Relaxation of certain integral functionals arising in imaging models

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Abstract

We provide a relaxation result in $BV \times L^q$, $1 \leq q < +\infty$ for integral decoupled energies arising in image analysis, in the spirit of the total variation decomposition models.
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1 Introduction

Recently new concepts and new targets entered in image analysis for decomposition and restoration: decompose in the RGB model a given image $\phi : \Omega \to \mathbb{R}^3$ ($\Omega \subset \mathbb{R}^2$) into two components u and v, such that $\phi = u + v$ where u is a 'cartoon' representation of ϕ , a simplified approximation describing the three colors decomposition, while v is the oscillatory component, consisting of 'texture' or 'noise'. This new image analysis task has been first formulated in theory by Meyer [19] and the approach, classically adopted in literature (see [20], [22]) is a total variation minimization model first proposed in [21], coupled with the presence of an oscillatory function.

To capture the characteristics of each part in which the given image ϕ is decomposed, the u component, representing a 'cartoon' of the image ϕ is modeled by a function of bounded variation, while the v component representing 'texture' or 'noise' is modeled by an oscillatory function, bounded in some norm, suitable for numerical purposes. Often v is assumed to be an L^2 function.

In particular, in [19] the following minimization problem has been formulated

$$\inf_{(u,v) \in BV \times G \text{ s.t. } \phi = u + v} \{|Du|(\Omega) + \|v\|_G\} \tag{1.1}$$

where G denotes some suitable Banach space, in fact many choices for the space G have been successfully considered by Osher, Solé and Vese (see [20], [22], [23]). A different approach has been considered in [4], [5], and [6] i.e.

$$\inf_{u \in BV, \|v\|_G \leq \mu} \left\{ |Du|(\Omega) + \frac{1}{2\lambda} \|\phi - u - v\|^2_{L^2} \right\} \tag{1.2}$$

where μ represents a bound for the norm of v and λ a scaling factor for the L^2 norm of the fidelity term $\phi - (u + v)$.

Our note is devoted to generalize the above mentioned decomposition models, by considering a formulation more general than (1.1) and (1.2), replacing the total variation term $|Du|(\Omega)$ by the integral $\int_\Omega W(\nabla u(x))\, dx$, with W not necessarily convex, and $u \in W^{1,1}(\Omega; \mathbb{R}^3)$. To this term we add the integral $\int_\Omega \varphi(x, u(x), v(x))\, dx$ which may recover the fidelity term $\|\phi - u - v\|^2_{L^2}$. Considering also more general

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dimensions and including some heterogeneous behaviour in the density W, i.e. dependence on x and u, let Ω be a bounded open set of $\mathbb{R}^N$ and let the functional $I : BV(\Omega;\mathbb{R}^d) \times L^q(\Omega;\mathbb{R}^m) \longrightarrow \mathbb{R} \cup \{+\infty\}$ be defined by

$$I(u,v) = \begin{cases} \displaystyle\int_\Omega W(x,u(x),\nabla u(x))\,dx + \int_\Omega \varphi(x,u(x),v(x))\,dx \\ \qquad\qquad \text{if } (u,v) \in (W^{1,1}(\Omega;\mathbb{R}^d) \cap L^p(\Omega;\mathbb{R}^d)) \times L^q(\Omega;\mathbb{R}^m), \\ +\infty \quad \text{otherwise,} \end{cases} \tag{1.3}$$

where $W : \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times N} \to \mathbb{R}$ is a continuous function with linear growth from above and below in the gradient variable, $\varphi : \Omega \times \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}$ is a Carathéodory function (that is $\varphi(\cdot,u,v)$ is measurable for all $(u,v) \in \mathbb{R}^d \times \mathbb{R}^m$ and $\varphi(x,\cdot,\cdot)$ is continuous for a.e. $x \in \Omega$), with growth p and q respectively in the variables u and v.

In order to deal with such minimization problems, since there may be a lack of lower semicontinuity, it is necessary to pass to the relaxed functional defined in $BV(\Omega;\mathbb{R}^d) \times L^q(\Omega;\mathbb{R}^m)$

$$\overline{I}(u,v) := \inf\left\{\liminf_{n\to+\infty} I(u_n,v_n) : (u_n,v_n) \in BV(\Omega;\mathbb{R}^d) \times L^q(\Omega;\mathbb{R}^m) : u_n \to u \text{ in } L^1, v_n \rightharpoonup v \text{ in } L^q\right\}, \tag{1.4}$$

and prove a representation result for $\overline{I}$.

It is worthwhile to remark that for $q = 1$, the functional $\overline{I}$ may fail to be sequentially lower semicontinuous. However, as we will observe below, this can be achieved provided that φ is uniform continuous, cf. (1.11).

We prove the following theorem.

Theorem 1.1. *Let $p \geq 1$ and $q \geq 1$ and let $\Omega \subset \mathbb{R}^N$ be a bounded open set. Assume that $W : \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times N} \to \mathbb{R}$ is a continuous function, satisfying*

(i) $\exists\, C > 0 : \ \dfrac{1}{C}|\xi| - C \leq W(x,u,\xi) \leq C(1+|\xi|), \ \forall\ (x,u,\xi) \in \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times N};$

(ii) for every compact subset K of $\Omega \times \mathbb{R}^d$ there is a continuous function $\omega_K : [0,+\infty) \to \mathbb{R}$ with $\omega_K(0) = 0$, and such that

$$|W(x,u,\xi) - W(x',u',\xi)| \leq \omega_K(|x-x'| + |u-u'|)(1+|\xi|), \ \forall\ (x,u,\xi),(x',u',\xi) \in K \times \mathbb{R}^{d\times N};$$

(iii) for every $x_0 \in \Omega$ and for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|x - x_0| < \delta \ \Rightarrow \ W(x,u,\xi) - W(x_0,u,\xi) \geq -\varepsilon(1+|\xi|), \ \forall\ (u,\xi) \in \mathbb{R}^d \times \mathbb{R}^{d\times N};$$

(iv) there exist $\alpha \in (0,1)$, and $C, L > 0$ such that

$$t|\xi| > L \ \Rightarrow \ \left|W^\infty(x,u,\xi) - \frac{W(x,u,t\xi)}{t}\right| \leq C\frac{|\xi|^{1-\alpha}}{t^\alpha}, \ \forall\ (x,u,\xi) \in \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times N}, t \in \mathbb{R}.$$

Moreover let $\varphi : \Omega \times \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}$ be a Carathéodory function, satisfying

(v) $\exists\, C > 0 : \ \dfrac{1}{C}(|u|^p + |v|^q) - C \leq \varphi(x,u,v) \leq C(1+|u|^p+|v|^q), \ \forall\ (x,u,v) \in \Omega \times \mathbb{R}^d \times \mathbb{R}^m.$

If I is defined by (1.3) and $\overline{I}$ is defined by (1.4) then, for every $u \in BV(\Omega;\mathbb{R}^d) \cap L^p(\Omega;\mathbb{R}^d)$ and $v \in L^q(\Omega;\mathbb{R}^m)$, the following identity holds:

$$\begin{aligned} \overline{I}(u,v) \ &= \int_\Omega QW(x,u(x),\nabla u(x))\,dx + \int_{J_u} \gamma(x,u^+,u^-,\nu_u)\,d\mathcal{H}^{N-1} + \\ &+ \int_\Omega (QW)^\infty\left(x,u(x),\frac{dD^c u}{d|D^c u|}\right) d|D^c u| + \int_\Omega C\varphi(x,u(x),v(x))\,dx. \end{aligned} \tag{1.5}$$

We also observe that the above result can be recasted also in the framework of relaxation results dealing with integral functionals arising in Nonlinear Elasticity for materials whose behaviour depends on the strain and on the chemical composition, cf. [14, 15].

In order to describe the right hand side of (1.5) we recall that for every $x \in \Omega$, $QW(x,u,\cdot)$ stands for the quasiconvexification of W, cf. (2.1), while $(QW)^\infty$ denotes the recession function of QW with respect to the last variable as introduced in Definition 2.4, and γ stands for the surface integral density, defined in (2.8). Finally for every $(x,u) \in \Omega \times \mathbb{R}^d$, $C\varphi$ stands for the convex envelope (or convexification) of $\varphi(x,u,\cdot)$, namely

$$C\varphi(x,u,\cdot) := \sup\{g : \mathbb{R}^m \to \mathbb{R} : g \text{ convex}, \, g(v) \leq \varphi(x,u,v) \, \forall \, v\}. \tag{1.6}$$

Classical results in Calculus of Variations ensure that, if φ takes only finite values then $C\varphi$ coincides with the bidual of φ, φ^{**}, whose characterization is given below

$$\varphi^{**}(x,u,\cdot) := \sup\{g : \mathbb{R}^m \to \mathbb{R} : g \text{ convex and lower semicontinuous}, \; g(v) \leq \varphi(x,u,v) \, \forall \, v\}. \tag{1.7}$$

Remark 1.2.

- *We observe that in the Sobolev setting, Theorem 1.1 can be proven without coercivity assumptions on φ, indeed let $f : \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times N} \times \mathbb{R}^m \to \mathbb{R}$ be a Carathéodory function satisfying*

  $$0 \leq f(x,u,\xi,v) \leq C\left(1 + |u|^p + |\xi|^p + |v|^q\right)$$

  for a.e. $x \in \Omega$, for every $(u,\xi,v) \in \mathbb{R}^d \times \mathbb{R}^{d\times N} \times \mathbb{R}^m$ and for some $C > 0$. Consider for every $1 \leq p, q < +\infty$ the following relaxed localized energy

  $$\mathcal{F}(u,v;A) :=$$

  $$\inf\left\{\liminf_{n\to\infty} \int_A f(x, u_n(x), \nabla u_n(x), v_n(x))\, dx : u_n \rightharpoonup u \text{ in } W^{1,p}\left(A;\mathbb{R}^d\right), \; v_n \rightharpoonup v \text{ in } L^q\left(A;\mathbb{R}^m\right)\right\}. \tag{1.8}$$

  Then, in [10, Theorem 1.1](cf. also [9]) it has been proven that, for every $u \in W^{1,p}\left(\Omega;\mathbb{R}^d\right)$, $v \in L^q\left(\Omega;\mathbb{R}^m\right)$ and $A \in \mathcal{A}(\Omega)$,

  $$\mathcal{F}(u,v;A) = \int_A QCf(x,u(x),\nabla u(x),v(x))\, dx,$$

  where QCf stands for the quasiconvex-convex envelope of f with respect to the last two variables, namely

  $$QCf(x,u,\xi,v) =$$

  $$\inf\left\{\frac{1}{|D|}\int_D f(x,u,\xi+\nabla\varphi(y), v+\eta(y))\, dy : \; \varphi \in W_0^{1,\infty}(D;\mathbb{R}^d), \eta \in L^\infty(D;\mathbb{R}^m), \int_D \eta(y)\, dy = 0\right\}, \tag{1.9}$$

  D being any bounded open set. Clearly this equality recovers our setting, since it suffices to define for every $(x,u,\xi,v) \in \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times N} \times \mathbb{R}^m$, $f(x,u,\xi,v) := W(x,u,\xi) + \varphi(x,u,v)$. In fact it is easily seen that if f satisfies the above growth assumptions, then

  $$QCf(x,u,\xi,v) = QW(x,u,\xi) + C\varphi(x,u,v).$$

- *We emphasize that the arguments adopted to prove the previous theorem strongly rely on the fact that the energy densities are decoupled. In particular, in the case $q = 1$, we will approximate the* *functional I by adding an extra term with superlinear growth at ∞ in the v variable. This will ensure the sequentially weak lower semicontinuity of the relaxed approximating functional*

$$\overline{I_\varepsilon}(u,v) := \inf\left\{\liminf_n \int_\Omega W(x,u_n,\nabla u_n)\,dx + \int_\Omega (\varphi(x,u,v)+\varepsilon\theta(|v|))\,dx : \right.$$
$$\left. u_n \to u \text{ in } L^1, v_n \rightharpoonup v \text{ in } L^1\right\},$$

allowing us to adopt arguments similar to those exploited in the proof for the case $q > 1$. These techniques are well suited for the convex setting but we are not aware if a similar procedure is possible in the quasiconvex-convex framework.

