AN ASYMPTOTICALLY UNBIASED MOMENT ESTIMATOR OF A NEGATIVE EXTREME VALUE INDEX

Frederico Caeiro

Departamento de Matemática Faculdade de Ciências e Tecnologia Universidade Nova de Lisboa 2829-516 Caparica, Portugal email: fac@fct.unl.pt M. IVETTE GOMES DEIO Faculdade de Ciências Universidade de Lisboa 1749–016 Lisboa, Portugal email: ivette.gomes@fc.ul.pt

Abstract

In this paper we consider a new class of consistent semi-parametric estimators of a negative extreme value index, based on the set of the klargest observations. This class of estimators depends on a control or tuning parameter, which enables us to have access to an estimator with a null second-order component of asymptotic bias, and with a rather interesting mean squared error, as a function of k. We study the consistency and asymptotic normality of the proposed estimators. Their finite sample behaviour is obtained through Monte Carlo simulation.

Keywords: Extreme value index, Semi-parametric estimation; Moment Estimator.

Mathematics Subject Classification (2010): Primary 62G32, 62E20; Secondary 65C05

1 Introduction and Outline

Let $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n}$ denote the order statistics associated with the sequence X_i , $i = 1, \ldots, n$, of independent random variables with common distribution function (d.f.) F. Suppose that F belongs to the max-domain of attraction of a non-degenerate d.f. G and use the notation $F \in \mathcal{D}_{\mathcal{M}}(G)$. Then G is the Extreme Value distribution (Gnedenko, [5]):

$$G_{\gamma}(x) \equiv \exp(-(1+\gamma x)^{-1/\gamma}), \quad 1+\gamma x > 0, \quad \gamma \in \mathbb{R}.$$
 (1)

The generalized shape parameter γ , also know as the *extreme value index* (EVI), is the parameter we want to estimate. A necessary and sufficient

condition for $F \in \mathcal{D}_{\mathcal{M}}(G)$ is (de Haan, [10]):

$$\lim_{t \to \infty} \frac{U(tx) - U(t)}{a(t)} = D_{\gamma}(x) := \begin{cases} \frac{x^{\gamma} - 1}{\gamma}, & \gamma \neq 0\\ \ln x, & \gamma = 0 \end{cases}, \quad \forall x > 0, \qquad (2)$$

for some measurable positive function a(t) and with U(t) standing for the reciprocal quantile function defined by $U(t) := F^{\leftarrow}(1 - 1/t), t \ge 1$ with $F^{\leftarrow}(x) := \inf\{y : F(y) \ge x\}$, the generalized inverse function of F.

Among the most popular EVI estimators, based on a set of positive upper order statistics, we refer the Hill estimator [12], defined by

$$\hat{\gamma}_n^H(k) \equiv M_n^{(1)}(k) := \frac{1}{k} \sum_{i=1}^k \left(\ln X_{n-i+1:n} - \ln X_{n-k:n} \right), \tag{3}$$

valid for $\gamma > 0$ only. For a general $\gamma \in \mathbb{R}$, Dekkers et al. [2] proposed the Moment estimator, with the functional expression

$$\hat{\gamma}_n^{MOM}(k) = \hat{\gamma}_n^H(k) + \hat{\gamma}_n^{NM}(k), \tag{4}$$

where

$$\hat{\gamma}_n^{NM}(k) := 1 - \frac{1}{2} \left(1 - \frac{\left(M_n^{(1)}(k)\right)^2}{M_n^{(2)}(k)} \right)^{-1},\tag{5}$$

and

$$M_n^{(\alpha)}(k) := \frac{1}{k} \sum_{i=1}^k \left(\ln X_{n-i+1:n} - \ln X_{n-k:n} \right)^{\alpha}, \quad \alpha > 0.$$
 (6)

The Moment estimator, in (4), combines two estimators: the Hill estimator, in (3), and the estimator in (5) which will be called, in this paper, the "negative Moment" estimator. For intermediate k, i.e., a sequence of integers $k = k_n$, $1 \le k < n$, such that

$$k = k_n \to \infty \quad \text{and} \quad k/n \to 0, \qquad \text{as} \quad n \to \infty,$$
 (7)

it is well known that $\hat{\gamma}_n^H(k)$, $\hat{\gamma}_n^{MOM}(k)$ and $\hat{\gamma}_n^{NM}(k)$ in (3), (4) and (5), are consistent for $\gamma^+ := \max(0, \gamma)$, γ and $\gamma^- := \min(0, \gamma)$, respectively.

