Dividends and ruin problems in finite time

Rui M R Cardoso
CMA and Departamento de Matemática
Faculdade de Ciências e Tecnologia
Universidade Nova de Lisboa
Quinta da Torre
P-2829-516 Caparica, Portugal
e-mail: rrc@fct.unl.pt

Abstract

In this work we consider three different classical risk models modified by the introduction of a constant dividend barrier, that is, when the surplus exceeds this threshold the insurer pays dividends to shareholders.

For our first model the dividend rate is equal to the premium income, the so called dividend barrier strategy. We derive the finite time version of recursions due to Dickson and Waters [Dickson, D. C. M. and Waters, H. R. (2004). Some optimal dividends problems. Astin Bulletin, 34(1):49–74] and we present a numerical procedure based on the Markov chain approach (Cardoso and Egídio dos Reis [Cardoso, R. M. R. and Egídio dos Reis, A. D. (2002). Recursive calculation of time to ruin distributions. Insurance: Mathematics and Economics, 30(2):219–230] and Cardoso and Waters [Cardoso, R. M. R. and Waters, H. R. (2003). Recursive calculation of finite time ruin probabilities under interest force. Insurance: Mathematics and Economics, 33(3):659–676; Cardoso, R. M. R. and Waters, H. R. (2005). Calculation of finite time ruin probabilities for some risk models. Insurance: Mathematics and Economics, 37(2):197–215]) for the calculation of the expected discounted value of dividend payouts, until ruin occurs or up to time t, and its net value when shareholders provide the initial surplus and pay the deficit at ruin. Also, using the Markov chain approach, we produce bounds for the former expected value.

We extend this risk model by allowing the process to continue after ruin, the second risk model analysed in this paper and introduced by Dickson and Waters [Dickson, D. C. M. and Waters, H. R. (2004).

Some optimal dividends problems. Astin Bulletin, 34(1):49–74]. The Markov chain approach is again used to obtain approximations to the net expected present value of dividends paid.

A threshold dividend strategy is also covered under which the dividend rate is lower than the premium rate. For this risk model we present bounds for the expected present value of dividend payments.

We also present numerical algorithms for the calculation of the finite time ruin probability.

Keywords: Ruin probability, Numerical algorithms, Markov chains, Dividend payments, Compound Poisson model, Finite time.

1 Introduction

In this work we consider the classical risk model modified by the introduction of a constant dividend barrier. Let $U(t)$ be the classical risk surplus at time $t(\geq 0)$, defined by

$$U(t) = u + ct - S(t) \tag{1}$$

where u is the initial surplus, c is the rate of premium income and $S(t)$ is the aggregate claims up to time t, which follows a compound Poisson process with Poisson parameter λ. We denote by G the individual claim distribution, assuming that all claims are positive with mean m_1. Without loss of generality we assume $\lambda = m_1 = 1$ so that $c = (1+\theta)$, where θ is the premium loading factor. The surplus process (1) can be written as a stochastic differential equation:

$$dU(t) = I(U(t))dt - dS(t) \tag{2}$$

where, in this case, $I(x) = c$.

The finite time ruin probability is defined by:

$$\psi(u,t) = \Pr[U(\tau) < 0, \text{ for } 0 \leq \tau \leq t]$$

and the ultimate ruin probability is $\psi(u) = \lim_{t\to\infty} \psi(u,t)$.

In our case we are interested in risk processes for which each time the surplus exceeds a threshold, b, dividends are paid to the stockholders. Let D denote the present value of the total dividend payments until ruin occurs, at interest force δ, with $V(u,b) = \mathbb{E}[D]$.

Over the last decade there has been an increasing interest in dividend strategies and several papers have been produced considering the compound

Poisson model, some other specific risk models and even different types of dividend strategies, see e.g. Dickson and Waters (2004), Gerber and Shiu (2006) and the references therein. In some cases the aim is not only the calculation of the expectation of the discounted dividends but also its maximization and consequently the determination of the optimal dividend barrier b^*. For a general approach see Gerber et al. (2006a). Mainly the focus is on infinite time and so it is the aim of this work to provide numerical methods concerning the discounted expectation of dividends paid during a finite time interval. Therefore, let $D(t)$ denote the present value of the total dividend payments up to time t or until ruin occurs, at interest force δ, with $V(u,b,t) = \mathbb{E}[D(t)]$ so that $V(u,b) = \lim_{t\to\infty} V(u,b,t)$.

In our case three different situations are analysed. For the first model studied, we consider that when the surplus reaches b it remains there until a claim occurs and the insurance company pays dividends at rate c. Hence this process is defined by (2) together with

$$I(x) = \begin{cases} c & \text{if } 0 \le x < b \\ 0 & \text{if } x \ge b \end{cases}$$

Therefore ultimate ruin is certain. This dividend strategy is known as a barrier strategy. If $u > b$ then the shareholders immediately receive, at time 0, the difference $u - b$.

Apart from receiving the dividend income we may reasonably expect that the shareholders pay for the deficit at ruin and also provide the initial surplus u. Here we assume, as before, that there is no further business after the time of ruin. In this situation the expected present value of (net) income to shareholders up to time t is:

$$L(u,b,t) = V(u,b,t) - u - \mathbb{E}[e^{-\delta T}\, Y\, \mathbb{I}(T \le t)] \tag{3}$$

where T is the time of ruin, Y is the deficit at ruin and $\mathbb{I}(A)$ is the indicator function of an event A. Note that we are only interested in the behavior of the surplus process until time t. We denote the last term in the expression above by $\varphi(u,b,t)$. This modification, for an infinite time horizon, was suggested by Dickson and Waters (2004) where they also obtained the optimal value b^0 that maximizes $L(u,b) = \lim_{t\to\infty} L(u,b,t)$. This new mathematical problem was studied by Gerber et al. (2006b) where they compare the optimal values b^* and b^0. The determination of the optimal value b^0 is also covered in Gerber et al. (2006a).

We may also consider that each time ruin happens the shareholders immediately pay the amount of the deficit at ruin and the surplus is restored to 0 allowing, then, the risk process to continue up to time t. Consequently the surplus moves between 0 and b and it can remain at b for a period, but immediately moves away from 0. This modified risk model is denoted in our paper by model II and our aim for such a process is the calculation of the expected present value of shareholders' net income up to time t which, in turn, we denote by $M(u, b, t)$. This dividend scenario was also introduced by Dickson and Waters (2004), although they considered, again, an infinite horizon.