Having in mind the continuous embedding of $BV(\Omega;\mathbb{R}^d)$ in $L^{\frac{N}{N-1}}(\Omega;\mathbb{R}^d)$ (assuming $\Omega \subset \mathbb{R}^N$), we can obtain, in an easier way, the relaxation result as above. Indeed we can prove the following result.

Theorem 1.3. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set, and let $1 \le p \le \frac{N}{N-1}$ and $q \ge 1$. Let $W : \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times N} \to \mathbb{R}$ be a continuous function satisfying $(i)\div(iv)$ of Theorem 1.1. Moreover let $\varphi : \Omega\times\mathbb{R}^d\times\mathbb{R}^m \to \mathbb{R}$ be a Carathéodory function satisfying (v) of Theorem 1.1 in the weaker form*

$$\exists\, C>0 : \ \frac{1}{C}|v|^q - C \le \varphi(x,u,v) \le C(1+|u|^p+|v|^q), \ \forall\,(x,u,v) \in \Omega\times\mathbb{R}^d\times\mathbb{R}^m. \tag{1.10}$$

Then, for every $(u,v) \in BV(\Omega;\mathbb{R}^d)\times L^q(\Omega;\mathbb{R}^m)$, (1.5) holds.

Remark 1.4. *The continuous embedding of $BV(\Omega;\mathbb{R}^d)$ into $L^{\frac{N}{N-1}}(\Omega;\mathbb{R}^d)$ (with $\Omega \subset \mathbb{R}^N$) allows us to have the just established result, also replacing (1.10) by the following condition:*

$$\exists\, C>0 : \ \frac{1}{C}|v|^q - C \le \varphi(x,u,v) \le C(1+|u|^r+|v|^q), \ \forall\,(x,u,v) \in \Omega\times\mathbb{R}^d\times\mathbb{R}^m$$

and for some $r \in \left[1, \frac{N}{N-1}\right]$.

We observe that under assumptions $(i)\div(iv)$ of Theorem 1.1, [17, Theorem 2.16] ensures that the functional

$$\int_\Omega QW(x,u,\nabla u)\,dx + \int_{J_u} \gamma(x,u^+,u^-,\nu_u)\,d\mathcal{H}^{N-1} + \int_\Omega (QW)^\infty\left(x,u,\frac{dD^cu}{d|D^cu|}\right)d|D^cu|$$

is lower semicontinuous with respect to the strong-L^1 topology. Moreover [16, Theorem 7.5] guarantees that

$$\int_\Omega C\varphi(x,u,v)\,dx$$

is sequentially weakly lower semicontinuous with respect to $L^1_{\text{strong}}\times L^1_{\text{weak}}$-topology provided the function $C\varphi$ is convex in the last variable, satisfies suitable growth conditions, as those in (2.5) and (2.6), and that the function $C\varphi(x,\cdot,\cdot)$ is lower semicontinuous. We will observe in Remark 2.3 below that this latter condition may not be verified just under the assumptions of Theorems 1.1 and 1.3. On the other hand an argument entirely similar to [11, Theorem 9.5] guarantees that $C\varphi(x,\cdot,\cdot)$ is lower semicontinuous (even continuous) by assuming additionally that

$$|\varphi(x,u,\xi)-\varphi(x,u',\xi)| \le \omega'(|u-u'|)(|\xi|+1) \tag{1.11}$$

for a suitable modulus of continuity ω', i.e. $\omega' : \mathbb{R}^+\cup\{0\} \to \mathbb{R}^+\cup\{0\}$ continuous and such that $\omega'(0)=0$.

Consequently the superadditivity of $\liminf$ entails the sequentially strong-weak lower semicontinuity of the right hand side of (1.5) even for $q=1$.

2 Notations and General Facts

2.1 Properties of the integral density functions

In this subsection we recall several notions applied to functions like quasiconvexity, envelopes and recession function, etc. We also recall or prove properties of those functions that will be useful through the paper. Such notions and related properties will apply to the density functions that will appear in the relaxed functionals that we characterize. We start recalling the notion of quasiconvex function due to Morrey.

Definition 2.1. *A Borel measurable function $h : \mathbb{R}^{d\times N} \to \mathbb{R}$ is said to be quasiconvex if there exists a bounded open set D of $\mathbb{R}^N$ such that*

$$h(\xi) \leq \frac{1}{|D|}\int_D h(\xi + \nabla\varphi(x))\, dx,$$

for every $\xi \in \mathbb{R}^{d\times N}$, and for every $\varphi \in W^{1,\infty}_0(D;\mathbb{R}^d)$.

If $h : \mathbb{R}^{d\times N} \to \mathbb{R}$ is any given Borel measurable function bounded from below, it can be defined the quasiconvex envelope of h, that is the largest quasiconvex function below h:

$$Qh(\xi) := \sup\{g(\xi) :\ g \leq h,\ g \text{ quasiconvex}\}.$$

Moreover, as well known (see the monograph [11]),

$$Qh(\xi) := \inf\left\{\frac{1}{|D|}\int_D h(\xi + \nabla\varphi(x))\, dx :\ \varphi \in W^{1,\infty}_0(D;\mathbb{R}^d)\right\}, \tag{2.1}$$

for any bounded open set $D \subset \mathbb{R}^N$.

Proposition 2.2. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set and*

$$W : \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times N} \to [0, +\infty)$$

be a continuous function. Let QW be the quasiconvexification of W (see (2.1)). Then the validity of (i) in Theorem 1.1 guarantees that there exists a constant $C > 0$ such that

$$\frac{1}{C}|\xi| - C \leq QW(x,u,\xi) \leq C(1 + |\xi|),\ \forall\ (x,u,\xi) \in \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times N}. \tag{2.2}$$

The validity of (i) and (ii) of Theorem 1.1 ensures that for every compact set $K \subset \Omega \times \mathbb{R}^d$, there exists a continuous function $\omega'_K : \mathbb{R} \to [0, +\infty)$ such that $\omega'_K(0) = 0$ and

$$|QW(x,u,\xi) - QW(x',u',\xi)| \leq \omega'_K(|x - x'| + |u - u'|)(1 + |\xi|),\ \forall\ (x,u), (x',u') \in K,\ \forall\ \xi \in \mathbb{R}^{d\times N}. \tag{2.3}$$

Conditions (i) and (iii) of Theorem 1.1 entail that, for every $x_0 \in \Omega$ and $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|x - x_0| \leq \delta\ \Rightarrow\ \forall\ (u,\xi) \in \mathbb{R}^d \times \mathbb{R}^{d\times N}\ \ QW(x,u,\xi) - QW(x_0,u,\xi) \geq -\varepsilon(1 + |\xi|). \tag{2.4}$$

Moreover, if W satisfies conditions (i) and (ii) of Theorem 1.1, QW is a continuous function.

Remark 2.3. *Analogous arguments entail that, under hypothesis (v) of Theorems 1.1 and 1.3, respectively,*

$$\exists\, C > 0 :\ \frac{1}{C}(|u|^p + |v|^q) - C \leq C\varphi(x,u,v) \leq C(1 + |u|^p + |v|^q),\ \ \forall\, (x,u,v) \in \Omega \times \mathbb{R}^d \times \mathbb{R}^m, \tag{2.5}$$

and

$$\exists\, C > 0 :\ \frac{1}{C}|v|^q - C \leq C\varphi(x,u,v) \leq C(1 + |u|^p + |v|^q),\ \ \forall\, (x,u,v) \in \Omega \times \mathbb{R}^d \times \mathbb{R}^m. \tag{2.6}$$

On the other hand we emphasize that being φ as in Theorems 1.1 and 1.3, namely a Carathéodory function, this is not enough to guarantee that $C\varphi$ is still a Carathéodory function, cf. Example 9.6 in [11] and Example 7.14 in [16]. In particular $C\varphi$ turns out to be measurable in x, upper semicontinuous in u, convex and hence continuous in ξ. Furthermore if $q > 1$, [12, Lemma 4.3] guarantees that $C\varphi(x,\cdot,\cdot)$ is lower semicontinuous.

Proof. By definition of the quasiconvex envelope of W, it is easily seen that (i) of Theorem 1.1 entails (2.2) with the same constant appearing in (i).

Next we prove (2.3). Let K be a compact set in $\Omega \times \mathbb{R}^d$ and take $(x,u),(x',u') \in K$. Let $\varepsilon > 0$, then using condition (2.1), we find $\varphi_\varepsilon \in W_0^{1,\infty}(Q;\mathbb{R}^d)$, Q being the unitary cube, such that

$$QW(x,u,\xi) \geq -\varepsilon + \int_Q W(x,u,\xi + \nabla\varphi_\varepsilon(y))\,dy.$$

Now, we observe that, by virtue of the coercivity condition expressed by (i) of Theorem 1.1 and by (2.2), it follows that

$$\|\xi + \nabla\varphi_\varepsilon\|_{L^1} \leq c(1+|\xi|).$$

By condition (ii) of Theorem 1.1, for every $(x,u),(x',u') \in K$ and for every $\xi \in \mathbb{R}^{d\times N}$, it results

$$|W(x,u,\xi) - W(x',u',\xi)| \leq \omega_K(|x-x'| + |u-u'|)(1+|\xi|).$$

Then we can write the following chain of inequalities:

$$\begin{aligned} QW(x,u,\xi) &\geq -\varepsilon + \int_Q W(x,u,\xi+\nabla\varphi_\varepsilon(y))\,dy \\ &\geq -\varepsilon - \int_Q \lambda(y)\,dy + \int_Q W(x',u',\xi+\nabla\varphi_\varepsilon(y))\,dy, \end{aligned}$$

where $\lambda(y) := |W(x,u,\xi+\nabla\varphi_\varepsilon(y)) - W(x',u',\xi+\nabla\varphi_\varepsilon(y))|$. We therefore get, from the definition of $QW(x',u',\xi)$, that,

$$\begin{aligned} QW(x',u',\xi) - QW(x,u,\xi) &\leq \varepsilon + \omega_K(|x-x'|+|u-u'|)(1+\|\xi+\nabla\varphi_\varepsilon\|_{L^1}) \\ &\leq \varepsilon + \omega_K(|x-x'|+|u-u'|)(1+c(1+|\xi|)). \end{aligned}$$

Since ε is arbitrarily chosen, and since we can obtain in a similar way the same inequality with x in the place of x', and u in the place of u', we get (2.3).

In order to prove condition (2.4), we fix $x_0 \in \Omega$ and $\varepsilon > 0$. As before, for every $x \in \Omega$ and for every $\sigma > 0$, by (2.1), the coercivity condition expressed by (i) of Theorem 1.1, and by (2.2), there exist a constant $c > 0$ and a function $\varphi_\sigma \in W_0^{1,\infty}(Q;\mathbb{R}^d)$ such that

$$QW(x,u,\xi) \geq -\sigma + \int_Q W(x,u,\xi+\nabla\varphi_\sigma(y))\,dy,$$

with $\|\xi+\nabla\varphi_\sigma\|_{L^1} \leq c(1+|\xi|)$.

