X_n \overset{\mathrm{d}}{\approx} Y_n$ as $\mu_n \gg 1$ we will also have $g(V_n) = g(\sigma_n)X_n \overset{\mathrm{d}}{\approx} g(\sigma_n)Y_n$ as $\mu_n \gg 1$.

As a particular case of this last train of thought, we consider that X has a standardized normal distribution and we get the law of of a polynomial on a standardized normal random variable approximated by the normal law of its linear part. This amounts in fact to a linearization procedure. We will write $X \in \mathcal{N}(m,\sigma)$ to denote a Gaussian random variable with mean m and standard deviation σ.

Corolary 2. *Let $X_n := (\mu_n + X)^\alpha$ with $X \in \mathcal{N}(0,1)$, $\alpha \geq 2$ and $Y_n = \mu_n^\alpha + \alpha\mu_n^{\alpha-1}X \in \mathcal{N}(\mu_n^\alpha, \alpha\mu_n^{\alpha-1})$ and so, even when α is integer and*

$$Z_n = \sum_{k=2}^{\alpha}\binom{\alpha}{k}\mu_n^{\alpha-k}X^k \ ,$$

Z_n is a random variable with a non trivial distribution. We have that the law of $(\mu_n+X)^\alpha$ is approximately normal for large values of μ_n. As a Gaussian has moments of all orders, the rate of convergence is $(1/(\mu_n)^{(1/2)+\epsilon})$ for all $\epsilon > 0$.

Proof. Apply theorem 3.1 considering $g(x) = x^\alpha$. □

The following multidimensional particular case is also relevant for STATIS data analysis methodology (see [Ramos 07]).

Corolary 3. *With the notations of theorem 3.2 and of subsection 3.2, consider:*

$$X_n := X(\boldsymbol{\mu}_n) := \prod_{i=1}^{r}(\mu_{n,i} + X_i)^{\alpha_i}$$

Then with $Y_n := Y(\boldsymbol{\mu}_n)$ given by:

$$\prod_{i=1}^{r}\mu_{n,i}^{\alpha_i} + \sum_{i=1}^{r}\left(\prod_{j\neq i, j=1}^{r}\mu_{n,j}^{\alpha_j}\right)\alpha_i\mu_{n,i}^{\alpha_i-1}X_i \in \mathcal{N}\left(\prod_{i=1}^{r}\mu_{n,i}^{\alpha_i}\ ,\ \sum_{i=1}^{r}\left(\prod_{j\neq i, j=1}^{r}\mu_{n,j}^{\alpha_j}\right)\alpha_i\mu_{n,i}^{\alpha_i-1}\right)$$

[1] Any continuous solution of this functional equation is of the form $g(x) = |x|^\alpha$ or $g(x) = \mathrm{sign}(x)\,|x|^\alpha$, for some α (see [Itô 93, p. 1443]). For having $g \in \mathcal{C}^2(\mathbb{R})$ we may infer that $g(x) = x^\alpha$, for some $\alpha \in \mathbb{N}$, is an acceptable solution

we have the uniform approximation $\lim_{n\to+\infty}\sup_x |F_{X_n}(x) - F_{Y_n}(x)| = 0$*, that is, for sufficient large* n *the law of* X_n *is approximately normal. The rate of convergence is, at least,* $(\sum_{i=1}^r (1/\mu_{n,i}^2))^{(1/4)+\epsilon})$ *for all* $\epsilon > 0$.

Proof. See the appendix for a straightforward but lengthy calculation. □

4 Non linear approximations

In previous sections, theorem 2.3 was used in instances where the main tool was the Taylor formula applied to get an asymptotic approximation by a linear transform of the initial random variable. In this section, we show in some examples that other types of asymptotic expansions may be treated with the same methodology.