For the third risk model studied in this work the rate of dividend income is $\hat{c} < c$, which contrasts with the previous modified risk models since now the surplus can go above the level b. This means that the insurance company still receive premiums at rate c but if the surplus is above b they only keep for themselves the rate $c - \hat{c}$. As a consequence, the ultimate ruin probability is now not equal to one, if $c - \hat{c} > \lambda m_1$. Besides, it is not certain that, after going above b, the surplus process would even fall below b. For this so called threshold strategy, Dickson and Drekic (2006) obtained exact expressions for the expected present value of the total dividend payments until ruin occurs, when individual claim amounts are exponentially or mixed exponentially distributed. This dividend strategy is also studied in Badescu et al. (2007) exploiting the connection between a surplus process and an embedded fluid queue process. This model is also under consideration in Lin and Pavlova (2006) studying the Gerber-Shiu discounted penalty function, deriving two integro-differential equations and solving them. Here we shall discuss the calculation of such expectations although with a finite horizon. In Leung et al. (2008) the focus is also on finite time, although modelling the surplus process as a restricted geometric Brownian motion. They obtained an analytical solution for the value function of the surplus process and, as subproduct, explicit expressions for the survival probability and the expected present value of dividends. Generalisations of the latter modified risk model have been also considered: see e.g. Lin and Sendova (2007) and Albrecher and Hartinger (2007) where both consider the Gerber-Shiu discounted penalty function under a risk model with multiple barriers.

As far as the net surplus is concerned, below b the rate of premium income is c and above that threshold the rate is $c - \hat{c}$. Thus this surplus process is governed by (2) with

$$I(x) = \begin{cases} c & \text{if } 0 \leq x < b \\ c - \hat{c} & \text{if } x \geq b \end{cases}$$

This variable premium rate surplus process was studied in Cardoso and Waters (2005) (see their Section 4), where they derived numerical methods to obtain approximations to and bounds for ruin probabilities in finite time using a Markov chain approach.

We note that the above risk models are time-homogeneous Markov processes which is an essential feature for developing our numerical algorithms. For the modified risk processes I and III we denote the probability of ruin up to time t by $\psi(u, b, t)$.

In the following section we present discrete (in time and amount) processes that are related to the continuous risk processes described above. These discrete models allow us to set up the numerical methods for the calculation of the quantities of interest. Then, we establish the finite time version of some recursions derived by Dickson and Waters (2004) to calculate approximations to $V(u, b, t)$ and $L(u, b, t)$, concerning the first modified model. In Section 4, we explain how the Markov chain approach can be adapted to incorporate the features of the three modified risk models and produce approximations to or bounds for the expected values defined in the present section and for the ruin probability within finite time. Numerical values are shown in the last section.

2 The discrete models

Let us consider again the surplus process defined by equation (1). The basic idea to construct our numerical algorithms is to replace the surplus process by a discrete process, which we denote U_d, starting from u, having the same rate of premium income and taking values x_j only at the time points $h, 2h, \ldots, Kh$, where h is a positive small number and K is such that $K = t/h$. The values x_j are defined in the following way: $x_0 = 0$ and

$$x_j = x_{j-1} + \int_0^h I(U(t) \mid U(0) = x_{j-1}, S(h) = 0)\, dt$$

for $j = 1, 2 \ldots$. This means that if the surplus process takes the value x_{j-1}, it would reach the value x_j in a time interval of length h if there were no claims during that time period, that is $x_j - x_{j-1} = ch$, which is the insurer's income in that time interval. Therefore the positive quadrant of the (*time* $\times$ *monetary amount*) plane is covered by a rectangular grid of points $\{nh\}_{n=0}^{\infty} \times \{x_j\}_{j=0}^{\infty}$. The process $\{U_d(nh);\ n = 0, 1, \ldots\}$ approximates the continuous surplus process $\{U(t);\ t \geq 0\}$ and the smaller the value of h, the better the approximation.

From the above this process may be defined as follows:

$$U_d(nh) = u + cnh - S_d(nh), \quad n \in \mathbb{N},$$

where $S_d(nh)$ is a compound Poisson process with Poisson parameter λ, taking, as a consequence of the definition of U_d, the values $\{x_j\}_{j=0}^{\infty}$. Intuitively S_d represents the discrete version of S on the monetary amount axis. Hence, U_d, verifies the following:

$$U_d(nh) = U_d((n-1)h) + ch - [S_d(nh) - S_d((n-1)h)], \quad n \in \mathbb{N},$$

and consequently, it is a Markov chain, since its value at any time nh depends on the value at time $(n-1)h$ and on the cdf of $S_d(nh) - S_d((n-1)h)$, that is on the cdf of $S_d(h)$. This distribution function, which we denote F_d, can be evaluated using the Panjer recursion formula and constructing an approximate discrete distribution function G_d of G, using the mean preserving method of De Vylder and Goovaerts (1988), with masses only at the points $\{x_j\}_{j=0}^{\infty}$. For simplicity we write $f_j = F_d(x_j) - F_d(x_j^-)$, for $j = 0, 1, \ldots$.

Moreover, two bounding distributions can also be constructed by concentrating the mass of probability $G(x_j) - G(x_{j-1})$ at x_j or x_{j-1}. The smaller the value of h, the closer the approximating and the bounding distributions are likely to be to G. Applying, again, the Panjer recursion formula, we can compute two discrete distribution functions, $\overline{F}_d$ and $\underline{F}_d$, such that,

$$\underline{F}_d(x) \leq F_d(x) \leq \overline{F}_d(x), \ \forall x \in \mathbb{R}_0^+$$

For this discrete risk process we define ruin as the event when the surplus goes to zero ($= x_0$) or below at some time $n > 0$, although we allow the initial surplus to be zero at time zero.

From the bounding cdf's $\underline{F}_d(x)$ and $\overline{F}_d(x)$ it is possible to construct two Markov chains $\overline{U}_d$ and $\underline{U}_d$ which bound, above and below, respectively, the continuous surplus process (1). These chains also behave as discrete surplus processes and are defined similarly to $\{U_d(nh),\ n \in \mathbb{N}_0\}$, with the same premium income and initial surplus.

The methodology described above goes back to Dickson and Waters (1991) and has been used by many authors, see e.g. Cardoso and Waters (2003) and Cardoso and Waters (2005), who constructed discrete bounding chains for different types of surpluses. The adaptation of these methods to the modified risk models is straightforward. For the first two modified risk models, the surpluses are bounded by a horizontal line b. Hence, the set of values x_j is bounded by x_B where B is such that $b = Bch$. Concerning the third modified model, there is no such bound but the points x_j are not equally spaced: below x_B the difference $x_j - x_{j-1}$ is equal to ch and above the difference is $(c - \hat{c})h$.