Thus arguing as above, and exploiting condition (iii) of Theorem 1.1, we have the following chain of inequalities, for $|x-x_0| < \delta$ with δ as in condition (iii) of Theorem 1.1,

$$\begin{aligned} QW(x_0,u,\xi) &\leq \int_Q W(x_0,u,\xi+\nabla\varphi_\sigma(y))\,dy \\ &\leq \int_Q W(x,u,\xi+\nabla\varphi_\sigma(y))\,dy + \varepsilon\int_Q (1+|\xi+\nabla\varphi_\sigma(y)|)\,dy \\ &\leq QW(x,u,\xi) + \sigma + \varepsilon(1+c(1+|\xi|)). \end{aligned}$$

Thus it suffices to let σ go to 0 in order to achieve the statement.

Finally we prove the continuity of QW. We need to show that, for every $\varepsilon > 0$ and $(x_0,u_0,\xi_0) \in \Omega\times\mathbb{R}^d\times\mathbb{R}^{d\times N}$, there is $\delta \equiv \delta(\varepsilon,x_0,u_0,\xi_0) > 0$ such that

$$|x-x_0| + |u-u_0| + |\xi-\xi_0| \leq \delta \Rightarrow |QW(x,u,\xi) - QW(x_0,u_0,\xi_0)| \leq \varepsilon. \tag{2.7}$$

Let $\varepsilon > 0$ be fixed. Since QW is quasiconvex on ξ, $QW(x_0, u_0, \cdot)$ is continuous and thus we can find $\delta_1 = \delta_1(\varepsilon, x_0, u_0, \xi_0) > 0$ such that

$$|\xi - \xi_0| \leq \delta_1 \;\Rightarrow\; |QW(x_0, u_0, \xi) - QW(x_0, u_0, \xi_0)| \leq \frac{\varepsilon}{2}.$$

Moreover, by virtue of (2.3), defining $K := \overline{B}_\sigma(x_0, u_0)$ for some $\sigma > 0$ such that $K \subset \Omega \times \mathbb{R}^d$, one has

$$|\xi - \xi_0| \leq \delta_1 \;\Rightarrow\; |QW(x, u, \xi) - QW(x_0, u_0, \xi)| \leq \omega'_K(|x - x_0| + |u - u_0|)(1 + |\xi_0| + \delta_1).$$

Since ω'_K is continuous and $\omega'_K(0) = 0$, there is $\delta_2 = \delta_2(\varepsilon, K, \xi_0) > 0$ such that

$$|x - x_0| + |u - u_0| \leq \delta_2 \;\Rightarrow\; \omega'_K(|x - x_0| + |u - u_0|) \leq \frac{\varepsilon}{2(1 + |\xi_0| + 1)}.$$

Consequently, by choosing δ as $\min\{\delta_1, \delta_2\}$, the above inequalities, and the triangular inequality give indeed (2.7). $\square$

We also recall the definition of the recession function.

Definition 2.4. *Let $h : \mathbb{R}^{d\times N} \to [0, +\infty)$. The recession function of h is denoted by $h^\infty : \mathbb{R}^{d\times N} \to [0, +\infty)$, and defined as*

$$h^\infty(\xi) := \limsup_{t\to+\infty} \frac{h(t\xi)}{t}.$$

Remark 2.5. (i) Recall that the recession function is a positively one homogeneous function, that is $g(t\xi) = tg(\xi)$ for every $t \geq 0$ and $\xi \in \mathbb{R}^{d\times N}$.
(ii) Through this paper we will work with functions $W : \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times N} \to [0, +\infty)$ and W^∞ is the recession function with respect to the last variable:

$$W^\infty(x, u, \xi) := \limsup_{t\to+\infty} \frac{W(x, u, t\xi)}{t}.$$

We trivially observe that, if W satisfies the growth condition (i) in Theorem 1.1, then W^∞ satisfies $\frac{1}{C}|\xi| \leq W^\infty(x, u, \xi) \leq C|\xi|$.
(iii) As showed in [17, Remark 2.2 (ii)], if a function $h : \mathbb{R}^{d\times N} \longrightarrow [0, +\infty)$ is quasiconvex and satisfies the growth condition $h(\xi) \leq c(1 + |\xi|)$, for some $c > 0$, then, its recession function is also quasiconvex.

We now describe the surface energy density γ appearing in the characterization of $\overline{I}$. Let $W : \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times N} \to \mathbb{R}$. By the notation above $(QW)^\infty$ is the recession function of the quasiconvex envelope of W. Then $\gamma : \Omega \times \mathbb{R}^n \times \mathbb{R}^n \times S^{N-1} \longrightarrow \mathbb{R}$ is defined by

$$\gamma(x, a, b, \nu) = \inf\left\{\int_{Q_\nu} (QW)^\infty(x, \phi(y), \nabla\phi(y))\, dy : \phi \in \mathcal{A}(a, b, \nu)\right\}, \tag{2.8}$$

where Q_ν is the unit cube centered at the origin with faces parallel to $\nu, \nu_1, \ldots, \nu_{N-1}$, for some orthonormal basis of $\mathbb{R}^N$, $\{\nu_1, \ldots, \nu_{N-1}, \nu\}$, and where

$$\mathcal{A}(a, b, \nu) := \Big\{\phi \in W^{1,1}(Q_\nu, \mathbb{R}^d) : \phi(y) = a \text{ if } < y, \nu >= \frac{1}{2},\ \phi(y) = b \text{ if } < y, \nu >= -\frac{1}{2},$$
$$\phi \text{ is } 1-\text{periodic in the } \nu_1, \ldots, \nu_{N-1} \text{ directions } \Big\}.$$

We observe that the function γ is the same whether we consider in the set $\mathcal{A}(a, b, \nu)$, $W^{1,1}(Q_\nu, \mathbb{R}^d)$ functions (like in [18] and [17]) or $W^{1,\infty}(Q_\nu, \mathbb{R}^d)$ functions (like in [2, page 312]). Moreover, if W doesn't depend on u, $W : \Omega \times \mathbb{R}^{d\times N} \to \mathbb{R}$, then $\gamma(x, a, b, \nu) = (QW)^\infty(x, (a - b) \otimes \nu)$ (see [2, page 313]).

Properties of the function $(QW)^\infty$ will be important to get the integral representation of the relaxed functionals under consideration. In particular, a proof entirely similar to [7, Proposition 3.4] ensures that for every $(x, u, \xi) \in \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times N}$, $Q(W^\infty)(x, u, \xi) = (QW)^\infty(x, u, \xi)$.

Proposition 2.6. *Let $W : \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times N} \to [0, +\infty)$ be a continuous function satisfying (i) and (iv) of Theorem 1.1. Then*

$$Q(W^\infty)(x, u, \xi) = (QW)^\infty(x, u, \xi) \quad \text{for every } (x, u, \xi) \in \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times N}. \tag{2.9}$$

Proof. The proof will be achieved by double inequality.

By definition of the quasiconvex envelope and the recession function, one gets $(QW)^\infty \leq W^\infty$ and thus $Q(QW)^\infty \leq Q(W^\infty)$. Since the recession function of a quasiconvex one is still quasiconvex, under hypothesis (i) of Theorem 1.1 (cf. Remark 2.5 (iii)) it follows that $(QW)^\infty \leq Q(W^\infty)$.

In order to prove the opposite inequality we start noticing that, since by (i), the function W is bounded from below, we can assume without loss of generality that $W \geq 0$. Then fix $(x, u, \xi) \in \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times N}$ and, for every $t > 1$, take $\varphi_t \in W_0^{1,\infty}(Q; \mathbb{R}^d)$ such that

$$\int_Q W(x, u, t\xi + \nabla\varphi_t(y))\, dy \leq QW(x, u, t\xi) + 1. \tag{2.10}$$

By (i) and (2.2) we have that $\left\|\nabla(\frac{1}{t}\varphi_t)\right\|_{L^1(Q)} \leq C$ for a constant independent of t but just on ξ.

Defining $\psi_t = \frac{1}{t}\varphi_t$, one has $\psi_t \in W_0^{1,\infty}(Q; \mathbb{R}^d)$ and thus

$$Q(W^\infty)(x, u, \xi) \leq \int_Q W^\infty(x, u, \xi + \nabla\psi_t(y))\, dy.$$

Let L be the constant appearing in condition (iv) of Theorem 1.1, we split the cube Q in the set $\{y \in Q : \ t|\xi + \nabla\psi_t(y)| \leq L\}$ and its complement in Q. Then we apply condition (iv) and the growth of W^∞ observed in Remark 2.5 (ii) to get

$$Q(W^\infty)(x, u, \xi) \quad \leq \quad \int_Q \left(C\frac{|\xi + \nabla\psi_t|^{1-\alpha}}{t^\alpha} + \frac{W(x, u, t\xi + \nabla\varphi_t(y))}{t} + C\frac{L}{t} \right) dy.$$

Applying Hölder inequality and (2.10), we get

$$Q(W^\infty)(x, u, \xi) \quad \leq \quad \frac{C}{t^\alpha}\left(\int_Q |\xi + \nabla\psi_t|\, dy\right)^{1-\alpha} + \frac{QW(x, u, t\xi) + 1}{t} + C\frac{L}{t},$$

and the desired inequality follows by definition of $(QW)^\infty$ and using the fact that $\nabla\psi_t$ has bounded L^1 norm, letting t go to $+\infty$. $\square$

The property of $(QW)^\infty$ stated next ensures that QW together with $(QW)^\infty$ satisfy the analogous condition to (iv) of Theorem 1.1. To this end we first observe, as emphasized in [17], that (iv) in Theorem 1.1 is equivalent to say that there exist $C > 0$ and $\alpha \in (0, 1)$ such that

$$|W^\infty(x, u, \xi) - W(x, u, \xi)| \leq C(1 + |\xi|^{1-\alpha}) \tag{2.11}$$

for every $(x, u, \xi) \in \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times N}$. Precisely we have the following result.

Proposition 2.7. *Let $W : \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times N} \to [0, +\infty)$ be a continuous function satisfying (i) and (iv) of Theorem 1.1. Then, there exist $\alpha \in (0, 1)$, and $C' > 0$ such that*

$$|(QW)^\infty(x, u, \xi) - QW(x, u, \xi)| \leq C(1 + |\xi|^{1-\alpha}), \ \ \forall\ (x, u, \xi) \in \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times N}.$$

Proof. The thesis will be achieved by double inequality. Let $\alpha \in (0, 1)$ be as in (iv) of Theorem 1.1, see also (2.11). Let $\xi \in \mathbb{R}^{d\times N}$, let Q be the unit cube in $\mathbb{R}^N$ and let c be a positive constant varying from line to line. For every $\varepsilon > 0$ by (2.1), find $\varphi \in W_0^{1,\infty}(Q; \mathbb{R}^d)$ such that

$$QW(x, u, \xi) > \int_Q W(x, u, \xi + \nabla\varphi(y))\, dy - \varepsilon.$$

By (i) of Theorem 1.1 and by (2.2) there exists $c > 0$ such that

$$\|\xi + \nabla\varphi\|_{L^1} \leq c(1 + |\xi|). \tag{2.12}$$

Since by Proposition 2.6 it results

$$(QW)^\infty(x, u, \xi) \leq \int_Q W^\infty(x, u, \xi + \nabla\varphi(y))\, dy,$$

we have

$$(QW)^\infty(x, u, \xi) - QW(x, u, \xi) \leq \int_Q (W^\infty(x, u, \xi + \nabla\varphi(y)) - W(x, u, \xi + \nabla\varphi(y)))\, dy + \varepsilon.$$

Applying (2.11), we obtain

$$\begin{aligned}
(QW)^\infty(x, u, \xi) - QW(x, u, \xi) \quad & \leq \int_Q c\left(1 + |\xi + \nabla\varphi(y)|^{1-\alpha}\right)\, dy + \varepsilon \\
& \leq c\left(1 + \int_Q |\xi + \nabla\varphi(y)|^{1-\alpha} dy\right) + \varepsilon \\
& \leq c + c\left(\int_Q |\xi + \nabla\varphi(y)| dy\right)^{1-\alpha} + \varepsilon \\
& \leq c + c^2(1 + |\xi|^{1-\alpha}) + \varepsilon \\
& \leq C'(1 + |\xi|^{1-\alpha}) + \varepsilon.
\end{aligned}$$

where in the last lines we have applied Holder inequality, (2.12) and we have estimated the term $(1+|\xi|)^{1-\alpha}$ by separating the cases $|\xi| \leq 1$ and $|\xi| > 1$ and summing them up. To conclude this part it suffices to send ε to 0.