We will write $X \in \mathcal{G}(p, \delta, b)$ if the density of X is given by:

$$\forall x \in \mathbb{R}\ f_{(p,\delta,b)}(x) = \frac{1}{\delta^p \Gamma(p)}(x-b)^{(p-1)} e^{-\frac{x-b}{\delta}} \mathbb{1}_{[b,+\infty[}(x)\ ,$$

$\mathbb{1}_{[b,+\infty[}$ being the indicator function of the interval $[b, +\infty[$. Recall that for such a random variable we have $\mathbb{E}[X] = p\delta + b$, $\mathbb{V}[X] = p\delta^2$ and

$$\sup_{x\in\mathbb{R}} f_{(p,\delta,b)}(x) = \sup_{x\in\mathbb{R}} f_{(p,\delta,0)}(x) = \frac{1}{\delta\Gamma(p)}(p-1)^{(p-1)} e^{-(p-1)}\ .$$

In our first example. we study an integral transform of a rescaled random variable having such a gamma distribution.

Proposition 1. *Let* $X \in \mathcal{G}(p, \delta, b)$ *with* $b > 0$, $p \geq 2$, $(\mu_n)_{n\in\mathbb{N}}$ *a sequence of non negative numbers such that* $\lim_{n\to+\infty} \mu_n = +\infty$ *and consider for some fixed* $a > 0$*:*

$$X_n := \int_0^a \frac{e^{-\mu_n X t}}{1+t^2} dt\ .$$

Then, for $\mu_n > 0$ *we have that for the constant* C_1 *in formula* (2.2) *and for some constant* $e = e(p, \delta, b, a)$

$$\sup_x \left| F_{X_n}(x) - F_{\frac{1}{\mu_n X}}(x) \right| \leq C_1 \frac{\sqrt{e}}{\mu_n^{\frac{1}{2}}}$$

which implies that, for μ_n *sufficiently large, the law of* X_n *may be approximated by the law of* $1/(\mu_n X)$*, uniformly, with rate of convergence order* $\mu_n^{1/2}$.

Proof. By an integration by parts we have the first terms of the asymptotic expansion for the relevant integral.

$$\int_0^a \frac{e^{-\lambda t}}{1+t^2} dt = \frac{1}{\lambda} - \frac{e^{-\lambda a}}{\lambda(1+a^2)} + \frac{1}{\lambda}\int_0^a e^{-\lambda t}\frac{2t}{(1+t^2)^2})dt$$

As a consequence and as the the rational function in the integral is bounded by $3\sqrt{3}/8$ we have, with $c(a) := 3\sqrt{3}/8 + 1/(ae(1+a^2))$ that:

$$\left| \int_0^a \frac{e^{-\lambda t}}{1+t^2} dt - \frac{1}{\lambda} \right| \leq \frac{e^{-\lambda a}}{\lambda(1+a^2)} + \frac{3\sqrt{3}}{8\lambda^2} \leq \frac{c(a)}{\lambda^2} \ .$$

the bound on the right being justified as the rational term term containing a verifies, for all $\lambda > 0$:

$$\frac{1}{\lambda^2}\left(\frac{\lambda e^{-\lambda a}}{(1+a^2)}\right) \leq \frac{1}{\lambda^2}\frac{1}{ae(1+a^2)}$$

and that the term inside the parenthesis, as a function of $\lambda > 0$, is bounded by a constant depending only on a. Now, as $\mathbb{E}[1/X^2] \leq 1/b^2 + 1/(\delta^2 p(p-1))$, we get $\mathbb{E}[|Z_n|] \leq c(a) \cdot c(a,\delta,p,b)/\mu_n^2$, with $c(a,\delta,p,b)$ being another constant depending only on a, δ, p and b. The distribution function of the random variable $Y_n = 1/(\mu_n X)$ is given by:

$$F_{Y_n}(x) = 1 - \frac{1}{\delta^p \Gamma(p)} \int_b^{\frac{1}{\mu_n x}} (t-b)^{p-1} e^{-\frac{t-b}{\delta}} dt \ .$$