Most of the classical EVI estimators have usually a high variance for small values of k and a high bias when k is large. This problem affects both the Hill and the Moment estimator and leads to a difficult choice of the "optimal" k, i.e., the value k that minimizes the asymptotic mean squared error. For heavy tails ($\gamma > 0$), the adequate accommodation of the bias of Hill's estimator has been extensively addressed by several authors. Recently, Caeiro et al. [1], Gomes et al. [7] and [6] introduced, in different ways, second-order minimum-variance reduced-bias estimators that reduce the bias of Hill's estimator without increasing the asymptotic variance, which is kept equal to γ^2 .

As already noticed in Fraga Alves [4], when $\gamma > 0$, $\hat{\gamma}_n^H(k)$ has smaller asymptotic variance than $\hat{\gamma}_n^{MOM}(k)$, and when $\gamma < 0$, $\hat{\gamma}_n^{NM}(k)$ and $\hat{\gamma}_n^{MOM}(k)$ have the same asymptotic variance. This remark led us to study a semi-parametric class of consistent estimators for $\gamma < 0$, which generalizes the negative Moment estimator in (5). Such a class, given by

$$\hat{\gamma}_n^{NM(\theta)}(k) := \hat{\gamma}_n^{NM}(k) + \theta M_n^{(1)}(k), \quad \theta \in \mathbb{R}.$$
(8)

depends on a tuning parameter $\theta \in \mathbb{R}$, and we get the estimator in (5) for $\theta = 0$. With the appropriate choice of θ , $\hat{\gamma}_n^{NM(\theta)}(k)$ enables us to have access to an estimator of a negative EVI with a smaller asymptotic bias and the same asymptotic variance as the Moment estimator.



Figure 1: Sample paths of $\hat{\gamma}_n^{NM(\theta)}(k)$ in (8) against k, for one sample of size n = 10000 from the $G_{-0.5}$ (left) and Arcsin (right) models.

Figure 1 illustrates, for several values of θ , the behaviour of $\hat{\gamma}_n^{NM(\theta)}(k)$ versus k for a sample of size n = 10000 from the Extreme Value distribution in (1) with $\gamma = -0.5$ and the ArcSin distribution with d.f. $F(x) = (2/\pi) \arcsin(\sqrt{x}), \ 0 < x < 1, \ (\gamma = -2).$

In Section 2 of this paper, we state a few results already proved in the literature and derive the asymptotic properties of the new class of EVI estimators, in (8). Finally, in Section 3, we perform a small-scale Monte-Carlo simulation, in order to compare the behaviour of the estimators under study for finite samples.

2 Main results

2.1 Second order regular variation conditions

In order to derive the asymptotic behaviour of several semi-parametric EVI estimators, we need the following second order condition:

$$\lim_{t \to \infty} \frac{\frac{U(tx) - U(t)}{a(t)} - \frac{x^{\gamma} - 1}{\gamma}}{A(t)} = H_{\gamma,\rho}(x) := \frac{1}{\rho} \left(\frac{x^{\gamma + \rho} - 1}{\gamma + \rho} - \frac{x^{\gamma} - 1}{\gamma} \right), \quad (9)$$

for all x > 0, where $\rho \leq 0$ is a second order parameter controlling the speed of convergence in (2) and $|A(t)| \in RV_{\rho}$, with RV_a standing for the class of regularly varying functions with an index of regular variation a, i.e. positive measurable functions g such that $\lim_{t\to\infty} g(tx)/g(t) = x^a$, for all x > 0. Since we need a second order condition for $\ln U(t)$, we state the following theorem:

Theorem 2.1 (Draisma et al., [3], de Haan and Ferreira [11], Appendix. B3). Assume $U(\infty) > 0$ and (9) holds with $\rho \leq 0$ and $\gamma \neq \rho$. Then,

$$\overline{A}(t) := \left(\frac{a(t)}{U(t)} - \gamma_+\right) \underset{t \to \infty}{\longrightarrow} 0, \tag{10}$$

and with

$$l:=\lim_{t\to\infty}\left(U(t)-\frac{a(t)}{\gamma}\right)\quad for \ \gamma+\rho<0,$$

$$\frac{\overline{A}(t)}{A(t)} \underset{t \to \infty}{\longrightarrow} c = \begin{cases} 0, & \gamma < \rho \le 0; \\ \frac{\gamma}{\gamma + \rho}, & 0 \le -\rho < \gamma \lor (0 < \gamma < -\rho, \ l = 0); \\ \pm \infty, & \gamma = -\rho \lor (0 < \gamma < -\rho, \ l \ne 0) \lor \rho < \gamma \le 0. \end{cases}$$

Furthermore, we have

$$\lim_{t \to \infty} \frac{\frac{\ln U(tx) - \ln U(t)}{\tilde{a}(t)} - \frac{x^{\gamma_-} - 1}{\gamma_-}}{\widetilde{A}(t)} = H_{\gamma_-, \tilde{\rho}}(x), \tag{11}$$

for all x > 0, with $\tilde{a}(t) := a(t)/U(t)$, H defined in (9),

$$\tilde{\rho} := \begin{cases} \gamma, & \rho < \gamma \leq 0\\ -\gamma, & 0 < \gamma < -\rho, \ l \neq 0\\ \rho, & (0 < \gamma < -\rho \land l = 0) \lor \gamma < \rho \leq 0 \lor \gamma \geq -\rho > 0 \end{cases}$$

and

$$\widetilde{A}(t) := \begin{cases} A(t), & c = 0\\ \gamma_{+} - \frac{a(t)}{U(t)}, & c = \pm \infty\\ \frac{\rho}{\gamma + \rho} A(t), & c = \frac{\gamma}{\gamma + \rho} \end{cases}$$
(12)

with $|\tilde{A}(t)| \in RV_{\tilde{\rho}}$.

Remark 2.1 (de Haan and Ferreira [11], Remark B.3.18). It follows that

$$q_{\gamma,\rho} := \lim_{t \to \infty} \frac{\overline{A}(t)}{\widetilde{A}(t)} = \begin{cases} 0 & \gamma < \rho \le 0; \\ \frac{\gamma}{\rho} & (0 < \gamma < -\rho \land l = 0) \lor \gamma \ge -\rho > 0; \\ -1 & (0 < \gamma < -\rho, \ l \ne 0) \lor \rho < \gamma \le 0. \end{cases}$$
(13)

Since $\rho < 0$ for a variety of models we add the following proposition with a similar proof to Proposition 1 from [8]:

Proposition 2.1. Let us assume that (9) holds with $\rho < 0$. Then, there exists $\tilde{a}_0(\cdot)$ and $\tilde{A}_0(\cdot)$ such that

$$\lim_{t \to \infty} \frac{\frac{\ln U(tx) - \ln U(t)}{\tilde{a}_0(t)} - \frac{x^{\gamma_-} - 1}{\gamma_-}}{\tilde{A}_0(t)} = \frac{x^{\gamma_- + \tilde{\rho}} - 1}{\gamma_- + \tilde{\rho}}, \qquad \forall x > 0,$$
(14)

with
$$\widetilde{A}_0(t) = \frac{\widetilde{A}(t)}{\widetilde{\rho}}$$
 and $\widetilde{a}_0(t) = \widetilde{a}(t)(1 - \widetilde{A}_0(t)).$

2.2 Auxiliary results

More generally than Lemma 3.5.5 in de Haan and Ferreira [11], but with a similar proof, we now state the following:

Lemma 2.1. Under the second order framework in (11) and for intermediate k, i.e. whenever (7) holds we have for any $\alpha > 0$, and $M_n^{(\alpha)}(k)$ defined in (6),

$$\frac{M_n^{(\alpha)}(k)}{(\tilde{a}(t))^{\alpha}} \stackrel{d}{=} \mu_{\alpha,\gamma} + \frac{\sigma_{\alpha,\gamma}}{\sqrt{k}} Z_k^{(\alpha)} + b_{\alpha,\gamma,\tilde{\rho}} \widetilde{A}(n/k) \left(1 + o_p(1)\right),$$

where with Y_i , $1 \leq i \leq n$ a sequence of independent, identically distributed Pareto random variables with d.f. $F_Y(x) = 1 - 1/x$, $x \geq 1$,

$$Z_k^{(\alpha)} := \frac{\sqrt{k}}{\sigma_{\alpha,\gamma}} \times \frac{1}{k} \sum_{i=1}^k \left(\left(\frac{Y_i^{\gamma_-} - 1}{\gamma_-} \right)^{\alpha} - \mu_{\alpha,\gamma} \right), \tag{15}$$

is a sequence of asymptotically standard normal random variable. Moreover,

$$\mu_{\alpha,\gamma} := E\left(\left(\frac{Y^{\gamma_{-}}-1}{\gamma_{-}}\right)^{\alpha}\right) = \begin{cases} \frac{\Gamma(\alpha+1)\Gamma(-1/\gamma)}{(-\gamma)^{\alpha+1}\Gamma(\alpha+1-1/\gamma)}, & \gamma < 0;\\ \Gamma(\alpha+1), & \gamma \ge 0, \end{cases}$$
(16)

$$\sigma_{\alpha,\gamma}^2 := V\left(\left(\frac{Y^{\gamma_-} - 1}{\gamma_-}\right)^{\alpha}\right) = \mu_{2\alpha,\gamma} - \mu_{\alpha,\gamma}^2,\tag{17}$$

and

$$b_{\alpha,\gamma,\tilde{\rho}} := E\left(\alpha\left(\frac{Y^{\gamma_{-}}-1}{\gamma_{-}}\right)^{\alpha-1}H_{\gamma_{-},\tilde{\rho}}(Y)\right).$$

Remark 2.2. If $\tilde{\rho} < 0$ one can easily get an explicit expression for $b_{\alpha,\gamma,\tilde{\rho}}$, which holds for any $\alpha > 0$ and $\gamma \in \mathbb{R}$. It is given by

$$b_{\alpha,\gamma,\tilde{\rho}} = \begin{cases} \frac{\alpha}{\tilde{\rho}} \left(\frac{\mu_{\alpha-1,\gamma/(1-(\gamma+\tilde{\rho}))}}{(1-(\gamma+\tilde{\rho}))^{\alpha}(\gamma+\tilde{\rho})} - \frac{\mu_{\alpha-1,\gamma}}{\gamma+\tilde{\rho}} - \mu_{\alpha,\gamma} \right), & \gamma < 0, \quad \tilde{\rho} < 0\\ \frac{\Gamma(\alpha+1)}{\tilde{\rho}} \left(\frac{1-(1-\tilde{\rho})^{\alpha}}{\tilde{\rho}(1-\tilde{\rho})^{\alpha}} - \alpha \right), & \gamma \ge 0, \quad \tilde{\rho} < 0 \end{cases}$$

,

with $\mu_{\alpha,\gamma}$ given in (16).

The next lemma follows closely Corollary 3.5.6 from [11].

Lemma 2.2. Under the conditions of Lemma 2.1,

$$\hat{\gamma}_n^{NM}(k) \stackrel{d}{=} \gamma_- + \frac{\sigma_{_{NM}}}{\sqrt{k}} Z_k^{NM} + b_{NM} \ \widetilde{A}(n/k) \left(1 + o_p(1)\right),$$

with

$$b_{NM} := \frac{(1-\gamma_{-})(1-2\gamma_{-})}{(1-\gamma_{-}-\tilde{\rho})(1-2\gamma_{-}-\tilde{\rho})},$$

$$\sigma_{_{NM}}^{2} := \frac{(1-\gamma_{-})^{2}(1-2\gamma_{-})(1-\gamma_{-}+6\gamma_{-}^{2})}{(1-3\gamma_{-})(1-4\gamma_{-})}$$

and with $\overline{\sigma}_{\alpha,\gamma} := \sigma_{\alpha,\gamma}/\mu_{\alpha,\gamma}$, $\mu_{\alpha,\gamma}$ and $\sigma_{\alpha,\gamma}$ given in (16) and (17), respectively, and $Z_k^{(\alpha)}$ defined in (15),