In order to prove the opposite inequality we can argue in the same way. Let $\xi \in \mathbb{R}^{d\times N}$. For every $\varepsilon > 0$, by (2.1) and Proposition 2.6 there exists $\psi \in W_0^{1,\infty}(Q; \mathbb{R}^d)$ such that

$$(QW)^\infty(x, u, \xi) > \int_Q W^\infty(x, u, \xi + \nabla\psi(y))\, dy - \varepsilon.$$

Clearly, by (2.2), (i) of Theorem 1.1 and (ii) of Remark 2.5 there exists $C > 0$ such that

$$\|\xi + \nabla\psi\|_{L^1} \leq C|\xi| + \varepsilon. \tag{2.13}$$

By (2.1) it results

$$QW(x, u, \xi) \leq \int_Q W(x, u, \xi + \nabla\psi(y))\, dy,$$

hence

$$QW(x, u, \xi) - (QW)^\infty(x, u, \xi) \leq \int_Q (W(x, u, \xi + \nabla\psi(y)) - W^\infty(x, u, \xi + \nabla\psi(y)))\, dy + \varepsilon.$$

Now, (iv) of Theorem 1.1 in the form (2.11) provide

$$\begin{aligned}
QW(x, u, \xi) - (QW)^\infty(x, u, \xi) \quad & \leq C\int_Q (1 + |\xi + \nabla\psi(y)|^{1-\alpha})\, dy + \varepsilon \leq \\
& \leq C'(1 + |\xi|^{1-\alpha}) + \varepsilon,
\end{aligned}$$

where in the last line it has been used Holder inequality, (2.13) and an argument entirely similar to the first part of the proof. By sending ε to 0 we conclude the proof. $\square$

2.2 Some Results on Measure Theory and BV Functions

Let Ω be a generic open subset of $\mathbb{R}^N$, we denote by $\mathcal{M}(\Omega)$ the space of all signed Radon measures in Ω with bounded total variation. By the Riesz Representation Theorem, $\mathcal{M}(\Omega)$ can be identified to the dual of the separable space $\mathcal{C}_0(\Omega)$ of continuous functions on Ω vanishing on the boundary $\partial\Omega$. The N-dimensional Lebesgue measure in $\mathbb{R}^N$ is designated as $\mathcal{L}^N$ while $\mathcal{H}^{N-1}$ denotes the $(N-1)$-dimensional Hausdorff measure. If $\mu \in \mathcal{M}(\Omega)$ and $\lambda \in \mathcal{M}(\Omega)$ is a nonnegative Radon measure, we denote by $\frac{d\mu}{d\lambda}$ the Radon-Nikodým derivative of μ with respect to λ. By a generalization of the Besicovich Differentiation Theorem (see [1, Proposition 2.2]), it can be proved that there exists a Borel set $E \subset \Omega$ such that $\lambda(E) = 0$ and

$$\frac{d\mu}{d\lambda}(x) = \lim_{\rho\to 0^+} \frac{\mu(x+\rho\, C)}{\lambda(x+\rho\, C)} \quad \text{for all } x \in \text{Supp }\ \mu \setminus E \tag{2.14}$$

and any open convex set C containing the origin. (Recall that the set E is independent of C.)

We say that $u \in L^1(\Omega;\mathbb{R}^d)$ is a function of bounded variation, and we write $u \in BV(\Omega;\mathbb{R}^d)$, if all its first distributional derivatives $D_j u_i$ belong to $\mathcal{M}(\Omega)$ for $1 \le i \le d$ and $1 \le j \le N$. We refer to [2] for a detailed analysis of BV functions. The matrix-valued measure whose entries are $D_j u_i$ is denoted by Du and $|Du|$ stands for its total variation. By the Lebesgue Decomposition Theorem we can split Du into the sum of two mutually singular measures $D^a u$ and $D^s u$ where $D^a u$ is the absolutely continuous part of Du with respect to the Lebesgue measure $\mathcal{L}^N$, while $D^s u$ is the singular part of Du with respect to $\mathcal{L}^N$. By ∇u we denote the Radon-Nikodým derivative of $D^a u$ with respect to the Lebesgue measure so that we can write

$$Du = \nabla u \mathcal{L}^N + D^s u.$$

The set S_u of points where u does not have an approximate limit is called the approximated discontinuity set, while $J_u \subseteq S_u$ is the so called jump set of u defined as the set of points $x \in \Omega$ such that there exist $u^\pm(x) \in \mathbb{R}^d$ (with $u^+(x) \neq u^-(x)$) and $\nu_u(x) \in \mathbb{S}^{N-1}$ satisfying

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^N} \int_{\{y\in B_\varepsilon(x):(y-x)\cdot\nu_u(x)>0\}} |u(y) - u^+(x)|\, dy = 0,$$

and

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^N} \int_{\{y\in B_\varepsilon(x):(y-x)\cdot\nu_u(x)<0\}} |u(y) - u^-(x)|\, dy = 0.$$

It is known that J_u is a countably $\mathcal{H}^{N-1}$-rectifiable Borel set. By Federer-Vol'pert Theorem (see Theorem 3.78 in [2]), $\mathcal{H}^{N-1}(S_u \setminus J_u) = 0$ for any $u \in BV(\Omega;\mathbb{R}^d)$. The measure $D^s u$ can in turn be decomposed into the sum of a jump part and a Cantor part defined by $D^j u := D^s u \llcorner J_u$ and $D^c u := D^s u \llcorner (\Omega \setminus S_u)$. We now recall the decomposition of Du:

$$Du = \nabla u \mathcal{L}^N + (u^+ - u^-) \otimes \nu_u \mathcal{H}^{N-1} \llcorner J_u + D^c u.$$

The three measures above are mutually singular. If $\mathcal{H}^{N-1}(B) < +\infty$, then $|D^c u|(B) = 0$ and there exists a Borel set E such that

$$\mathcal{L}^N(E) = 0,\ |D^c u|(X) = |D^c u|(X \cap E)$$

for all Borel sets $X \subseteq \Omega$.

3 Relaxation

This section is devoted to the proof of the integral representation results dealing with the decoupled models described in the introduction.

To prove Theorems 1.1 and 1.3 we will use the characterization for the relaxed functional of $I_W : L^1(\Omega;\mathbb{R}^d) \longrightarrow \mathbb{R} \cup \{+\infty\}$ defined by

$$I_W(u) := \begin{cases} \displaystyle\int_\Omega W(x, u(x), \nabla u(x))\, dx & \text{if } u \in W^{1,1}(\Omega;\mathbb{R}^d), \\ +\infty & \text{otherwise.} \end{cases} \tag{3.1}$$

The relaxed functional of I_W is defined by

$$\overline{I_W}(u) := \inf \Big\{ \liminf_n I_W(u_n) : \ u_n \in BV(\Omega;\mathbb{R}^d), \ u_n \to u \text{ in } L^1 \Big\}$$

and it was characterized by Fonseca-Müller in [17], provided (among other hypotheses) that W is quasiconvex. In the next lemma we establish conditions to obtain the representation of I_W in the general case, that is, with W not necessarily quasiconvex.

We will also use the following notation. The functional $I_{QW} : L^1(\Omega;\mathbb{R}^d) \longrightarrow \mathbb{R} \cup \{+\infty\}$ is defined by

$$I_{QW}(u) := \begin{cases} \displaystyle\int_\Omega QW(x,u(x),\nabla u(x))\,dx & \text{if } u \in W^{1,1}(\Omega;\mathbb{R}^d), \\ +\infty & \text{otherwise,} \end{cases} \tag{3.2}$$

and its relaxed functional is

$$\overline{I_{QW}}(u) := \inf \Big\{ \liminf_n I_{QW}(u_n) : \ u_n \in BV(\Omega;\mathbb{R}^d), \ u_n \to u \text{ in } L^1 \Big\}.$$

We are now in position to establish the mentioned lemma and we notice that we make no assumptions on the quasiconvexified function QW.

Lemma 3.1. *Let $W : \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times N} \longrightarrow [0,+\infty)$ be a continuous function and consider the functionals I_W and I_{QW} and their corresponding relaxed functionals defined as above. Then, if W satisfies conditions (i) $\div$ (iv) of Theorem 1.1, the two relaxed functionals coincide in $BV(\Omega,\mathbb{R}^d)$ and moreover*

$$\overline{I_W}(u) = \overline{I_{QW}}(u) = \int_\Omega QW(x,u(x),\nabla u(x))\,dx + \int_{J_u} \gamma(x,u^+,u^-,\nu_u)\,d\mathcal{H}^{N-1} + \\ + \int_\Omega (QW)^\infty\left(x,u(x),\frac{dD^c u}{d|D^c u|}\right) d|D^c u|.$$

Proof. First we observe that $\overline{I_W}(u) = \overline{I_{QW}}(u)$, for every $u \in BV(\Omega;\mathbb{R}^d)$. Indeed, since $QW \le W$, it results $\overline{I_{QW}} \le \overline{I_W}$. Next we prove the opposite inequality in the nontrivial case that $\overline{I_{QW}}(u) < +\infty$. For fixed $\delta > 0$, we can consider $u_n \in W^{1,1}(\Omega;\mathbb{R}^d)$ with $u_n \to u$ strongly in $L^1(\Omega;\mathbb{R}^d)$ and such that

$$\overline{I_{QW}}(u) \ge \lim_n \int_\Omega QW(x,u_n(x),\nabla u_n(x))\,dx - \delta.$$

Applying [11, Theorem 9.8], for each n there exists a sequence $\{u_{n,k}\}$ converging to u_n weakly in $W^{1,1}(\Omega;\mathbb{R}^d)$ such that

$$\int_\Omega QW(x,u_n(x),\nabla u_n(x))\,dx = \lim_k \int_\Omega W(x,u_{n,k}(x),\nabla u_{n,k}(x))\,dx.$$

Consequently

$$\overline{I_{QW}}(u) \ge \lim_n \lim_k \int_\Omega W(x,u_{n,k}(x),\nabla u_{n,k}(x))\,dx - \delta, \tag{3.3}$$

and

$$\lim_n \lim_k \|u_{n,k} - u\|_{L^1} = 0.$$

Via a diagonal argument, there exists a sequence $\{u_{n,k_n}\}$ satisfying $u_{n,k_n} \to u$ in $L^1(\Omega;\mathbb{R}^d)$ and realizing the double limit in the right hand side of (3.3). Thus, it results

$$\overline{I_{QW}}(u) \ge \lim_n \int_\Omega W(x,u_{n,k_n}(x),\nabla u_{n,k_n}(x))\,dx - \delta \ge \overline{I_W}(u) - \delta.$$

Letting δ go to 0 the conclusion follows.