This implies that the corresponding density is given by:

$$f_{Y_n}(x) = F'_{Y_n}(x) = \frac{1}{\delta^p \Gamma(p)} \left(\frac{1}{\mu_n x} - b\right)^{p-1} e^{-\frac{1}{\delta}\left(\frac{1}{\mu_n x}-b\right)} \frac{1}{\mu_n x^2} \ .$$

Let us determine the stationary points of this density. Considering $u(x) := 1/(\mu_n x) - b$ we have with $M = 1/(\delta^p\Gamma(p))$ and $g(u) = u^{p-1}(u+b)^2 e^{-\frac{u}{\delta}}$ that $f_{Y_n}(x) = M\mu_n g(u(x))$ and so $f'_{Y_n}(x) = M\mu_n g'(u(x))u'(x)$. As we have always $u'(x) = -1/\mu_n x^2 \neq 0$ we have that the stationary points of f_{Y_n} are those of $g(u)$, that is, those points satisfying $u^2 - (\delta(p+1)-b)u - (p-1)b = 0$, that is the points

$$u_\pm = u_\pm u(p,\delta,b) = \frac{\delta(p+1) - b \pm \sqrt{(\delta(p+1)-b)^2 + 4(p-1)b}}{2}$$

Now considering the regularity of $f_{Y_n} = g(u)$ it is clear that

$$\sup_{x\in[0,+\infty[} f_{Y_n}(x) = M\mu_n \max(g(u_+), g(u_-)) = \mu_n d(p,\delta,b) \ ,$$

with $d(p,\delta,b)$ a constant depending only on p, δ and b. Considering now the constant $e = c(a) \cdot c(a,\delta,p,b) \cdot d(p,\delta,b)$ we have the result stated. □

Our second example has its source in the general theme of Laplace integrals. Often, the asymptotic behavior of models in the applied sciences is studied using this kind of integrals. The example presented here amounts to a randomization of these models.

Proposition 2. *Let* $\alpha > 0$, $\beta > 0$, $a \in]0,+\infty[$, $f \in \mathcal{C}^0([0,a[)$ *such that* $f(0) \neq 0$ *and* $f(x) =_{x\to 0} f(0) + \mathcal{O}(x)$ *and* $X \in \mathcal{G}(p,\delta,b)$ *with* $b > 0$ *and* $p > (\beta+1)/\alpha$. *Suppose* *that $(\mu_n)_{n\in\mathbb{N}}$ is a sequence of non negative numbers such that $\lim_{n\to+\infty}\mu_n = +\infty$ and consider:*

$$X_n := \int_0^a t^{\beta-1} f(t) e^{-\mu_n X t^\alpha} dt \text{ and } Y_n := \frac{f(0)}{\alpha(\mu_n X)^{\frac{\beta}{\alpha}}}\Gamma(\frac{\beta}{\alpha}) \, .$$

we have that for any $\lambda_0 > 0$ fixed, for any $\mu_n > \lambda_0/b$, for the constant C_1 in formula (2.2) and for some constant $c = c(\alpha, \beta, p, \delta, b, a, \lambda_0, f)$:

$$\sup_x |F_{X_n}(x) - F_{Y_n}(x)| \le C_1 \frac{\sqrt{c}}{\mu_n^{\frac{1}{2\alpha}}}$$

which implies that, for μ_n sufficiently large, the law of X_n may be approximated by the law of Y_n, uniformly, with rate of convergence of order $\mu_n^{1/(2\alpha)}$.