$$Z_k^{NM} := (1 - \gamma_-)(1 - 2\gamma_-) \left(\overline{\sigma}_{2,\gamma} Z_k^{(2)} - 2\overline{\sigma}_{1,\gamma} Z_k^{(1)}\right)$$

is an asymptotically standard normal random variable. Consequently, if $\sqrt{k}\widetilde{A}(n/k) \underset{n \to \infty}{\to} \lambda$ finite, possibly non null,

$$\sqrt{k}(\hat{\gamma}_n^{NM}(k) - \gamma_-) \stackrel{d}{\underset{n \to \infty}{\longrightarrow}} N\left(\lambda \ b_{NM}, \sigma_{NM}^2\right).$$

2.3 Asymptotic behaviour of the new class of estimators

Theorem 2.2. Under the conditions of Lemma 2.1:

$$\hat{\gamma}_{n}^{NM(\theta)}(k) \stackrel{d}{=} (\gamma_{-} + \theta \gamma_{+}) + \frac{1}{\sqrt{k}} \left(\sigma_{NM} Z_{k}^{NM} + \gamma_{+} \sigma_{1,\gamma} Z_{k}^{(1)} \right) + \left(b_{NM} + \theta \gamma_{+} b_{1,\gamma,\tilde{\rho}} + \frac{\theta}{1 - \gamma_{-}} \right) \widetilde{A}(n/k) \left(1 + o_{p}(1) \right),$$

$$(18)$$

with $q_{\gamma,\rho}$ defined in (13).

Proof. Using Lemma 2.1,

$$M_n^{(1)}(k) \stackrel{d}{=} \tilde{a}(t) \left(\frac{1}{1 - \gamma_-} + \frac{\sigma_{1,\gamma}}{\sqrt{k}} Z_k^{(1)} + b_{1,\gamma,\tilde{\rho}} \, \widetilde{A}(n/k) \, (1 + o_p(1)) \right),$$

Next, since $\tilde{a}(t) = \gamma_+ + q_{\gamma,\rho} \tilde{A}(t)$ and $\gamma_+/(1-\gamma_-) = \gamma_+$,

$$M_n^{(1)}(k) \stackrel{d}{=} \gamma_+ + \frac{\gamma_+ \sigma_{1,\gamma}}{\sqrt{k}} Z_k^{(1)} + \left(\gamma_+ b_{1,\gamma,\tilde{\rho}} + \frac{q_{\gamma,\rho}}{1 - \gamma_-}\right) \widetilde{A}(n/k)(1 + o_p(1)),$$

Finally, using Lemma 2.2, (18) follows.

Remark 2.3. If $\gamma < 0$, and $\sqrt{k}\widetilde{A}(n/k) \xrightarrow[n \to \infty]{} \lambda$, finite,

$$\sqrt{k}(\hat{\gamma}_n^{NM(\theta)}(k) - \gamma) \xrightarrow[n \to \infty]{d} N\left(\lambda \left(b_{NM} + \frac{\theta q_{\gamma,\rho}}{1 - \gamma}\right), \sigma_{NM}^2\right).$$

Remark 2.4. With the adequate choice of $\theta_0 = -(1 - \gamma)b_{NM}/q_{\gamma,\rho}$, if $q_{\gamma,\rho} \neq 0$, we have access to an asymptotically unbiased second-order EVI estimator. The adaptively choice of the "optimal" θ is outside the scope of this work.

Remark 2.5. We advise not to choose blindly the value of θ in (8). At this stage we think sensible to draw a few sample paths of $\hat{\gamma}_n^{NM(\theta)}(k)$, for a few values of θ , and elect the value θ_0 which provides higher stability in the region of k values, for which we get admissible estimates. However, we hope to be able to adaptively choose θ on the basis of the bootstrap methodology and the consideration of an auxiliary statistic like $\hat{\gamma}_n^{NM(\theta)}(k) - \hat{\gamma}_n^{NM(\theta)}(\lfloor k/2 \rfloor)$, of the type of the one used in Gomes and Oliveira [9] for the adaptive choice of k through the Hill estimator in (3). This is however outside the scope of this paper.