Finally we prove the integral representation for $\overline{I_{QW}}$ and consequently for $\overline{I_W}$. To this end we invoke [17, Theorem 2.16] (see also [2, Theorem 5.54]).

By the hypotheses, and by Proposition 2.2 above, QW satisfies conditions (H1), (H2), (H3) and (H4) in [17], and condition $(H5)$ follows from Proposition 2.7. Applying [17, Theorem 2.16] we conclude the proof. $\square$

Let $I_{QW+\varphi} : BV(\Omega;\mathbb{R}^d) \times L^q(\Omega;\mathbb{R}^m) \to \mathbb{R} \cup \{+\infty\}$ be the functional defined by

$$I_{QW+\varphi}(u,v) := \begin{cases} \displaystyle\int_\Omega QW(x,u(x),\nabla u(x))\,dx + \int_\Omega \varphi(x,u(x),v(x))\,dx \\ \qquad\qquad \text{if } (u,v) \in (W^{1,1}(\Omega;\mathbb{R}^d) \cap L^p(\Omega;\mathbb{R}^d)) \times L^q(\Omega;\mathbb{R}^m), \\ +\infty \quad \text{otherwise,} \end{cases} \tag{3.4}$$

and its relaxed functional as

$$\begin{aligned} &\overline{I_{QW+\varphi}}(u,v) := \\ &\inf\{\liminf_n I_{QW+\varphi}(u_n,v_n) : (u_n,v_n) \in BV(\Omega;\mathbb{R}^d) \times L^q(\Omega;\mathbb{R}^m), u_n \to u \text{ in } L^1, v_n \rightharpoonup v \text{ in } L^q\}. \end{aligned} \tag{3.5}$$

We can obtain, as in the first part of the proof of Lemma 3.1, the following result.

Corollary 3.2. *Let $p \geq 1$, $q \geq 1$ and let $\Omega \subset \mathbb{R}^N$. Assume $W : \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times N} \to [0,+\infty)$ and $\varphi : \Omega \times \mathbb{R}^d \times \mathbb{R}^m \to [0,+\infty)$ satisfying (i) $\div$ (iv) of Theorem 1.1 and (v) of Theorem 1.1 respectively. Let I and $\overline{I}$ be defined by* (1.3) *and* (1.4) *respectively. Let $I_{QW+\varphi}$ and $\overline{I_{Qw+\varphi}}$ be as in* (3.4) *and* (3.5) *respectively, then*

$$\overline{I}(u,v) = \overline{I_{QW+\varphi}}(u,v)$$

for every $(u,v) \in BV(\Omega;\mathbb{R}^d) \times L^q(\Omega;\mathbb{R}^m)$.

Remark 3.3. *We observe that, in the case $1 \leq p < +\infty$, $1 < q < \infty$, given $W : \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times N} \to \mathbb{R}$ and $\varphi : \Omega \times \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}$, Carathéodory functions satisfying (i) and (v) of Theorem 1.1 respectively, then, if one can provide that $C\varphi$ is still Carathéodory, an argument entirely similar to the first part of Lemma 3.1, entails that*

$$\overline{I}(u,v) = \inf\left\{\liminf_{n\to+\infty} \int_\Omega (QW(x,u_n,\nabla u_n) + C\varphi(x,u_n,v_n))\,dx : \right.$$
$$\left. (u_n,v_n) \in BV(\Omega;\mathbb{R}^d) \times L^q(\Omega;\mathbb{R}^m), u_n \to u \text{ in } L^1, v_n \rightharpoonup v \text{ in } L^q\right\}$$

where $\overline{I}$ is the functional defined by (1.4)*, QW and $C\varphi$ are defined in* (2.1) *and* (1.6)*. But we emphasize that since, assuming only (v) of Theorem 1.1 there may be a lack of continuity of $C\varphi(x,\cdot,\cdot)$ as observed in Remark 2.3, we focus just on the relaxation of the term $\int_\Omega W(x,u,\nabla u)\,dx$ and we prove Lemma 3.1 (see also Corollary 3.2) in order to be allowed to assume W quasiconvex without loosing generality.*

We are now in position to prove Theorem 1.1.

Proof of Theorem 1.1. The proof is divided in two parts. First we consider the case $q > 1$ and then we consider $q = 1$. In both cases we first prove a lower bound for the relaxed energy $\overline{I}$ and then we prove that the lower bound obtained is also an upper bound for $\overline{I}$.

Preliminarly we observe that by virtue of Corollary 3.2, Propositions 2.2, 2.6, 2.7 we can assume without loss of generality, that W is quasiconvex in the last variable.

Part 1: $q > 1$.
Lower bound. Let $u \in BV(\Omega;\mathbb{R}^d) \cap L^p(\Omega;\mathbb{R}^d)$ and let $v \in L^q(\Omega;\mathbb{R}^m)$. We will prove that, for any sequences $u_n \in BV(\Omega;\mathbb{R}^d)$ and $v_n \in L^q(\Omega;\mathbb{R}^m)$ such that $u_n \to u$ in L^1 and $v_n \rightharpoonup v$ in L^q,

$$\liminf_{n\to+\infty} I(u_n,v_n) \geq \int_\Omega W(x,u,\nabla u)\,dx + \int_{J_u} \gamma(x,u^+,u^-,\nu_u)\,d\mathcal{H}^{N-1} + \int_\Omega W^\infty\left(x,u,\frac{dD^c u}{d|D^c u|}(x)\right)d|D^c u| +$$
$$+ \int_\Omega C\varphi(x,u,v)\,dx.$$

Let u_n and v_n be two sequences in the conditions described above. Then, by [17, Theorem 2.16]

$$\begin{aligned}&\int_\Omega W(x,u,\nabla u)\,dx+\int_{J_u}\gamma(x,u^+,u^-,\nu_u)\,d\mathcal{H}^{N-1}+\int_\Omega W^\infty\left(x,u,\frac{dD^cu}{d|D^cu|}\right)d|D^cu|\le\\&\le\liminf_{n\to+\infty}\int_\Omega W(x,u_n,\nabla u_n)\,dx.\end{aligned}\tag{3.6}$$

Moreover, since we can assume $\liminf_n\int_\Omega\varphi(x,u_n,v_n)\,dx<+\infty$, the bound on $\|u_n\|_{L^p}$ provided by (v), the fact that $u_n\to u$ in $L^1(\Omega)$ and consequently pointwise, guarantee that $u_n\to u$ strongly in L^p. Furthermore $v_n\rightharpoonup v$ weakly in L^q and because of the lower semi-continuity of $C\varphi(x,\cdot,\cdot)$ (cf. [12, Lemma 4.3]), it results (cf. [16, Theorem 7.5] or [13])

$$\int_\Omega C\varphi(x,u,v)\,dx\le\liminf_{n\to+\infty}\int_\Omega C\varphi(x,u_n,v_n)\,dx\le\liminf_{n\to+\infty}\int_\Omega\varphi(x,u_n,v_n)\,dx.\tag{3.7}$$

Consequently, the superadditivity of the lim inf, gives the desired lower bound.

Upper bound. Let $u\in BV(\Omega;\mathbb{R}^d)\cap L^p(\Omega;\mathbb{R}^d)$ and $v\in L^q(\Omega;\mathbb{R}^m)$. We will prove that

$$\begin{aligned}\overline{I}(u,v)\quad&\le\int_\Omega W(x,u,\nabla u)\,dx+\int_{J_u}\gamma(x,u^+,u^-,\nu_u)\,d\mathcal{H}^{N-1}+\\&+\int_\Omega W^\infty\left(x,u,\frac{dD^cu}{d|D^cu|}\right)d|D^cu|+\int_\Omega C\varphi(x,u,v)\,dx\end{aligned}\tag{3.8}$$

constructing convenient sequences $u_n\in BV(\Omega;\mathbb{R}^d)$ such that $u_n\to u$ in L^1 and $v_n\in L^q(\Omega,\mathbb{R}^m)$ such that $v_n\rightharpoonup v$ in L^q.

We can assume, without loss of generality, that

$$\begin{aligned}&\int_\Omega W(x,u,\nabla u)\,dx+\int_{J_u}\gamma(x,u^+,u^-,\nu_u)\,d\mathcal{H}^{N-1}+\int_\Omega W^\infty\left(x,u,\frac{dD^cu}{d|D^cu|}\right)d|D^cu|+\\&+\int_\Omega C\varphi(x,u,v)\,dx<+\infty.\end{aligned}\tag{3.9}$$

In particular, from (v) it follows that $u\in L^p(\Omega;\mathbb{R}^d)$.

Moreover we suppose, without loss of generality, that $W\ge0$ and $\varphi\ge0$. We will consider two cases.

Case 1: $u\in L^\infty(\Omega;\mathbb{R}^d)$.

Fix $M\in\mathbb{N}$. We will prove that, for some constant c (independent of M),

$$\begin{aligned}\overline{I}(u,v)\quad&\le\int_\Omega W(x,u,\nabla u)\,dx+\int_{J_u}\gamma(x,u^+,u^-,\nu_u)\,d\mathcal{H}^{N-1}+\int_\Omega W^\infty\left(x,u,\frac{dD^cu}{d|D^cu|}\right)d|D^cu|+\\&+\int_\Omega C\varphi(x,u,v)\,dx+\frac{c}{M}.\end{aligned}$$

Then we get the desired inequality by letting M go to $+\infty$.

We proceed in three steps.

Case 1, step 1: construction of a convenient sequence converging to u in $L^1(\Omega;\mathbb{R}^d)$.