Proof. Let $R(x)$ be such that for some $\epsilon > 0$ and for $x \in [0, \epsilon]$ we have $f(x) = f(0)+R(x)$ and $|R(x)| \le Cx$ for some constant C. By observing that

$$\int_0^{+\infty} t^{\beta-1} e^{-\lambda t^\alpha} dt = \frac{1}{\alpha\lambda^{\frac{\beta}{\alpha}}}\Gamma(\frac{\beta}{\alpha}) \tag{4.1}$$

we have that:

$$\left|\int_0^a t^{\beta-1} f(t) e^{-\lambda t^\alpha} dt - \frac{f(0)}{\alpha\lambda^{\frac{\beta}{\alpha}}}\Gamma(\frac{\beta}{\alpha})\right| =$$
$$\left|\int_0^\epsilon t^{\beta-1} R(t) e^{-\lambda t^\alpha} dt + \int_\epsilon^a t^{\beta-1} f(t) e^{-\lambda t^\alpha} dt - f(0)\int_\epsilon^{+\infty} t^{\beta-1} e^{-\lambda t^\alpha} dt\right| \, .$$

We also have

$$\int_\epsilon^a t^{\beta-1} |f(t)| e^{-\lambda t^\alpha} dt \le e^{-\lambda\epsilon^\alpha}\int_0^a t^{\beta-1}|f(t)|\, dt =: e^{-\lambda\epsilon^\alpha} A_1$$

and, given $\lambda_0 > 0$, for all $\lambda \ge \lambda_0 > 0$,

$$\int_\epsilon^{+\infty} t^{\beta-1} e^{-\lambda t^\alpha} dt = \int_\epsilon^{+\infty} t^{\beta-1} e^{-\lambda_0 t^\alpha} e^{-(\lambda-\lambda_0)t^\alpha} dt \le e^{-(\lambda-\lambda_0)\epsilon^\alpha}\int_\epsilon^{+\infty} t^{\beta-1} e^{-\lambda_0 t^\alpha} dt \le$$
$$\le \left[e^{\lambda_0\epsilon^\alpha}\int_0^{+\infty} t^{\beta-1} e^{-\lambda_0 t^\alpha} dt\right] e^{\lambda\epsilon^\alpha} = \left[\frac{e^{\lambda_0\epsilon^\alpha}}{\alpha\lambda_0^{\frac{\beta}{\alpha}}}\Gamma(\frac{\beta}{\alpha})\right] e^{-\lambda\epsilon^\alpha} =: A_2 e^{-\lambda\epsilon^\alpha} \, .$$

Using again (4.1) we have that

$$\int_0^\epsilon t^{\beta-1}|R(t)| e^{-\lambda t^\alpha} dt \le C\frac{1}{\alpha\lambda^{\frac{\beta+1}{\alpha}}}\Gamma(\frac{\beta+1}{\alpha})$$

thus allowing us to conclude that for some fixed $\lambda_0 > 0$, and for all $\lambda > \lambda_0 > 0$

$$\left|\int_0^a t^{\beta-1} f(t) e^{-\lambda t^\alpha} dt - \frac{f(0)}{\alpha\lambda^{\frac{\beta}{\alpha}}}\Gamma(\frac{\beta}{\alpha})\right| \le \frac{1}{\lambda^{\frac{\beta+1}{\alpha}}}\left(\frac{C}{\alpha}\Gamma(\frac{\beta+1}{\alpha}) + (A_1 + A_2)\sup_{\lambda>\lambda_0}\lambda^{\frac{\beta+1}{\alpha}} e^{-\lambda\epsilon^\alpha}\right)$$

Now for future convenience we will define

$$A := \frac{C}{\alpha}\Gamma(\frac{\beta+1}{\alpha}) + (A_1 + A_2)\left(\frac{\beta+1}{\alpha\epsilon^\alpha}\right)^{\frac{\beta+1}{\alpha}} e^{-\frac{\beta+1}{\alpha}}$$

We will now proceed as in the proof of proposition 1. Define $Z_n := X_n - Y_n$ and observe that, by the estimations above we have that, $|Z_n| \leq A/(\mu_n X)^{(\beta+1)/\alpha}$. As a consequence, we have that $\mathbb{E}[|Z_n|] \leq (A/\mu_n^{(\beta+1)/\alpha})\mathbb{E}[1/X^{(\beta+1)/\alpha}]$. Now, as $X \in \mathcal{G}(p, \delta, b)$ and $p > (\beta+1)/\alpha$ we have that $\mathbb{E}[1/X^{(\beta+1)/\alpha}]$ is a real constant B depending only on $\alpha, \beta, p, \delta, b$ and so $\mathbb{E}[|Z_n|] \leq (AB/\mu_n^{(\beta+1)/\alpha})$.