3 Finite sample properties of the new class of EVI-estimators

We now study the finite sample behaviour of the extreme value index estimator $\hat{\gamma}_n^{NM(\theta)}(k)$ and compare it with the classical Moment estimator. We have chosen two different values of θ : one of them is $\theta = 0$ and the other one was chosen by *trial* until we had a stable sample path of the estimates (for small up to moderate values of k). We generated 5000 pseudorandom samples of size n, with $n \in \{50, 100, 200, 500, 1000, 2000, 5000, 10000\}$, from the following distributions:

- The EV_{γ} distribution in (1), with $\gamma = -0.5, -0.9$ ($\rho = -1, \tilde{\rho} = \gamma$);
- The Arcsin distribution with d.f. $F(x) = (2/\pi) \arcsin(\sqrt{x}), 0 < x < 1$, for which we have $\gamma = \rho = \tilde{\rho} = -2$;
- The Half-normal distribution, i.e., the absolute value of a standard normal distribution ($\gamma = \rho = \tilde{\rho} = 0$).

Although the asymptotic results, from the previous section, exclude the case $\gamma = \rho$, we have decided to include the Arcsin and Half-normal distributions in the simulation study. To ilustraste the finite sample behaviour of the EVI estimators for such models, we present, in Figure 3, the simulated mean values (E) and root mean squared error (RMSE) patterns of the above mentioned estimators, as functions of k, for a sample size n = 200, 1000 and 5000. Figures 3 and 4 have the same simulated quantities for the $EV_{-0.5}$ and Half-normal distribution with n = 1000.

In Tables 1 and 2 we present the simulated mean values and root mean squared errors of the estimators under study, at their simulated optimal levels.

Based on the simulated results here presented, we may draw the following conclusions:

- 1. With the proper choice of θ , the new class of estimators, $\hat{\gamma}_n^{NM(\theta)}(k)$, appears to be asymptotically unbiased. The variance is kept unchanged and is equal to the variance of the Moment estimator.
- 2. The new class of EVI estimators has, in general, reasonably stable sample paths, which makes less troublesome the choice of the level k.
- 3. Although outside of the theoretical framework, we get the same properties for the Arcsin and Half-normal distribution.

E	50	100	200	500	1000	2000	5000	10000				
$\operatorname{Arcsin} (\gamma = -2)$												
MOM	-1.8764	-1.8268	-1.8438	-1.8787	-1.9037	-1.9228	-1.9637	-1.9603				
NM(.45)	-2.3850	-2.1796	-2.1276	-2.0524	-2.0330	-2.0314	-2.0099	-2.0094				
NM(0)	-2.7167	-2.3748	-2.2618	-2.1719	-2.1172	-2.0868	-2.0584	-2.0360				
MOM	-0.9217	-0.7243	-0.6508	-0.5892	-0.5647	-0.5451	-0.5343	-0.5251				
NM(1.3)	-0.7789	-0.6212	-0.5485	-0.5124	-0.4991	-0.4970	-0.4958	-0.4983				
NM(0)	-1.3135	-1.0231	-0.8571	-0.7297	-0.6665	-0.6264	-0.5968	-0.5789				
MOM	-1.5462	-1.2523	-1.1339	-1.0432	-1.0189	-0.9889	-0.9615	-0.9484				
NM(1.75)	-1.2870	-1.0660	-0.9699	-0.9233	-0.9103	-0.9032	-0.8996	-0.8979				
NM(0)	-1.8055	-1.4752	-1.2401	-1.1300	-1.0655	-1.0218	-0.9931	-0.9640				
Half-Normal $(\gamma = -0)$												
MOM	-0.3424	-0.2511	-0.2068	-0.1683	-0.1399	-0.1233	-0.1073	-0.0979				
NM(1.2)	-0.2199	-0.1495	-0.1151	-0.0979	-0.0918	-0.0857	-0.0787	-0.0699				
NM(0)	-0.7854	-0.5834	-0.4669	-0.3652	-0.3104	-0.2737	-0.2392	-0.2138				