Let $\{u_n\}$ be a sequence in $W^{1,1}(\Omega;\mathbb{R}^d)$ such that $u_n\to u$ in L^1 and

$$\begin{aligned}&\lim\int_\Omega W(x,u_n,\nabla u_n)\,dx=\\&=\int_\Omega W(x,u,\nabla u)\,dx+\int_{J_u}\gamma(x,u^+,u^-,\nu_u)\,d\mathcal{H}^{N-1}+\int_\Omega W^\infty\left(x,u,\frac{dD^cu}{d|D^cu|}\right)d|D^cu|.\end{aligned}$$

This exists by [17, Theorem 2.16]. Next we will truncate the sequence u_n.
Fix k such that $e^k - 1 > 2||u||_{L^\infty}$. Then, hypothesis (3.9) together with the coercivity condition of W on ξ, cf. (i), and the fact that $\varphi \geq 0$, imply that $\sup ||\nabla u_n||_{L^1}$ is bounded by a constant independent of the sequence u_n. Thus

$$\begin{aligned}\sum_{i=0}^{M-1} \int_{\{x\in\Omega:\ k+i\leq \ln(1+|u_n|)<k+i+1\}} (1+|\nabla u_n|)\,dx &= \int_{\{x\in\Omega:\ k\leq \ln(1+|u_n|)<k+M\}} (1+|\nabla u_n|)\,dx \\ &\leq |\Omega| + \sup_n ||\nabla u_n||_{L^1}\end{aligned}$$

and so, for each $n \in \mathbb{N}$, we can find $i = i(n) \in \{0, ..., M-1\}$ such that

$$\int_{\{x\in\Omega:\ k+i\leq \ln(1+|u_n|)<k+i+1\}} (1+|\nabla u_n(x)|)\,dx \leq \frac{|\Omega| + \sup_n ||\nabla u_n||_{L^1}}{M}. \tag{3.10}$$

For each n, and accordingly to the previous choice of $i(n)$, consider $\tau_n : \mathbb{R}_0^+ \longrightarrow [0,1]$ such that $\tau_n \in C^1(\mathbb{R}_0^+)$, $|\tau_n'| \leq 1$,

$$\tau_n(t) = 1, \text{ if } 0 \leq t < k+i(n) \qquad \text{and} \qquad \tau_n(t) = 0, \text{ if } t \geq k+i(n)+1.$$

We can now define the truncated sequence. Let $g_n(z) := \tau_n(\ln(1+|z|))\,z$, and $\overline{u}_n(x) = g_n(u_n)$. Since in a neighborhood of 0 the function $\tau_n(\ln(1+|\cdot|))$ is identically 1, g_n is a Lipschitz, C^1 function with

$$\nabla g_n(z) = \begin{cases} \tau_n(\ln(1+|z|))\,\mathbb{I} + \tau_n'(\ln(1+|z|))\frac{1}{1+|z|}\frac{z\otimes z}{|z|}, \text{ if } z \neq 0 \\ \mathbb{I}, \text{ if } z = 0 \end{cases}$$

and $|\nabla g_n(z)| \leq c$. So, by Theorem 3.96 in [2], $\overline{u}_n \in W^{1,1}(\Omega;\mathbb{R}^d)$, $\nabla\overline{u}_n = \nabla g_n(u_n)\nabla u_n \mathcal{L}^N$ and $|\nabla\overline{u}_n| \leq c|\nabla u_n|$ which is bounded in L^1 as observed above. Moreover $||\overline{u}_n||_{L^\infty} \leq e^{k+i(n)+1} - 1 \leq e^{k+M} - 1$ and $\overline{u}_n \to u$ in L^1. Indeed, if $u \equiv 0$ then $||\overline{u}_n||_{L^1} \leq ||u_n||_{L^1} \to 0$. If not

$$\begin{aligned} ||\overline{u}_n - u||_{L^1(\Omega)} &= \int_{\{x\in\Omega:\ 0\leq\ln(1+|u_n|)<k+i(n)\}} |u_n(x) - u(x)|\,dx + \\ &\quad + \int_{\{x\in\Omega:\ k+i(n)\leq\ln(1+|u_n|)<k+i(n)+1\}} |\overline{u}_n(x) - u(x)|\,dx + \\ &\quad + \int_{\{x\in\Omega:\ \ln(1+|u_n|)\geq k+i(n)+1\}} |u(x)|\,dx \\ &\leq ||u_n - u||_{L^1(\Omega)} + \int_{\{x\in\Omega:\ k+i(n)\leq\ln(1+|u_n|)<k+i(n)+1\}} |\overline{u}_n(x) - u_n(x)|\,dx + \\ &\quad + ||u_n - u||_{L^1(\Omega)} + ||u||_{L^\infty(\Omega)}\,|\{x\in\Omega : \ln(1+|u_n|) \geq k+i(n)+1\}| \\ &\leq 2||u_n - u||_{L^1(\Omega)} + \int_{\{x\in\Omega:\ k+i(n)\leq\ln(1+|u_n|)<k+i(n)+1\}} |u_n(x)|\,dx + \\ &\quad + ||u||_{L^\infty(\Omega)}\,|\{x\in\Omega :\ \ln(1+|u_n|) \geq k+i(n)+1\}| \end{aligned}$$

these last terms converging to zero because $u_n \to u$ in L^1 and because of the following estimates:

$$\begin{aligned} \int_{\{x\in\Omega:\ k+i(n)\leq\ln(1+|u_n|)<k+i(n)+1\}} |u_n(x)|\,dx &\leq \int_{\{x\in\Omega:\ k+i(n)\leq\ln(1+|u_n|)<k+i(n)+1\}} e^{k+M} - 1\,dx \\ &\leq (e^{k+M} - 1)\left|\{x\in\Omega : |u_n| \geq e^{k+i(n)} - 1\}\right| \\ &\leq (e^{k+M} - 1)\left|\{x\in\Omega :\ |u_n - u| \geq ||u||_{L^\infty(\Omega)}\}\right| \\ &\leq (e^{k+M} - 1)\frac{||u_n - u||_{L^1(\Omega)}}{||u||_{L^\infty(\Omega)}}, \end{aligned}$$

$$
\begin{aligned}
|\{x \in \Omega : \ln(1+|u_n|) \geq k+i(n)+1\}| &= \left|\{x \in \Omega : |u_n| \geq e^{k+i(n)+1}-1\}\right| \\
&\leq \left|\{x \in \Omega : |u_n - u| \geq ||u||_{L^\infty(\Omega)}\}\right| \\
&\leq \frac{||u_n - u||_{L^1(\Omega)}}{||u||_{L^\infty(\Omega)}}.
\end{aligned}
$$

So, we have, in particular, that $\overline{u}_n$ converges to u in L^1 and $\overline{u}_n$ clearly belongs to $L^p(\Omega;\mathbb{R}^d)$.

Case 1, step 2: construction of a convenient sequence $\{v_n\}$ weakly converging to v in L^q.

We have, by (v), [16, Theorem 6.68 and Remark 6.69 (ii)], for any $w \in L^1(\Omega;\mathbb{R}^d)$

$$
\int_\Omega C\varphi(x,w,v)\,dx = \inf\left\{\liminf_{n\to+\infty}\int_\Omega \varphi(x,w,v_n)\,dx : \{v_n\} \subset L^q(\Omega;\mathbb{R}^m), v_n \rightharpoonup v \text{ in } L^q\right\}
$$

whenever the second term is finite.

Since $q > 1$ and thus $L^{q'}(\Omega;\mathbb{R}^m)$ is separable, we can consider a sequence $\{\psi_l\}$ of functions, dense in $L^{q'}(\Omega;\mathbb{R}^m)$.

Then, for each $n \in \mathbb{N}$ let $v_j^n \in L^q(\Omega;\mathbb{R}^m)$ be such that

$$
\int_\Omega C\varphi(x,\overline{u}_n,v)\,dx = \lim_{j\to+\infty}\int_\Omega \varphi(x,\overline{u}_n,v_j^n)\,dx
$$

and

$$
\lim_{j\to+\infty}\int_\Omega (v_j^n - v)\psi_l\,dx = 0,\ \forall\ l \in \mathbb{N}.
$$

We then extract a diagonalizing sequence v_n in the following way: for each $n \in \mathbb{N}$ consider $j(n)$ increasing and verifying

$$
\left|\int_\Omega \left(\varphi(x,\overline{u}_n,v^n_{j(n)}) - C\varphi(x,\overline{u}_n,v)\right)\,dx\right| \leq \frac{1}{n}
$$

$$
\left|\int_\Omega (v^n_{j(n)} - v)\psi_l\,dx\right| \leq \frac{1}{n},\ l = 1,\dots,n.
$$

Define then $v_n = v^n_{j(n)}$. We have v_n bounded in the L^q norm:

$$
\int_\Omega |v_n|^q\,dx \leq C\int_\Omega \varphi(x,\overline{u}_n,v_n)\,dx \leq \frac{C}{n} + C\int_\Omega C\varphi(x,\overline{u}_n,v)\,dx \leq C + C\int_\Omega \varphi(x,\overline{u}_n,v)\,dx
$$

this last term being bounded because $\overline{u}_n$ is a bounded sequence in L^∞ and because of the growth condition (v) on φ.

Moreover the density of ψ_l in $L^{q'}$ ensures that $v_n \rightharpoonup v$ in L^q. Indeed, let $\psi \in L^q(\Omega;\mathbb{R}^m)$ and let $\delta > 0$. Consider $l \in \mathbb{N}$ such that $||\psi_l - \psi||_{L^{q'}} \leq \delta$. Then, for sufficiently large n,

$$
\left|\int_\Omega (v_n - v)\psi\,dx\right| \leq \left|\int_\Omega (v_n - v)(\psi - \psi_l)\,dx\right| + \left|\int_\Omega (v_n - v)\psi_l\,dx\right| \leq ||v_n - v||_{L^q}||\psi_l - \psi||_{L^{q'}} + \delta \leq c\delta + \delta.
$$

Case 1, step 3: upper bound for $\overline{I}$.

Start remarking that

$$
\limsup_{n\to+\infty}\int_\Omega \varphi(x,\overline{u}_n,v_n)\,dx \leq \int_\Omega C\varphi(x,u,v)\,dx.
$$

Indeed,

$$
\begin{aligned}
\limsup_{n\to+\infty}\int_\Omega \varphi(x,\overline{u}_n,v_n)\,dx &= \limsup_{n\to+\infty}\int_\Omega \left(\varphi(x,\overline{u}_n,v_n) - C\varphi(x,\overline{u}_n,v) + C\varphi(x,\overline{u}_n,v)\right)\,dx \\
&\leq \limsup_{n\to+\infty}\left(\frac{1}{n} + \int_\Omega C\varphi(x,\overline{u}_n,v)\,dx\right).
\end{aligned}
$$

As observed in Remark 2.3, $C\varphi(x,\cdot,v)$ is upper semi-continuous. By the pointwise convergence of $\overline{u}_n$ towards u (up to a subsequence), we have

$$\limsup_{n\to+\infty} C\varphi(x,\overline{u}_n,v) \le C\varphi(x,u,v).$$

Moreover the fact that $\overline{u}_n$ is bounded in L^∞ and the hypothesis (v) allows to apply the "inverted" Fatou's lemma and get the desired inequality.
Now we have

$$\begin{aligned}
\int_\Omega W(x,\overline{u}_n,\nabla\overline{u}_n)\,dx &= \int_{\{x\in\Omega:\ 0\le\ln(1+|u_n|)<k+i(n)\}} W(x,u_n,\nabla u_n)\,dx+\\
&\quad+\int_{\{x\in\Omega:\ k+i(n)\le\ln(1+|u_n|)<k+i(n)+1\}} W(x,\overline{u}_n,\nabla\overline{u}_n)\,dx+\\
&\quad+\int_{\{x\in\Omega:\ \ln(1+|u_n|)\ge k+i(n)+1\}} W(x,0,0)\,dx\\
&\le \int_\Omega W(x,u_n,\nabla u_n)\,dx+\int_{\{x\in\Omega:\ k+i(n)\le\ln(1+|u_n|)<k+i(n)+1\}} C(1+|\nabla\overline{u}_n|)\,dx+\\
&\quad+C\,|\{x\in\Omega:\ \ln(1+|u_n|)\ge k+i(n)+1\}|
\end{aligned}$$

(where it has been used the growth condition (i). Using the expression of $\overline{u}_n$, by [2, Theorem 3.96], we have $|\nabla\overline{u}_n|\le c|\nabla u_n|$ and so, using (3.10), we get

$$\limsup_{n\to+\infty}\int_{\{x\in\Omega:\ k+i(n)\le\ln(1+|u_n|)<k+i(n)+1\}} C(1+|\nabla\overline{u}_n|)\,dx \le c\,\frac{|\Omega|+\sup||\nabla u_n||_{L^1}}{M}=\frac{c}{M}$$