We will firstly suppose, as in Dickson and Waters (2004), that a dividend of value ch is payable at time nh if the surplus reached at time $(n-1)h$

the value x_B and there were no claims during that period. We will also present, later, another less conservative definition concerning the payment of dividends

Throughout this paper, we will identify the quantities which approximate or bound, as appropriate, $\psi(u,b,t)$, $V(u,b,t)$, $L(u,b,t)$ and $M(u,b,t)$, and the chains related to the above discrete models by adding the subscript "d". In order to alleviate problems with notation, without ambiguity, we use the same representation in each of the modified risk models. We also assume, for convenience, that $u = x_\omega$, for some $\omega \in \{0, 1, \ldots\}$.

For all the numerical examples in this paper, we consider $h = \frac{1}{c\beta}$ where β is some large positive number. This means that, for instance concerning the first modified model, that the corresponding chain U_d has a state space with values x_j equal to $\frac{j}{\beta}$, with $j = 0, 1, \ldots, B$.

3 Some recursions based on Dickson and Waters (2004)

We now present finite time versions of some recursions due to Dickson and Waters (2004) (see their section 5). Here we only consider the first modified discrete risk process. We begin by deriving a recursion for the calculation of approximations to $V(u,b,t)$. Conditioning on the aggregate claim amount in the first period we get, for $i = \max(0, B-n+1), \ldots, B-1$,

$$V_d(x_i, b, nh) = e^{-\delta h} \sum_{j=0}^{\min(i-B+n-1,i)} f_j\, V_d(x_{i+1-j}, b, (n-1)h) \tag{4}$$

This result follows by considering what happens in the first time period of length h. Note that starting from x_i, with $i < B$, the surplus takes the value x_{i+1}, if there were no claims during the first period, or it takes the value x_i, if the aggregate claim amount over the same period is equal to ch, and so on. If ruin happens then no further dividends are paid. Otherwise the surplus, at time h, is positive, leveled to x_{i+1-j}, with $j = 0, 1, \ldots, \min(i-B+n-1, i)$, and then we should look for the dividend payouts in the next $(n-1)h$ periods.

If the surplus starts from x_B, the surplus could be, time h later, at x_B if either there were no claims, and thus dividends are paid, or the aggregate claim amount is equal to ch. Therefore we have

$$V_d(x_B, b, nh) = e^{-\delta h} \left(f_0\,(ch + V_d(x_B, b, (n-1)h)) + \sum_{j=1}^{\min(B,n-1)} f_j\, V_d(x_{B+1-j}, b, (n-1)h) \right) \tag{5}$$

In the above expressions we take into account that

$$V_d(x_l, b, mh) = 0 \quad \text{for } l = 0, \ldots, B - m,$$

which is reflected in the values of i and in the upper limit of the summations. These equations give us too the starting values for formulae (4) and (5).

Using the same arguments, we also obtain a recursion for the calculation of approximations to $\varphi(u, b, t)$. Hence, we have, for $i = 0, 1, \ldots, B - 1$,

$$\begin{aligned}
\varphi_d(x_i, b, nh) &= e^{-\delta h}\left(\sum_{j=0}^{i} f_j \varphi_d(x_{i+1-j}, b, (n-1)h) + \sum_{j=i+1}^{\infty} f_j\,(x_j - x_{i+1})\right) \\
&= e^{-\delta h}\left(\sum_{j=0}^{i} f_j \varphi_d(x_{i+1-j}, b, (n-1)h) + \lambda h - \sum_{j=0}^{i} x_j f_j - x_{i+1}(1 - F_d(x_i))\right)
\end{aligned}$$

and

$$\begin{aligned}
\varphi_d(x_B, b, nh) = e^{-\delta h}\Bigg(& f_0 \varphi_d(x_B, b, (n-1)h) + \sum_{j=1}^{B} f_j \varphi_d(x_{B+1-j}, b, (n-1)h) \\
& + \sum_{j=B+1}^{\infty} f_j\,(x_j - (x_B + ch))\Bigg) \\
= e^{-\delta h}\Bigg(& f_0 \varphi_d(x_B, b, (n-1)h) + \sum_{j=1}^{B} f_j \varphi_d(x_{B+1-j}, b, (n-1)h) + \lambda h \\
& - \sum_{j=0}^{B} x_j f_j - (x_B + ch)(1 - F_d(x_B))\Bigg)
\end{aligned}$$

Combining the above two recursions and considering formula (3) we then obtain approximations to $L(u, b, t)$.

4 Some recursions based on the Markov chain approach

We present in this section some recursions to obtain approximate values or bounds for the finite time ruin probability and the expected values defined earlier in Section 1, $V(u, b, t)$, $L(u, b, t)$ and $M(u, b, t)$, considering the methodology described in Cardoso and Egídio dos Reis (2002), Cardoso and Waters (2003) and Cardoso and Waters (2005). This approach was used in those papers to calculate approximations to or bounds for $\psi(u, t)$ for the classical risk model, the compound risk model, the risk model with possibility of investment, the risk model with variable premium rate and the time varying risk model.

We already concluded, in Section 2, that the discrete risk models, presented there, are Markov chains whose transition probabilities will now be derived completing in this way their definitions. We will apply the Markov chain approach to each of the modified risk models introduced in Section 1.

Throughout this section we use the following notation for the one step transition probabilities,

$$p_{i,j} = \Pr[U_d(nh) = x_j \mid U_d((n-1)h) = x_i], \qquad \text{for } n \in \mathbb{N}$$

4.1 For the modified risk model I

In this case the one step transition probabilities $p_{i,j}$ for the chain U_d are defined in the following way:

$$p_{i,j} = \begin{cases} 0 & \text{for } i > B \text{ or } j > i+1 \\ F_d((x_{i+1} - x_{j-1})^-) - F_d((x_{i+1} - x_j)^-) & \text{for } 1 \leq i \leq B-1 \text{ and } 1 \leq j \leq i+1 \\ 1 - F_d(x_{i+1}^-) & \text{for } 1 \leq i \leq B-1 \text{ and } j = 0 \\ F_d((x_B + ch - x_{j-1})^-) - F_d((x_B + ch - x_j)^-) & \text{for } i = B \text{ and } 1 \leq j \leq i-1 \\ F_d((x_B + ch - x_{B-1})^-) & \text{for } i = B \text{ and } j = B \\ 1 - F_d((x_B + ch)^-) & \text{for } i = B \text{ and } j = 0 \end{cases} \tag{6}$$

Since the chain U_d starts from u, we have $\Pr[U_d(0) = x_\omega] = 1$. The state x_0 represents ruin and therefore it is an absorbing state. However the chain is allowed to start from 0 and in such a case we have the following first step transition probabilities

$$\begin{aligned} \Pr[U_d(h) = x_1 \mid U_d(0) = x_0] &= F_d(x_1^-) \\ \Pr[U_d(h) = x_0 \mid U_d(0) = x_0] &= 1 - F_d(x_1^-) \end{aligned}$$

The n-step transition probabilities, from state $x_\omega \to x_j$, may be calculated recursively using the formula

$$p_{\omega,j}^{(n)} = \sum_{i=j^*}^{\min(\omega+n-1,B)} p_{\omega,i}^{(n-1)} \, p_{i,j} \tag{7}$$

which is derived from the Chapman-Kolmogorov equations, where $j^* = \max(\min(j,1), j-1)$.