Table 1: Simulated Mean Values at Optimal Levels

Table 2: Simulated Root Mean Squared Errors at Optimal Levels

RMSE	50	100	200	500	1000	2000	5000	10000				
Arcsin												
MOM	1.2220	0.7607	0.5326	0.3524	0.2654	0.1987	0.1369	0.1022				
NM(.45)	1.3029	0.7758	0.5067	0.3061	0.2129	0.1506	0.0962	0.0686				
NM(0)	1.4865	0.8982	0.6014	0.3793	0.2763	0.2021	0.1372	0.1006				
G_0.5												
MOM	0.9114	0.5078	0.3273	0.2006	0.1427	0.1053	0.0709	0.0531				
NM(1.3)	0.8392	0.4509	0.2819	0.1643	0.1132	0.0797	0.0503	0.0358				
NM(0)	1.1952	0.7351	0.4992	0.3333	0.2513	0.1949	0.1434	0.1139				
G_0.9												
MOM	1.3489	0.7406	0.4815	0.2986	0.2164	0.1619	0.1120	0.0860				
NM(1.75)	1.2155	0.6301	0.3895	0.2264	0.1570	0.1100	0.0690	0.0488				
NM(0)	1.5544	0.8873	0.5889	0.3726	0.2782	0.2090	0.1477	0.1133				
Half-Normal												
MOM	0.5261	0.3589	0.2679	0.2033	0.1690	0.1457	0.1234	0.1104				
NM(1.2)	0.4500	0.2859	0.1991	0.1400	0.1157	0.1004	0.0879	0.0796				
NM(0)	0.9222	0.6773	0.5308	0.4104	0.3490	0.3019	0.2594	0.2310				



Figure 2: Simulated mean values (above) and root mean squared errors (below) of MOM, NM(0) and NM(0.45) extreme value index estimators, for samples of size n = 200 (left), 1000 (center) and 5000 (right) from the Arcsin distribution.



Figure 3: Simulated mean values (left) and root mean squared errors (right) of MOM, NM(0) and NM(0.45) extreme value index estimators, for samples of size n = 1000 from the EV distribution with $\gamma = -0.5$.

References

 Caeiro, F., Gomes, M.I. and Pestana, D.D. (2005). Direct reduction of bias of the classical Hill estimator. *Revstat*, 3(2), 113-136.



Figure 4: Simulated mean values (left) and root mean squared errors (right) of MOM, NM(0) and NM(1.2) extreme value index estimators, for samples of size n = 1000 from the Half-normal distribution ($\gamma = 0$).

- [2] Dekkers, A.L.M., Einmahl, J.H.J. and de Haan, L. (1989). A moment estimator for the index of an extreme-value distribution. *The Annals* of Statistics, 17(4), 1833-1855.
- Draisma, G., de Haan, L., Peng, L. and Themido Pereira, T., (1999).
 A bootstrap-based method to achieve optimality in estimating the extreme value index. Extremes 2(4), 367-404.
- [4] Fraga Alves, M.I. (2001). "Weiss-Hill estimator". Test, 10, 203-224.
- [5] Gnedenko, B.V. (1943). Sur la distribution limite du terme maximum d'une série aléatoire. Ann. Math. 44, 423-453.
- [6] Gomes, M.I., de Haan, L. and Henriques Rodrigues, L. (2008). Tail Index estimation for heavy-tailed models: accommodation of bias in weighted log-excesses. J. R. Stat. Soc. Ser. B, 70(1), 31-52.
- [7] Gomes, M.I., Martins, M.J. and Neves, M.M. (2007). Improving second order reduced-bias tail index estimation. *Revstat*, 5(2), 177-207.
- [8] Gomes, M.I. and Neves, C. (2008). Asymptotic comparison of the mixed moment and classical extreme value index estimators. *Statistics & Probability Letters* 78, 643-653.
- [9] Gomes, M.I. and Oliveira, O. (2001). The bootstrap methodology in Statistics of Extremes - choice of the optimal sample fraction. Extremes 4(4), 331-358.

- [10] de Haan, L. (1970). On Regular Variation and its Application to the Weak Convergence of Sample Extremes. Mathematical Centre Tract 32, Amesterdam.
- [11] de Haan, L. and Ferreira, A. (2006). Extreme Value Theory: an Introduction. Springer, New York.
- [12] Hill, B.M. (1975). A Simple General Approach to Inference About the Tail of a Distribution. *The Annals of Statistics*, 3(5), 1163-1174.