(note that c is independent of n and of the sequence u_n, and it doesn't represent always the same constant).
Moreover, since $|\{x\in\Omega:\ \ln(1+|u_n|)\ge k+i(n)+1\}|\to 0$ as $n\to+\infty$ (as already seen in the case where $||u||_{L^\infty}\neq 0$) we get,

$$\limsup_{n\to+\infty}\int_\Omega W(x,\overline{u}_n,\nabla\overline{u}_n)\,dx \le \lim_{n\to+\infty}\int_\Omega W(x,u_n,\nabla u_n)\,dx+\frac{c}{M}.$$

Note that if $u=0$ we can still get $|\{x\in\Omega:\ \ln(1+|u_n|)\ge k+i(n)+1\}|\to 0$:

$$|\{x\in\Omega:\ \ln(1+|u_n|)\ge k+i(n)+1\}| \le \left|\{x\in\Omega:\ |u_n|\ge e^{k+1}-1\}\right| \le \frac{||u_n||_{L^1}}{e^{k+1}-1}\to 0$$

since $u_n\to 0$ in L^1.
Finally, we get, as desired,

$$\begin{aligned}
\overline{I}(u,v) &\le \liminf_{n\to+\infty}\int_\Omega W(x,\overline{u}_n,\nabla\overline{u}_n)\,dx+\int_\Omega \varphi(x,\overline{u}_n,v_n)\,dx\\
&\le \limsup_{n\to+\infty}\int_\Omega W(x,\overline{u}_n,\nabla\overline{u}_n)\,dx+\limsup_{n\to+\infty}\int_\Omega \varphi(x,\overline{u}_n,v_n)\,dx\\
&\le \int_\Omega W(x,u,\nabla u)\,dx+\int_{J_u}\gamma(x,u^+,u^-,\nu_u)\,d\mathcal{H}^{N-1}+\int_\Omega W^\infty\left(x,u,\frac{dD^cu}{d|D^cu|}\right)d|D^cu|+\\
&\quad+\frac{c}{M}+\int_\Omega C\varphi(x,u,v)\,dx.
\end{aligned}$$

Case 2: arbitrary $u\in BV(\Omega;\mathbb{R}^d)\cap L^p(\Omega;\mathbb{R}^d)$.

To achieve the upper bound on this case, we will reduce ourselves to Case 1 by means of a truncature argument developed in [17, Theorem 2.16, Step 4], in turn inspired by [3, Theorem 4.9]. We reproduce the same argument as in [17] for the reader's convenience.
Let $\phi_n \in C_0^1(\mathbb{R}^d;\mathbb{R}^d)$ be such that

$$\phi_n(y) = y \text{ if } y \in B_n(0), \quad \|\nabla\phi_n\|_{L^\infty} \leq 1,$$

and fix $u \in BV(\Omega;\mathbb{R}^d) \cap L^p(\Omega;\mathbb{R}^d)$.

As proven in [3, Theorem 4.9], directly from the definitions and properties for the approximate discontinuity set and the triplets (u^+, u^-, ν_u) (see Subection 2.2), it results that

$$\begin{aligned}
&J_{\phi_n(u)} \subset J_u, \\
&(\phi_n(u)^+, \phi_n(u)^-, \nu_{\phi_n(u)}) = (\phi_n(u^+), \phi_n(u^-), \nu_u) \text{ in } J_{\phi_n(u)}.
\end{aligned}$$

Moreover one has

$$|D\phi_n(u)|(B) \leq |D(u)|(B), \text{ for every Borel set } B \subset \Omega. \tag{3.11}$$

Consequently

$$\phi_n(u) \in BV(\Omega;\mathbb{R}^d) \cap L^\infty(\Omega;\mathbb{R}^d).$$

Since $\phi_n(u) \to u$ in L^1, by the lower semicontinuity of $\overline{I}$ (since $q > 1$) and by Case 1 we get

$$\begin{aligned}
\overline{I}(u,v) \leq \liminf_{n\to+\infty} &\left[\int_\Omega W(x,\phi_n(u),\nabla\phi_n(u))\,dx + \int_{J_{\phi_n(u)}} \gamma(x,\phi_n(u)^+,\phi_n(u)^-,\nu_{\phi_n(u)})\,d\mathcal{H}^{N-1} + \right.\\
&\left. + \int_\Omega W^\infty\left(x,\phi_n(u),\frac{dD^c(\phi_n(u))}{d|D^c(\phi_n(u))|}\right) d|D^c\phi_n(u)| + \int_\Omega C\varphi(x,\phi_n(u),v)\,dx\right].
\end{aligned}$$

By the upper semicontinuity of γ in all of its arguments as stated in [17, (c) of Lemma 2.15] and by the fact that $\gamma(x,a,b,\nu) \leq C|a-b|$ for every $(x,a,b,\nu) \in \Omega \times \mathbb{R}^d \times \mathbb{R}^d \times S^{N-1}$ (see [17, (d) Lemma 2.15]) and the properties of ϕ_n we have

$$\gamma(x,\phi_n(u^+),\phi_n(u^-),\nu_u) \leq C|u^+ - u^-|,$$

and so, by Fatou's Lemma we obtain

$$\limsup_{n\to+\infty} \int_{J_{\phi_n(u)}} \gamma(x,\phi_n(u)^+,\phi_n(u)^-,\nu_u)\,d\mathcal{H}^{N-1} \leq \int_{J_u} \gamma(x,u^+,u^-,\nu_u)\,d\mathcal{H}^{N-1}.$$

Moreover we have

$$\limsup_{n\to+\infty} \int_\Omega C\varphi(x,\phi_n(u),v)\,dx = \int_\Omega C\varphi(x,u,v)\,dx \tag{3.12}$$

Indeed, as already observed in step 2, $C\varphi(x,\cdot,v)$ is upper semicontinuous and $\phi_n(u)$ is pointwise converging to u and thus we can apply the inverted Fatou's lemma.
For what concerns the other terms, setting $\Omega_n := \{x \in \Omega \setminus J_u : |u(x)| \leq n\}$, we have

$$\begin{aligned}
&\limsup_{n\to+\infty} \int_\Omega W(x,\phi_n(u),\nabla\phi_n(u))\,dx = \\
&= \limsup_{n\to+\infty} \left[\int_{\Omega_n} W(x,\phi_n(u),\nabla\phi_n(u))\,dx + \int_{(\Omega\setminus\Omega_n)\setminus J_u} W(x,\phi_n(u),\nabla\phi_n(u))\,dx\right] \\
&\leq \int_\Omega W(x,u,\nabla u)\,dx + \limsup_{n\to+\infty} C\left[|\Omega\setminus\Omega_n| + |D\phi_n(u)|((\Omega\setminus\Omega_n)\setminus J_u)\right].
\end{aligned}$$

On the other hand by (3.11) we deduce that

$$\limsup_{n\to+\infty} |D\phi_n(u)|((\Omega \setminus \Omega_n) \setminus J_u) \leq \limsup_{n\to+\infty} |Du|(\Omega \setminus (\Omega_n \cup J_u)) = 0$$

and so

$$\limsup_{n\to+\infty} \int_\Omega W(x, \phi_n(u), \nabla\phi_n(u))\, dx \leq \int_\Omega W(x, u, \nabla u)\, dx.$$

Similarly

$$\limsup_{n\to+\infty} \int_\Omega W^\infty\left(x, \phi_n(u), \frac{dD^c\phi_n(u)}{d|D^c\phi_n(u)|}\right) d|D^c\phi_n(u)| =$$

$$= \limsup_{n\to+\infty} \int_{\Omega_n} W^\infty\left(x, \phi_n(u), \frac{dD^c\phi_n(u)}{d|D^c\phi_n(u)|}\right) d|D^c\phi_n(u)| +$$

$$+ \limsup_{n\to+\infty} \int_{(\Omega\setminus\Omega_n)\setminus J_u} W^\infty\left(x, \phi_n(u), \frac{dD^c\phi_n(u)}{d|D^c\phi_n(u)|}\right) d|D^c\phi_n(u)| \leq$$

$$\leq \int_\Omega W^\infty\left(x, u, \frac{dD^c u}{d|D^c u|}\right) d|D^c u| + C \limsup_{n\to+\infty} [|D\phi_n(u)|((\Omega \setminus \Omega_n) \setminus J_u)] =$$

$$= \int_\Omega W^\infty\left(x, u, \frac{dD^c u}{d|D^c u|}\right) d|D^c u|.$$

This finishes the proof.

Part 2: $q = 1$.

Lower bound. Let $u \in BV(\Omega;\mathbb{R}^d) \cap L^p(\Omega;\mathbb{R}^d)$, $v \in L^1(\Omega;\mathbb{R}^m)$, $u_n \in BV(\Omega;\mathbb{R}^d)$ and $v_n \in L^1(\Omega;\mathbb{R}^m)$ such $u_n \to u$ strongly in L^1 and $v_n \rightharpoonup v$ in L^1. Then by Lemma 3.1 exactly as in the case $q > 1$, (3.6) continues to hold. Moreover [8, Theorem 1.1], ensures that

$$\int_\Omega C\varphi(x, u, v)\, dx \leq \liminf_{n\to+\infty} \int_\Omega \varphi(x, u_n, v_n)\, dx.$$

Again the lower bound follows from the superadditivity of the liminf.

Upper bound. Let $u \in BV(\Omega;\mathbb{R}^d) \cap L^p(\Omega;\mathbb{R}^d)$ and $v \in L^1(\Omega;\mathbb{R}^m)$. We aim to prove (3.8), constructing convenient sequences $u_n \in BV(\Omega;\mathbb{R}^d)$ and $v_n \in L^1(\Omega;\mathbb{R}^m)$ with $u_n \to u$ in L^1 and $v_n \rightharpoonup v$ in L^1.

Case 1. As in the case $q > 1$ we first assume that $u \in L^\infty(\Omega;\mathbb{R}^d)$ and develop our proof in three steps.

Case 1, step 1. The step 1 is identical to Case 1, step 1 proven for $q > 1$.

Case 1, step 2. For what concerns this step, we preliminarly consider a continuous increasing function $\theta : [0, +\infty) \to [0, +\infty)$ such that

$$\lim_{t\to+\infty} \frac{\theta(t)}{t} = +\infty. \tag{3.13}$$

Then consider a decreasing sequence $\varepsilon \to 0$ and take the functional $I_\varepsilon : BV(\Omega;\mathbb{R}^d) \times L^1(\Omega;\mathbb{R}^m) \to \mathbb{R} \cup \{+\infty\}$, defined as

$$I_\varepsilon(u, v) := I(u, v) + \varepsilon \int_\Omega \theta(|v|)\, dx. \tag{3.14}$$

Let $C(\varphi(x, u, \cdot) + \varepsilon\theta(|\cdot|))$ be the convexification of $\varphi(x, u, \cdot) + \varepsilon\theta(|\cdot|)$ as in (1.6).