According to the assumption, set in Section 2, concerning the payment of dividends (in discrete time), we have the following approximation to

$V(u,b,t)$,

$$V_d(u,b,t) = \sum_{n=1}^{K} che^{-nh\delta} p_{\omega,B}^{(n-1)} f_0 \tag{8}$$

Intuitively, we sum up the expected present value of dividends for each period. A dividend of value ch is paid in the n-th period with probability $p_{\omega,B}^{(n-1)} f_0$. Formula (8) can be refined since, starting from x_ω, the surplus only attains the dividend level at time nh such that $\omega + n = B$ if no claims occurred before. Hence, the summation in (8) may start from $n = B - \omega + 1$.

One advantage of this algorithm is that we can also obtain, simultaneously, approximate values for the ruin probability in finite time:

$$\psi(u,b,t) \approx \psi_d(u,b,t) = p_{\omega,0}^{(K)}$$

The assumption related to the payment of dividends is rather conservative because the dividend of value ch is only distributed at time nh if the chain takes the value x_B at time $(n-1)h$ and there were no claims during the time period $[(n-1)h, nh]$. The surplus may, in fact, up cross b during this time interval and hence dividends should be paid and these payments are not considered in discrete time. A less restrictive assumption for the distribution of dividends is the following: a dividend of value ch is payable when the chain attains x_B. Thus, an approximation to the expected present value of the total of payments is now given by

$$V_d^*(u,b,t) = \sum_{n=0}^{K-1} che^{-nh\delta}\, p_{\omega,B}^{(n)}$$

which may lead to an overestimation of the discounted expectation of dividends payouts. A natural approximation to $V(u,b,t)$ is therefore the average of the two expressions above:

$$\tilde{V}_d(u,b,t) = \sum_{n=1}^{K} che^{-nh\delta} \left(\frac{f_0 + e^{h\delta}}{2} \right) p_{\omega,B}^{(n-1)}$$

However, from our numerical examples, these two approximations seems to be less accurate than the one given by formula (8). Therefore for the remainder of this paper we only make use of the approximation $V_d(u,b,t)$. Nevertheless, these definitions concerning the payment of dividends can lead us to bounds for $V(u,b,t)$. To get these quantities we use the bounding chains $\overline{U}_d$ and $\underline{U}_d$. The whole methodology involved can be seen in Cardoso and Waters (2005).

For the former chain its transition probabilities are obtained by replacing F_d by $\overline{F}_d$ in (6). In the case of $\underline{U}_d$, we have

$$\begin{aligned}\Pr[\underline{U}_d(h) = x_1 \mid \underline{U}_d(0) = x_0] =& \underline{F}_d(x_0) \\ \Pr[\underline{U}_d(h) = x_0 \mid \underline{U}_d(0) = x_0] =& 1 - \underline{F}_d(x_0)\end{aligned}$$

and

$$\underline{p}_{i,j} = \begin{cases} 0 & \text{for } i > B \text{ or } j > i+1 \\ \underline{F}_d(x_{i+1} - x_j) - \underline{F}_d(x_{i+1} - x_{j+1}) & \text{for } 1 \leq i \leq B-1 \text{ and } 1 \leq j \leq i+1 \\ 1 - \underline{F}_d(x_{i+1} - x_1) & \text{for } 1 \leq i \leq B-1 \text{ and } j = 0 \\ \underline{F}_d(x_B + ch - x_j) - \underline{F}_d(x_B + ch - x_{j+1}) & \text{for } i = B \text{ and } 1 \leq j \leq i-1 \\ \underline{F}_d(x_B + ch - x_B) & \text{for } i = B \text{ and } j = B \\ 1 - \underline{F}_d(x_B + ch - x_1) & \text{for } i = B \text{ and } j = 0 \end{cases}$$

Using similar recursions to (7) we are able to compute $\overline{p}^{(n)}_{\omega,j}$ and $\underline{p}^{(n)}_{\omega,j}$.

Hence, bounds for $V(u, b, t)$ are given by the following two expressions:

$$\begin{aligned}\overline{V}_d(u, b, t) &= \sum_{n=0}^{K-1} che^{-nh\delta} \overline{p}^{(n)}_{\omega,B} \\ \underline{V}_d(u, b, t) &= \sum_{n=1}^{K} che^{-nh\delta} \underline{p}^{(n-1)}_{\omega,B} \, \underline{f}_0\end{aligned}$$

so that $\underline{V}_d(u, b, t) \leq V(u, b, t) \leq \overline{V}_d(u, b, t)$. Furthermore, we can determine lower and upper bounds for the probability of ruin within finite time,

$$\overline{p}^{(K)}_{\omega,0} \leq \psi(u, b, t) \leq \underline{p}^{(K)}_{\omega,0}$$

We now turn to the calculation of approximations to

$$\varphi(u, b, t) = \mathbb{E}[e^{-\delta T} Y \, \mathbb{I}(T \leq t)]$$

Conditioning on the time of ruin and on the value of the surplus time h before the time of ruin, we get the approximation

$$\varphi_d(u, b, t) = \sum_{n=1}^{K} e^{-nh\delta} \sum_{i=1}^{\min\{\omega+n-1, B\}} \mathbb{E}_i \times p^{(n-1)}_{\omega,i} \times p_{i,0}, \qquad \text{for } \omega > 0$$

where

$$\mathbb{E}_i = \mathbb{E}[S_d(h) - x_{i+1} \mid S_d(h) > x_i], \qquad \text{for } i < B$$

and

$$\mathbb{E}_B = \mathbb{E}[S_d(h) - (x_B + ch) \mid S_d(h) > x_B]$$

We note that ruin happens at time nh, given that at time $(n-1)h$ the discrete surplus takes the value x_i, if the aggregate claim amount, from $(n-1)h$ up to nh, is bigger than or equal to $x_{i+1} = x_i + ch$ and, consequently, the deficit at ruin is the difference $S_d(h) - x_{i+1}$. By the law of total probability, we can evaluate $\mathbb{E}_i$, since

$$\mathbb{E}[S_d(h)] = \mathbb{E}[S_d(h) \mid S_d(h) > x_i](1 - F_d(x_i)) + \mathbb{E}[S_d(h) \mid S_d(h) \leq x_i]F_d(x_i)$$

where

$$\mathbb{E}[S_d(h) \mid S_d(h) \leq x_i] = \sum_{j=0}^{i} x_j \frac{f_j}{F_d(x_i)}$$