We now proceed as in the previous proposition. For simplicity, let $M := \Gamma(\beta/\alpha)f(0)/\alpha$. The distribution function of the random variable $Y_n = M/(\mu_n X)^{\beta/\alpha}$ is given by:

$$F_{Y_n}(x) = 1 - \frac{1}{\delta^p\Gamma(p)}\int_b^{\frac{1}{\mu_n}(\frac{M}{x})^{\frac{\alpha}{\beta}}} (t-b)^{p-1} e^{-\frac{t-b}{\delta}} dt \, .$$

This implies that the corresponding density is given by:

$$f_{Y_n}(x) = F'_{Y_n}(x) = \frac{1}{\delta^p\Gamma(p)}\left(\frac{M^{\frac{\alpha}{\beta}}}{\mu_n x^{\frac{\alpha}{\beta}}} - b\right)^{p-1} e^{-\frac{1}{\delta}\left(\frac{M^{\frac{\alpha}{\beta}}}{\mu_n x^{\frac{\alpha}{\beta}}} - b\right)} \frac{M^{\frac{\alpha}{\beta}}}{\mu_n x^{\frac{\alpha}{\beta}+1}}\frac{\alpha}{\beta} \, .$$

We have now to determine the stationary points of this density. Considering $u(x) := M^{\alpha/\beta}/(\mu_n x^{\alpha/\beta}) - b$ we have with $g(u) = u^{p-1}(u+b)^{1+\beta/\alpha}e^{-\frac{u}{\delta}}$ and also with $M' := \alpha/(\delta^p\Gamma(p)M\beta)$ that $f_{Y_n}(x) = M'\mu_n^{\beta/\alpha}g(u(x))$ and so $f'_{Y_n}(x) = M'\mu_n^{\beta/\alpha}g'(u(x))u'(x)$. As we have always $u'(x) = -(\alpha/\beta)(M^{\alpha/\beta}/\mu_n)(1/x^{1+(\alpha/\beta)}) \neq 0$ we have that the stationary points of f_{Y_n} are those of $g(u)$, that is, those points satisfying $u^2 - (\delta(p + (\beta/\alpha) - b)u - (p-1)b = 0$, that is the points

$$u_\pm = u_\pm u(p, \delta, b, \alpha, \beta) = \frac{\delta(p + (\beta/\alpha)) - b \pm \sqrt{(\delta(p + (\beta/\alpha)) - b)^2 + 4(p-1)b}}{2} \, .$$

Now, considering the regularity of $f_{Y_n} = M'\mu_n^{\beta/\alpha}g(u)$ it is clear that

$$\sup_{x\in[0,+\infty[} f_{Y_n}(x) = M'\mu_n^{\beta/\alpha}\max(g(u_+), g(u_-)) = \mu_n^{\beta/\alpha}D(p, \delta, b, \alpha, \beta) \, ,$$

with $D = D(p, \delta, b, \alpha, \beta)$ a constant depending only on p, δ, b, α and β,and so we may conclude as stated with the constant $c = ABD$. □

Remark 3. The examples presented above as well as other instances of application of the method here developed seems to point to the following heuristic principle: a classical asymptotic formula may be randomized provided that the source of randomization has a density admitting a sufficiently sharp maximum. This heuristic principle was further illustrated in [Esquivel et al. 09] with a simulation study.

5 Conclusion and final remarks

In this work we have shown that under mild technical hypothesis it is possible to establish the rate of convergence of asymptotic approximations, in the Kolmogorov's distance sense, to distributions of random variables tied by some non trivial almost sure asymptotic relation. Other simulation results, not presented in this work, make believe that the validity of these asymptotic approximations is much more comprehensive, thus indicating the need of further studies.