By [16, Theorem 6.68 and Remark 6.69], we have that for every $w \in L^1(\Omega;\mathbb{R}^m)$

$$\int_\Omega C(\varphi(x, w, v) + \varepsilon\theta(|v|))\, dx = \inf\left\{\liminf_{n\to+\infty} \int_\Omega (\varphi(x, w, v_n) + \varepsilon\theta(|v_n|))\, dx : v_n \rightharpoonup v \text{ in } L^1\right\}$$

whenever the second term is finite. Moreover the left hand side coincides with the sequentially weakly-L^1 lower semicontinuous envelope. Consequently for every $n \in \mathbb{N}$, let $\overline{u}_n$ be the sequence constructed in *Case 1, step 1* and let $v_j^n \in L^1(\Omega;\mathbb{R}^m)$ be such that $v_j^n \rightharpoonup v$ in L^1 as $j \to +\infty$ and

$$\int_\Omega C(\varphi(x, \overline{u}_n, v) + \varepsilon\theta(|v|))\, dx = \lim_{j\to+\infty} \int_\Omega (\varphi(x, \overline{u}_n, v_j^n) + \varepsilon\theta(|v_j^n|))\, dx.$$

The proof now develops as in [16, Proposition 3.18]. The growth condition (v) and the fact that $\overline{u_n}$ is bounded in L^∞ and thus in L^1, entails that there exists a constant M such that

$$\sup_{n,j\in\mathbb{N}} \int_\Omega \theta(|v_j^n|)\, dx \leq M. \tag{3.15}$$

We observe that the growth conditions on θ guarantee that $\sup_{n,j\in\mathbb{N}} \|v_j^n\|_{L^1(\Omega)} \leq C(M)$. Moreover the separability of $C_0(\Omega)$ allows us to consider a dense sequence of functions $\{\psi_l\}$.

Next, mimicking the argument used in the analogous step for $q > 1$, for every $\varepsilon > 0$ we construct a diagonalizing sequence v_n as follows. For each $n \in \mathbb{N}$ consider $j(n)$ increasing and such that

$$\left|\int_\Omega \Big(\varphi(x, \overline{u}_n, v_{j(n)}^n) + \varepsilon\theta(|v_{j(n)}^n|) - C(\varphi(x, \overline{u}_n, v) + \varepsilon\theta(|v|))\Big)\, dx\right| \leq \frac{1}{n}$$

$$\left|\int_\Omega (v_{j(n)}^n - v)\psi_l\, dx\right| \leq \frac{1}{n},\ l = 1, \dots, n.$$

Define $v_n := v_{j(n)}^n$. The bounds on θ, the fact that $\overline{u}_n$ is bounded in L^1 and the separability of $C_0(\Omega)$ guarantee that $v_n \overset{\star}{\rightharpoonup} v$ in $\mathcal{M}(\Omega)$ and moreover, (3.15), Dunford-Pettis' theorem entail that the convergence of v_n towards v is weak-L^1.

Case 1, step 3. Arguing as in the first part of *Case 1, step 3* for $q > 1$, we can prove that

$$\limsup_{n\to+\infty} \int_\Omega (\varphi(x, \overline{u}_n, v_n) + \varepsilon\theta(|v_n|))\ dx \leq \int_\Omega C(\varphi(x, u, v) + \varepsilon\theta(|v|))\, dx.$$

Next we define

$$\overline{I}_\varepsilon(u, v) := \inf\{\liminf_{n\to+\infty} I_\varepsilon(u_n, v_n) : (u_n, v_n) \in BV(\Omega; \mathbb{R}^d) \times L^1(\Omega; \mathbb{R}^m), u_n \to u \text{ in } L^1, v_n \rightharpoonup v \text{ in } L^1\}. \tag{3.16}$$

The same argument of the last part in *Case 1, step 3*, for $q > 1$, allows to prove that

$$\begin{aligned}\overline{I}_\varepsilon(u, v) \quad &\leq \int_\Omega W(x, u, \nabla u)\, dx + \int_{J_u} \gamma(x, u^+, u^-, \nu_u)\, d\mathcal{H}^{N-1} + \\ &+ \int_\Omega W^\infty\left(x, u, \frac{dD^c u}{d|D^c u|}\right) d|D^c u| + \int_\Omega C\left(\varphi(x, u, v) + \varepsilon\theta(|v|)\right)\, dx\end{aligned} \tag{3.17}$$

for every $u \in BV(\Omega; \mathbb{R}^d) \cap L^\infty(\Omega; \mathbb{R}^d)$ and $v \in L^1(\Omega; \mathbb{R}^m)$. On the other hand we observe that the sequence $\overline{I}_\varepsilon(u, v)$ is increasing in ε and $\overline{I} \leq \overline{I}_\varepsilon$ for every ε. Moreover by virtue of the increasing behaviour in ε of $\varphi + \varepsilon\theta$, invoking [16, Proposition 4.100] it results that for every $(x, u) \in \Omega \times \mathbb{R}^d$, we have

$$\inf_\varepsilon C\left(\varphi(x, u, v) + \varepsilon\theta(|v|)\right) = \lim_{\varepsilon\to 0} C\left(\varphi(x, u, v) + \varepsilon\theta(|v|)\right) = C\varphi(x, u, v).$$

Thus applying Lebesgue monotone convergence theorem we have

$$\begin{aligned}&\overline{I}(u, v) \leq \lim_{\varepsilon\to 0} \overline{I}_\varepsilon(u, v) = \lim_{\varepsilon\to 0}\left(\int_\Omega W(x, u, \nabla u)\, dx + \int_{J_u} \gamma(x, u^+, u^-, \nu_u)\, d\mathcal{H}^{N-1} + \right.\\ &\left. + \int_\Omega W^\infty\left(x, u, \frac{dD^c u}{d|D^c u|}\right) d|D^c u| + \int_\Omega C\left(\varphi(x, u, v) + \varepsilon\theta(|v|)\right)\, dx\right) = \\ &= \int_\Omega W(x, u, \nabla u)\, dx + \int_{J_u} \gamma(x, u^+, u^-, \nu_u) d\mathcal{H}^{N-1} + \int_\Omega W^\infty\left(x, u, \frac{dD^c u}{d|D^c u|}\right) d|D^c u| + \int_\Omega C\varphi(x, u, v)\, dx,\end{aligned} \tag{3.18}$$

for every $(u, v) \in (BV(\Omega; \mathbb{R}^d) \cap L^\infty(\Omega; \mathbb{R}^d)) \times L^1(\Omega; \mathbb{R}^m)$.

Case 2. Now we consider $u \in BV(\Omega; \mathbb{R}^d) \cap L^p(\Omega; \mathbb{R}^d)$ and $v \in L^1(\Omega; \mathbb{R}^m)$.

To achieve the upper bound we can preliminarily observe that, a proof entirely similar to [16, Proposition 3.18], guarantees that for every $\varepsilon > 0$, the functional $\overline{I}_\varepsilon(u,v)$, defined in (3.16) is sequentially weakly lower semicontinuous with respect to the topology $L^1(\Omega;\mathbb{R}^d)_{\text{strong}} \times L^1(\Omega;\mathbb{R}^m)_{\text{weak}}$. Thus, arguing exactly as in the *Case 2*, for $q > 1$, we have that

$$\begin{aligned}\overline{I}_\varepsilon(u,v) \quad &\leq \int_\Omega W(x,u,\nabla u)\,dx + \int_{J_u} \gamma(x,u^+,u^-,\nu_u)\,d\mathcal{H}^{N-1} + \int_\Omega W^\infty\left(x,u,\frac{dD^c u}{d|D^c u|}\right) d|D^c u| + \\ &+ \int_\Omega C(\varphi(x,u,v) + \varepsilon\theta(|v|))\,dx.\end{aligned} \tag{3.19}$$

Finally the monotonicity argument for ε invoked in the *Case 1, step 3* for $q = 1$ can be recalled also in this context leading to the same inequality in (3.18) for every $u \in BV(\Omega;\mathbb{R}^d) \cap L^p(\Omega;\mathbb{R}^d)$ and for every $v \in L^1(\Omega;\mathbb{R}^m)$, and that concludes the proof of (3.8). $\square$

Now we present the proof of Theorem 1.3, which is much easier than the latter one, since, by virtue of the continuous embedding of $BV(\Omega;\mathbb{R}^d)$ in $L^{\frac{N}{N-1}}(\Omega;\mathbb{R}^d)$, it does not involve any truncature argument.

Proof of Theorem 1.3. We omit the details of the proof since it develops in the same way as that of Theorem 1.1. First we invoke Corollary 3.2 and assume without loss of generality that W is quasiconvex in the last variable. Then we prove a lower bound for the relaxed energy and finally we show that the lower bound is also an upper bound. As in Theorem 1.1 we may consider two separate cases: $q > 1$ and $q = 1$.

Lower bound for the cases $q = 1$ and $q > 1$. The proof of the lower bound is identical to that of Theorem 1.1.

Upper bound, case $q > 1$. Let $u \in BV(\Omega;\mathbb{R}^d)$ and $v \in L^q(\Omega;\mathbb{R}^m)$. We can assume

$$\begin{aligned}&\int_\Omega W(x,u,\nabla u)\,dx + \int_{J_u} \gamma(x,u^+,u^-,\nu_u)\,d\mathcal{H}^{N-1} + \int_\Omega W^\infty\left(x,u,\frac{dD^c u}{d|D^c u|}\right) d|D^c u| + \\ &+ \int_\Omega C\varphi(x,u,v)\,dx < +\infty.\end{aligned} \tag{3.20}$$

Without loss of generality, we assume also that W and $\varphi \geq 0$. Applying Lemma [17, Theorem 2.16], we can get a sequence $\{u_n\}$ in $W^{1,1}(\Omega;\mathbb{R}^d)$ such that $u_n \to u$ in L^1 and

$$\begin{aligned}\lim \int_\Omega W(x,u_n,\nabla u_n)\,dx = &\int_\Omega W(x,u,\nabla u)\,dx + \int_{J_u} \gamma(x,u^+,u^-,\nu_u)\,d\mathcal{H}^{N-1} + \\ &+ \int_\Omega W^\infty\left(x,u,\frac{dD^c u}{d|D^c u|}\right) d|D^c u|.\end{aligned}$$

We observe that, by the coercivity condition on W and by (3.20), ∇u_n is bounded in L^1. Moreover, the continuous embedding of $BV(\Omega;\mathbb{R}^d)$ in $L^{\frac{N}{N-1}}(\Omega;\mathbb{R}^d)$, imply that u_n is bounded in $L^{\frac{N}{N-1}}(\Omega;\mathbb{R}^d)$ and thus in $L^p(\Omega;\mathbb{R}^d)$ since we are assuming $1 \leq p \leq \frac{N}{N-1}$.

Then, as in the proof of Theorem 1.1, *Case 1, step 2*, $q > 1$ we can construct a recovery sequence v_n using the relaxation theorem [16, Theorem 6.68] and the same diagonalizing argument. We emphasize that there is no need to make a preliminary truncature of the recovery sequence u_n. Indeed, to ensure that v_n is bounded in $L^q(\Omega;\mathbb{R}^m)$ (required to obtain the weak convergence of v_n towards v in L^q) it suffices to use the growth condition of φ and the fact that u_n is bounded in L^p.

Therefore it is possible to get $v_n \rightharpoonup v$ in L^q and such that

$$\limsup_{n\to+\infty} \int_\Omega \varphi(x,u_n,v_n)\,dx \leq \int_\Omega C\varphi(x,u,v)\,dx.$$

The upper bound then follows by the sub-additivity of the limsup.

Upper bound, case $q = 1$. In analogy with the case $q > 1$ there is no need of truncature because of the continuous embedding of BV in $L^{\frac{N}{N-1}}$. As for Theorem 1.1 it suffices to approximate the functional I by I_ε in (3.14) and consequently it is enough to use, for the correspective relaxed functional, the diagonalization argument adopted in Theorem 1.1, Case 1, step 2 for $q = 1$ via an application of Dunford-Pettis' theorem. Finally the monotonicity behaviour in ε of $\overline{I}_\varepsilon$, the approximation of the energy densities allowed by [16, Proposition 4.100] and the Lebegue monotone convergence theorem conclude the proof. $\square$

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