The special case $\omega = 0$ means that the initial surplus is zero. Then

$$\varphi_d(0, b, t) = e^{-h\delta}\mathbb{E}_0 \times (1 - F_d(x_1^-)) + \sum_{n=2}^{K} e^{-nh\delta} \sum_{i=1}^{\min\{n-1,B\}} \mathbb{E}_i \times p_{0,i}^{(n-1)} \times p_{i,0}$$

We can now write down the approximation $L_d(u, b, t)$:

$$L_d(u, b, t) = V_d(u, b, t) - u - \varphi_d(u, b, t)$$

4.2 For the modified risk model II

In this model the shareholders, apart from providing the initial surplus, pay the amount of the deficit at ruin each time ruin occurs, so that the surplus level is restored to zero and this time point is a renewal point. As far as the corresponding discrete model is concerned, the state $x_0 = 0$ is no longer an absorbing state and the transition probabilities with origin in x_0 are, then, the same as in the previous risk model when the process starts from 0, which is not surprising since now when ruins happens a new process begins with initial level 0. Thus, adding the following probabilities to those defined by expressions (6)

$$\begin{aligned} p_{0,0} &= 1 - F_d(x_1^-) \\ p_{0,1} &= F_d(x_1^-) \end{aligned}$$

we complete the definition of the one step transition probabilities. Another modification that comes along with this new risk process is the lower limit in the summation in recursion (7) which is now set to $\max(0, j-1)$. Hence, after calculating the nth transition probabilities, we get the following approximation to $M(u, b, t)$:

$$M_d(u, b, t) = V_d(u, b, t) - u - \varphi_d(u, b, t)$$

remarking that $V_d(u, b, t)$ is calculated using formula (8), but applying the transition probabilities concerned to this discrete modified risk model, and that

$$\varphi_d(u, b, t) = \sum_{n=1}^{K} e^{-nh\delta} \sum_{i=0}^{\min\{\omega+n-1,B\}} \mathbb{E}_i \times p_{\omega,i}^{(n-1)} \times p_{i,0}$$

since now the Markov chain can *start* from 0 at any time point nh.

4.3 For the modified risk model III

In what follows we consider the methodology built up by Cardoso and Waters (2005) for constructing two Markov chains, $\underline{U}_d$ and $\overline{U}_d$, that bound the time continuous modified surplus with variable premium rate. The transition probabilities corresponding to those chains can be found in that paper, which were used for the computation of bounds for $\psi(u, b, t)$. However, those authors could not show that the resulting probabilities $\overline{p}_{\omega,0}^{(K)}$ and $\underline{p}_{\omega,0}^{(K)}$ are in fact bounds for $\psi(u, b, t)$. The difficulty lies in the definition of the transition probabilities because it is assumed that, over a time interval of length h, the rate of premium income is constant, where for the continuous surplus it is not if, during that time interval, the surplus crosses the level b. However, in practice, $\overline{p}_{\omega,0}^{(K)}$ and $\underline{p}_{\omega,0}^{(K)}$ are unlikely, respectively, to be greater and smaller than the finite time ruin probability and therefore they can be regarded as lower and upper bounds. However, for our case, since the net premium rate above b is smaller than c, $\underline{p}_{\omega,0}^{(K)}$ is definitely an upper bound for $\psi(u, b, t)$. Apart from producing bounds for the probability of ruin, we are, here, interested in the computation of bounds for $V(u, b, t)$ using the Markov chain technique. The same argument still holds here and it does not allow us to obtain a proper upper bound for that expected value. The average of these bounds gives an approximation to the expected discounted value of dividends paid up to time t or until ruin happens. These dividends are payable, at rate $\hat{c}$, if the insurer's surplus is not below b.

To obtain a lower bound for $V(u, b, t)$, consider the discrete (in time and in amount) risk process $\underline{U}_d$ for which a dividend, of amount $\hat{c}h$, is payable at nh if $\underline{U}_d((n-1)h) \geq x_B$ and time h later the process $\underline{U}_d$ is still above x_B.

The probability of such an event is given by

$$\underline{p}_{\omega,B\nearrow}^{(n)} = \sum_{j=B}^{\omega+n-1} \underline{p}_{\omega,j}^{(n-1)} \times \underline{F}_d(x_{j+1} - x_{B+1}) \tag{9}$$

Note that $\underline{F}_d(x_{j+1} - x_{B+1})$ is the conditional probability of $\underline{U}_d$ being above x_B, at end of some time period of length h, given that, at the beginning of the same period, $\underline{U}_d$ had value x_j, with $j \geq B$. Using (9) we then obtain a lower bound for $V(u,b,t)$:

$$\underline{V}_d(u,b,t) = \sum_{n=1}^{K} \hat{c}he^{-nh\delta}\underline{p}_{\omega,B\nearrow}^{(n)}$$

As said before, the Markov chain approach does not produce an upper bound for $V(u,b,t)$. Nevertheless we can obtain an *approximate* value of that bound. We first consider the event "the chain $\overline{U}_d$ is above or equal to x_B at time nh", which has probability

$$\overline{p}_{\omega,B^+}^{(n)} = \sum_{j=B}^{\omega+n} \overline{p}_{\omega,j}^{(n)}$$

Hence, applying the second definition, presented in this paper, concerning the payment of dividends (see Subsection 4.1), we get the following value

$$\overline{V}_d(u,b,t) = \sum_{n=0}^{K-1} \hat{c}he^{-nh\delta}\overline{p}_{\omega,B^+}^{(n)}$$

We remark that we cannot prove that $\overline{V}_d(u,b,t) \geq V(u,b,t)$ because the premium income over a time interval of length h, for the chain $\overline{U}_d$, is not always greater than the premium income received for $U(t)$. If the surplus process falls below b in the time interval $[nh, nh+h]$, the total premium income will be greater than $(c-\hat{c})h$. A similar argument shows that $\overline{p}_{\omega,0}^{(K)}$ cannot be regarded as a lower bound for $\psi(u,b,t)$.

5 Numerical results and some comments

For the numerical examples we present in this section we set the discretisation parameter β to 100, which we found adequate for the purpose and also because some existing numerical values for $V_d(u,b)$ were calculated using the same parameter value. We use two contrasting distributions for the individual claim amounts in terms of the right tail: exponential(1) and Pareto(3,2) whose means are equal to one. The Poisson parameter is also set to 1.

Although we have assumed, for convenience, that $u = x_\omega$, for some $\omega \in \{0, 1, \ldots\}$, our numerical methods work when this condition is not verified. In such a case we just need to (re)define the first step transition probabilities from u to x_i as in Cardoso and Waters (2005).