Appendix

In this section we gather the proofs of theorem 2.1 and of corollary 3, for the reader's commodity.

Proof of theorem 2.1. Consider the set

$$A_n(\epsilon) := \bigcap_{m=n}^{+\infty} \left\{ \left| \frac{X_m}{Y_m} - 1 \right| < \epsilon \right\}$$

$p_n(\epsilon) := \mathbb{P}[A_n(\epsilon)]$ which converges to 1 on account of the hypothesis and $\delta_n(\epsilon) := 1 - p_n(\epsilon)$. Then we have that for $x > 0$ and over $A_n(\epsilon)$

$$1 - \epsilon < \frac{X_n}{Y_n} < 1 + \epsilon$$

which implies

$$Y_n < \frac{x}{1+\epsilon} \Rightarrow X_n < x \Rightarrow X_n < \frac{x}{1-\epsilon}$$

that is

$$A_n(\epsilon) \cap \left\{ Y_n < \frac{x}{1+\epsilon} \right\} \subseteq A_n(\epsilon) \cap \{X_n < x\} \subseteq A_n(\epsilon) \cap \left\{ Y_n < \frac{x}{1-\epsilon} \right\} .$$

This in turn implies:

$$p_n(\epsilon) + F_{Y_n}(\frac{x}{1+\epsilon}) - 1 \le \mathbb{P}\left[A_n(\epsilon) \cap \left\{ Y_n < \frac{x}{1+\epsilon} \right\} \right] \le \min\left(p_n(\epsilon), F_{Y_n}(\frac{x}{1+\epsilon}) \right)$$

$$p_n(\epsilon) + F_{X_n}(x) - 1 \le \mathbb{P}\left[A_n(\epsilon) \cap \{X_n < x\} \right] \le \min\left(p_n(\epsilon), F_{X_n}(x) \right)$$

$$p_n(\epsilon) + F_{Y_n}(\frac{x}{1-\epsilon}) - 1 \le \mathbb{P}\left[A_n(\epsilon) \cap \left\{ Y_n < \frac{x}{1-\epsilon} \right\} \right] \le \min\left(p_n(\epsilon), F_{Y_n}(\frac{x}{1-\epsilon}) \right) .$$

Finally we get:

$$F_{Y_n}(\frac{x}{1+\epsilon}) + p_n(\epsilon) - 1 \le F_{X_n}(x) \text{ and } F_{X_n}(x) + p_n(\epsilon) - 1 \le F_{Y_n}(\frac{x}{1-\epsilon})$$

which is exactly the desired result. □

Proof of corollary 3. The proof rests on theorem 3.2 with $\mathbf{g}(x_1,\ldots,x_r) = x_1^{\alpha_1}\ldots x_r^{\alpha_r}$. We then have:

$$\frac{\partial \mathbf{g}}{\partial x_i}(x_1,\ldots,x_r) = \alpha_i x_i^{\alpha_i-1}\prod_{\substack{p=1\\p\neq i}}^{r} x_p^{\alpha_p}, \qquad \forall i=1,\ldots,r$$

$$\frac{\partial^2 \mathbf{g}}{\partial x_i \partial x_j}(x_1,\ldots,x_r) = \begin{cases} \alpha_i\alpha_j x_i^{\alpha_i-1}x_j^{\alpha_j-1}\displaystyle\prod_{\substack{p=1\\p\neq i,j}}^{r} x_p^{\alpha_p} & \text{if} \quad i\neq j \\ \\ \alpha_i(\alpha_i-1)x_i^{\alpha_i-2}\displaystyle\prod_{\substack{p=1\\p\neq i}}^{r} x_p^{\alpha_p} & \text{if} \quad i=j \end{cases} \qquad \forall i,j=1,\ldots,r$$

and

$$\|D\mathbf{g}(x_1,\ldots,x_r)\|^2 = \alpha_1^2 x_1^{2\alpha_1-2}\prod_{\substack{p=1\\p\neq 1}}^{r} x_p^{2\alpha_p} + \ldots + \alpha_r^2 x_r^{2\alpha_r-2}\prod_{\substack{p=1\\p\neq r}}^{r} x_p^{2\alpha_p}.$$