5.1 For the modified risk model I

In Tables 1 and 2 we present approximations to $V(u, b, t)$, for several combinations of the initial surplus, the dividend barrier and time t. We set the interest force per period to be 0.1% and 0.5%. In both tables the individual claim amounts are exponentially distributed and the premium loading factor, θ, is 0.1, so that $c = 1.1$. We can regard these governing parameters in a different way: for $t = 1000$ we can think in a time horizon of ten years with 100 claims expected per annum and a force of interest of 10% (corresponding to $\delta = 0.1\%$). The values $V_d(u, b, t)$ were calculated using the recursion given by (4) and (5) and formula (8) which produces the same numerical values. We also show the exact values $V(u, b)$ given by formula (2.7) in Dickson and Waters (2004) and their approximations, $V_d(u, b)$, obtained using the numerical algorithm given by their formulae (5.1) and (5.2). We observe that the approximations $V_d(u, b, t)$ tend to the approximate values $V_d(u, b)$. Moreover they are identical, up to four decimal places, for $u = 0, 10$, $b = 10$ and for some large values of t. For the other combinations, the approximations $V_d(u, b, 1000)$ and $V_d(u, b)$ are not so close because the finite time ruin probabilities are not equal to one, meaning that the shareholders can still receive dividends from the insurance company. We remark that the above patterns are present for other numerical values we had calculated. Among the numerical algorithms derived in Subsection 4.1, it seems that formula (8) gives the closest values to $V(u, b, t)$. The approximate values $V_d^*(u, b, t)$ seem to be very much above the exact values and consequently they push up the values $\tilde{V}_d(u, b, t)$. Therefore, in the following examples we only show the approximations $V_d(u, b, t)$.

In terms of the finite time ruin probabilities Tables 1 and 2 do not show many interesting values. In Table 3 we present the values of $t_q = \max\{t : \psi_d(u, b, t) \leq q\}$, for $q = 0.05$ and 0.1, and the corresponding approximations $V_d(u, b, t_q)$. For the other combinations of u and b we do not show any values since t_q is almost zero. Intuitively, $V_d(u, b, t_q)$ gives an approximate value of the expected discounted value of future dividends subject to a constraint on the ruin probability. In general, the value t_q is attained very early meaning

Table 1: Approximations to $V(u, b, t)$ and to $\psi(u, b, t)$, for $\delta = 0.001$, $c = 1.1$, $\lambda = 1$ and exponential(1) claims.

	$t=50$	$t=100$	$t=200$	$t=500$	$t=700$	$t=1000$	$t=\infty$
$V(0,10)$	–	–	–	–	–	–	2.5032
$V_d(0,10,t)$	0.9778	1.7676	2.3327	2.5020	2.5040	2.5041	2.5041
$V_d^*(0,10,t)$	0.9866	1.7837	2.3539	2.5247	2.5268	2.5269	–
$\tilde{V}_d(0,10,t)$	0.9822	1.7756	2.3432	2.5133	2.5154	2.5155	–
$\psi_d(0,10,t)$	0.8911	0.9448	0.9858	0.9998	1.0000	1.0000	–
$V(10,10)$	–	–	–	–	–	–	17.8354
$V_d(10,10,t)$	10.0253	14.0703	16.9638	17.8309	17.8414	17.8420	17.8420
$V_d^*(10,10,t)$	10.1165	14.1983	17.1181	17.9931	18.0036	18.0042	–
$\tilde{V}_d(10,10,t)$	10.0708	14.1342	17.0408	17.9118	17.9223	17.9229	–
$\psi_d(10,10,t)$	0.4425	0.7172	0.9272	0.9988	0.9999	1.0000	–
$V(0,20)$	–	–	–	–	–	–	4.6285
$V_d(0,20,t)$	0.1373	0.7736	1.9146	3.6877	4.1645	4.4683	4.6296
$V_d^*(0,20,t)$	0.1385	0.7807	1.9320	3.7212	4.2024	4.5089	–
$\tilde{V}_d(0,20,t)$	0.1379	0.7772	1.9233	3.7044	4.1834	4.4885	–
$\psi_d(0,20,t)$	0.8717	0.8923	0.9167	0.9610	0.9765	0.9890	–
$V(10,20)$	–	–	–	–	–	–	32.9782
$V_d(10,20,t)$	2.7957	7.5239	15.0851	26.7753	29.9194	31.9220	32.9859
$V_d^*(10,20,t)$	2.8211	7.5923	15.2223	27.0188	30.1916	32.2123	–
$\tilde{V}_d(10,20,t)$	2.8084	7.5580	15.1536	26.8968	30.0552	32.0669	–
$\psi_d(10,20,t)$	0.1866	0.2922	0.4509	0.7429	0.8449	0.9274	–
$V(20,20)$	–	–	–	–	–	–	46.4960
$V_d(20,20,t)$	10.2960	16.2514	25.2531	39.1330	42.8660	45.2437	46.5068
$V_d^*(20,20,t)$	10.3896	16.3992	25.4828	39.4890	43.2559	45.6552	–
$\tilde{V}_d(20,20,t)$	10.3427	16.3252	25.3678	39.3106	43.0606	45.4490	–
$\psi_d(20,20,t)$	0.0545	0.1611	0.3481	0.6947	0.8159	0.9138	–

Table 2: Approximations to $V(u,b,t)$ and to $\psi(u,b,t)$, for $\delta = 0.005$, $c = 1.1$, $\lambda = 1$ and exponential(1) claims.

	$t=50$	$t=100$	$t=200$	$t=500$	$t=700$	$t=1000$	$t=\infty$
$V(0,10)$	–	–	–	–	–	–	1.8447
$V_d(0,10,t)$	0.8643	1.4575	1.7844	1.8446	1.8449	1.8449	1.8449
$V_d^*(0,10,t)$	0.8722	1.4708	1.8007	1.8615	1.8617	1.8617	–
$\tilde{V}_d(0,10,t)$	0.8682	1.4641	1.7925	1.8530	1.8532	1.8532	–
$V(10,10)$	–	–	–	–	–	–	14.3277
$V_d(10,10,t)$	9.3077	12.3454	14.0196	14.3280	14.3291	14.3291	14.3291
$V_d^*(10,10,t)$	9.3927	12.4582	14.1476	14.4588	14.4600	14.4600	–
$\tilde{V}_d(10,10,t)$	9.3497	12.4012	14.0829	14.3927	14.3939	14.3939	–
$V(0,20)$	–	–	–	–	–	–	1.7962
$V_d(0,20,t)$	0.1165	0.5870	1.2242	1.7362	1.7827	1.7946	1.7960
$V_d^*(0,20,t)$	0.1175	0.5924	1.2354	1.7521	1.7990	1.8110	–
$\tilde{V}_d(0,20,t)$	0.1170	0.5897	1.2298	1.7441	1.7908	1.8027	–
$V(10,20)$	–	–	–	–	–	–	13.9509
$V_d(10,20,t)$	2.4407	5.9542	10.1797	13.5556	13.8621	13.9404	13.9495
$V_d^*(10,20,t)$	2.4630	6.0086	10.2727	13.6794	13.9887	14.0677	–
$\tilde{V}_d(10,20,t)$	2.4517	5.9811	10.2257	13.6168	13.9248	14.0034	–
$V(20,20)$	–	–	–	–	–	–	23.4838
$V_d(20,20,t)$	9.5378	13.9734	19.0054	23.0136	23.3776	23.4705	23.4814
$V_d^*(20,20,t)$	9.6249	14.1010	19.1790	23.2238	23.5911	23.6849	–
$\tilde{V}_d(20,20,t)$	9.5810	14.0365	19.0913	23.1177	23.4833	23.5766	–

Table 3: Values of t_q, for q =0.05 and 0.1, and the corresponding approximations to $V(u,b,t_q)$, for $c = 1.1$, $\lambda = 1$ and exponential(1) claims.

u	b	$t_{0.05}$	$V_d(u,b,t_{0.05})$ $\delta = 0.001$	$V_d(u,b,t_{0.05})$ $\delta = 0.005$	$t_{0.1}$	$V_d(u,b,t_{0.1})$ $\delta = 0.001$	$V_d(u,b,t_{0.1})$ $\delta = 0.005$
10	10	9.3364	3.5310	3.4794	14.1273	4.5890	4.4891
10	20	13.5091	0.0376	0.0358	23.9636	0.5276	0.4889
20	20	47.7545	9.9923	9.2881	71.4545	13.0146	11.6725

that the ruin probability grows very quickly. For different forces of interest the values of $V_d(u,b,t_q)$ do not differ much, for the same combination of u and b, although for the values $V_d(u,b,1000)$ there is a much bigger difference. This is due to the small values of t_q.

Apart from the approximations to $V(u,b,t)$ and $\psi(u,b,t)$, numerical values resulting from applying the methods to obtain bounds to these quantities are shown in Tables 4 and 5. Here $\overline{\psi}_d(u,b,t)$ and $\underline{\psi}_d(u,b,t)$ represent, respectively, the upper and lower bounds for $\psi(u,b,t)$. From the tables we see that $\underline{V}_d(u,b,t)$ is closer to $V_d(u,b,t)$ than $\overline{V}_d(u,b,t)$ which is due to the corresponding assumptions about the dividend payouts used in the construction of these bounds. Note also that the expected discounted future dividends are, in general, larger for the case of the Pareto claim distribution since for this case the ruin probabilities are also bigger than in the case of exponential distributed claims although we observe some exceptions when $b = 20$, $u = 10, 20$ and $t = 500, 700, 1000$. For the same level of the probability of ruin, it seems, from our results, that for exponential claim amounts, the approximate values of $V(u,b,t)$ are larger than the corresponding values for the Pareto claim amounts, as we can see from Table 6, where we compare the two claim distributions in terms of the approximations $V_d(u,b,t^{\#})$ with $t^{\#} = \max\{t : \psi_d(u,b,t^{\#}) \leq 0.05\}$.

Figure 1 illustrates graphically the approximations and bounds we obtain for $V(u,b,t)$ considering exponential claims, a dividend barrier equal to 50 and four different levels for the initial surplus, $u = 10, 20, 40$ and 50.

In Figure 2 we show the values $V_d(u,b,t)$, $\overline{V}_d(u,b,t)$ and $\underline{V}_d(u,b,t)$ but now considering Pareto distributed claims, a constant initial surplus 10 and three different values for the dividend barrier. For $b = 20$ and 50, dividends begin to be paid much later than for $b = 10$ because for those cases the dividend barriers are reached later too since the surplus started from 10. After some time the expected discounted dividend payments for $b = 20$ is greater than for $b = 10$ and some time later these become smaller than for $b = 50$. We also observe that after some time point the lines of the approximations to and bounds for $V(u,b,t)$ become flat and, as expected,

Table 4: Approximations to and bounds for $V(u,b,t)$ and $\psi(u,b,t)$, for $\delta = 0.001$, $c = 1.1$, $\lambda = 1$ and exponential(1) claims.

	$t=50$	$t=100$	$t=200$	$t=500$	$t=700$	$t=1000$
$V_d(10,10,t)$	10.0253	14.0703	16.9638	17.8309	17.8414	17.8420
$\overline{V}_d(10,10,t)$	10.2578	14.4717	17.5464	18.5033	18.5159	18.5166
$\underline{V}_d(10,10,t)$	9.8860	13.8023	16.5479	17.3396	17.3484	17.3488
$\psi_d(10,10,t)$	0.4425	0.7172	0.9272	0.9988	0.9999	1.0000
$\overline{\psi}_d(10,10,t)$	0.4496	0.7251	0.9314	0.9990	1.0000	1.0000
$\underline{\psi}_d(10,10,t)$	0.4353	0.7092	0.9229	0.9985	0.9998	0.9999
$V_d(10,20,t)$	2.7957	7.5239	15.0851	26.7753	29.9194	31.9220
$\overline{V}_d(10,20,t)$	2.9077	7.8431	15.7995	28.3562	31.8387	34.1185
$\underline{V}_d(10,20,t)$	2.7114	7.2794	14.5235	25.4871	28.3430	30.1108
$\psi_d(10,20,t)$	0.1866	0.2922	0.4509	0.7429	0.8449	0.9274
$\overline{\psi}_d(10,20,t)$	0.1923	0.3017	0.4649	0.7584	0.8578	0.9358
$\underline{\psi}_d(10,20,t)$	0.1809	0.2828	0.4371	0.7270	0.8315	0.9183
$V_d(20,20,t)$	10.2960	16.2514	25.2531	39.1330	42.8660	45.2437
$\overline{V}_d(20,20,t)$	10.5344	16.7103	26.1313	40.9609	45.0737	47.7661
$\underline{V}_d(20,20,t)$	10.1530	15.9453	24.6174	37.7058	41.1150	43.2255
$\psi_d(20,20,t)$	0.0545	0.1611	0.3481	0.6947	0.8159	0.9138
$\overline{\psi}_d(20,20,t)$	0.0569	0.1679	0.3612	0.7115	0.8302	0.9233
$\underline{\psi}_d(20,20,t)$	0.0522	0.1544	0.3353	0.6776	0.8010	0.9035
$V_d(0,50,t)$	0.0000	0.0017	0.1772	1.8249	2.8508	4.0388
$\overline{V}_d(0,50,t)$	0.0000	0.0020	0.1992	2.0249	3.1581	4.4731
$\underline{V}_d(0,50,t)$	0.0000	0.0015	0.1589	1.6551	2.5891	3.6679
$\psi_d(0,50,t)$	0.8716	0.8900	0.9010	0.9080	0.9099	0.9124
$\overline{\psi}_d(0,50,t)$	0.8747	0.8934	0.9048	0.9123	0.9144	0.9172
$\underline{\psi}_d(0,50,t)$	0.8685	0.8865	0.8971	0.9036	0.9054	0.9077
$V_d(30,50,t)$	0.2907	2.6258	9.7020	30.0511	40.6518	52.7986
$\overline{V}_d(30,50,t)$	0.3093	2.7849	10.2791	31.8379	43.0936	56.0299
$\underline{V}_d(30,50,t)$	0.2756	2.4962	9.2302	28.5753	38.6255	50.1016
$\psi_d(30,50,t)$	0.0022	0.0110	0.0290	0.0620	0.0799	0.1056
$\overline{\psi}_d(30,50,t)$	0.0024	0.0120	0.0321	0.0697	0.0902	0.1195
$\underline{\psi}_d(30,50,t)$	0.0020	0.0100	0.0262	0.0551	0.0706	0.0931
$V_d(50,50,t)$	10.2962	16.2930	26.0917	48.1955	59.2102	71.7970
$\overline{V}_d(50,50,t)$	10.5346	16.7527	26.9856	50.2485	61.8976	75.2535
$\underline{V}_d(50,50,t)$	10.1532	15.9864	25.4470	46.6095	57.0975	69.0372
$\psi_d(50,50,t)$	0.0000	0.0004	0.0044	0.0289	0.0469	0.0735
$\overline{\psi}_d(50,50,t)$	0.0001	0.0005	0.0050	0.0331	0.0538	0.0843
$\underline{\psi}_d(50,50,t)$	0.0000	0.0003	0.0038	0.0251	0.0408	0.0639