Hence, if $i\neq j$ then

$$\frac{\left(\frac{\partial^2 \mathbf{g}}{\partial x_i\partial x_j}(x_1,\ldots,x_r)\right)^2}{\|D\mathbf{g}(x_1,\ldots,x_r)\|^2} \leq \frac{\alpha_i^2\alpha_j^2 x_i^{2\alpha_i-2}x_j^{2\alpha_j-2}\displaystyle\prod_{\substack{p=1\\p\neq i,j}}^{r} x_p^{2\alpha_p}}{\alpha_j^2 x_j^{2\alpha_j-2}\displaystyle\prod_{\substack{p=1\\p\neq j}}^{r} x_p^{2\alpha_p}} = \frac{\alpha_i^2\displaystyle\prod_{\substack{p=1\\p\neq j}}^{r} x_p^{2\alpha_p}}{x_i^2\displaystyle\prod_{\substack{p=1\\p\neq j}}^{r} x_p^{2\alpha_p}} = \frac{\alpha_i^2}{x_i^2}$$

which implies,

$$\frac{\left|\frac{\partial^2 \mathbf{g}}{\partial x_i\partial x_j}(\boldsymbol{\mu}_n)\right|}{\|D\mathbf{g}(\boldsymbol{\mu}_n)\|} \leq \frac{|\alpha_i|}{|\mu_{n,i}|} \longrightarrow 0, \quad n\to+\infty, \quad \forall i,j=1,\ldots r$$

Analogously,

$$\frac{\left(\frac{\partial^2 \mathbf{g}}{\partial x_i^2}(x_1,\ldots,x_r)\right)^2}{\|D\mathbf{g}(x_1,\ldots,x_r)\|^2} \leq \frac{\alpha_i^2(\alpha_i-1)^2 x_i^{2\alpha_i-4}\displaystyle\prod_{\substack{p=1\\p\neq i}}^{r} x_p^{2\alpha_p}}{\alpha_i^2 x_i^{2\alpha_i-2}\displaystyle\prod_{\substack{p=1\\p\neq i}}^{r} x_p^{2\alpha_p}} = \frac{(\alpha_i-1)^2}{x_i^2}$$

and thus,

$$\frac{\left|\frac{\partial^2 \mathbf{g}}{\partial x_i^2}(\boldsymbol{\mu}_n)\right|}{\|D\mathbf{g}(\boldsymbol{\mu}_n)\|} \leq \frac{|\alpha_i-1|}{|\mu_{n,i}|} \longrightarrow 0, \quad n\to+\infty, \quad \forall i=1,\ldots r$$

finally showing that:

$$\frac{\left|\left|\left|D^2\mathbf{g}(\boldsymbol{\mu}_n)\right|\right|\right|}{\|D\mathbf{g}(\boldsymbol{\mu}_n)\|} = \frac{1}{\|D\mathbf{g}(\boldsymbol{\mu}_n)\|}\left(\sum_{\substack{i,j=1\\ i\neq j}}^{r}\left|\frac{\partial^2\mathbf{g}}{\partial x_i\partial x_j}(\boldsymbol{\mu}_n)\right|^2 + \sum_{i=1}^{r}\left|\frac{\partial^2\mathbf{g}}{\partial x_i^2}(\boldsymbol{\mu}_n)\right|^2\right)^{\frac{1}{2}} \leq$$

$$\leq \left(\left(\max_{1\leq i\leq r}\alpha_i\right)(2r-1)\sum_{i=1}^{r}\frac{1}{\mu_{n,i}^2}\right)^{\frac{1}{2}}$$

□

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