Table 5: Approximations to and bounds for $V(u,b,t)$ and $\psi(u,b,t)$, for $\delta = 0.001$, $c = 1.1$, $\lambda = 1$ and Pareto(3,2) claims.

	$t=50$	$t=100$	$t=200$	$t=500$	$t=700$	$t=1000$
$V_d(10,10,t)$	11.6345	15.6770	17.9199	18.3382	18.3399	18.3399
$\overline{V}_d(10,10,t)$	11.8801	16.0600	18.4101	18.8591	18.8610	18.8610
$\underline{V}_d(10,10,t)$	11.4952	15.4387	17.5974	17.9902	17.9916	17.9917
$\psi_d(10,10,t)$	0.5572	0.8151	0.9678	0.9998	1.0000	1.0000
$\overline{\psi}_d(10,10,t)$	0.5613	0.8188	0.9691	0.9999	1.0000	1.0000
$\underline{\psi}_d(10,10,t)$	0.5531	0.8115	0.9664	0.9998	0.9999	0.9999
$V_d(10,20,t)$	4.2016	9.9078	17.4346	25.2508	26.3738	26.7905
$\overline{V}_d(10,20,t)$	4.3457	10.2559	18.1032	26.3827	27.6042	28.0678
$\underline{V}_d(10,20,t)$	4.0973	9.6538	16.9332	24.3704	25.4101	25.7873
$\psi_d(10,20,t)$	0.2822	0.4394	0.6553	0.9199	0.9697	0.9930
$\overline{\psi}_d(10,20,t)$	0.2864	0.4456	0.6626	0.9239	0.9718	0.9937
$\underline{\psi}_d(10,20,t)$	0.2780	0.4333	0.6482	0.9157	0.9675	0.9922
$V_d(20,20,t)$	12.1786	18.9909	27.8082	36.9614	38.2765	38.7645
$\overline{V}_d(20,20,t)$	12.4393	19.4682	28.6317	38.2969	39.7228	40.2639
$\underline{V}_d(20,20,t)$	12.0294	18.6887	27.2442	35.9819	37.2034	37.6465
$\psi_d(20,20,t)$	0.1647	0.3437	0.5964	0.9062	0.9645	0.9918
$\overline{\psi}_d(20,20,t)$	0.1671	0.3488	0.6036	0.9106	0.9669	0.9926
$\underline{\psi}_d(20,20,t)$	0.1622	0.3386	0.5893	0.9016	0.9620	0.9909
$V_d(0,50,t)$	0.0000	0.0150	0.5484	3.0364	4.2272	5.4403
$\overline{V}_d(0,50,t)$	0.0000	0.0169	0.5998	3.2939	4.5882	5.9150
$\underline{V}_d(0,50,t)$	0.0000	0.0135	0.5052	2.8191	3.9219	5.0382
$\psi_d(0,50,t)$	0.8468	0.8725	0.8900	0.9096	0.9194	0.9321
$\overline{\psi}_d(0,50,t)$	0.8499	0.8758	0.8935	0.9134	0.9232	0.9358
$\underline{\psi}_d(0,50,t)$	0.8438	0.8692	0.8864	0.9058	0.9155	0.9283
$V_d(30,50,t)$	0.7085	4.5235	13.5363	34.9588	44.6212	54.4573
$\overline{V}_d(30,50,t)$	0.7474	4.7438	14.1764	36.6774	46.8902	57.3520
$\underline{V}_d(30,50,t)$	0.6773	4.3494	13.0308	33.5847	42.7933	52.1054
$\psi_d(30,50,t)$	0.0303	0.0676	0.1276	0.2671	0.3463	0.4493
$\overline{\psi}_d(30,50,t)$	0.0310	0.0697	0.1323	0.2772	0.3588	0.4642
$\underline{\psi}_d(30,50,t)$	0.0296	0.0655	0.1231	0.2575	0.3343	0.4348
$V_d(50,50,t)$	12.1835	19.2334	30.3453	53.6037	63.9546	74.4897
$\overline{V}_d(50,50,t)$	12.4444	19.7200	31.2566	55.5903	66.4969	77.6675
$\underline{V}_d(50,50,t)$	12.0341	18.9243	29.7169	52.1245	62.0217	72.0281
$\psi_d(50,50,t)$	0.0076	0.0241	0.0705	0.2151	0.2998	0.4102
$\overline{\psi}_d(50,50,t)$	0.0078	0.0248	0.0729	0.2233	0.3110	0.4243
$\underline{\psi}_d(50,50,t)$	0.0075	0.0235	0.0682	0.2072	0.2892	0.3965

Table 6: Values of $t^{\#}$ and the corresponding approximations to $V(u, b, t^{\#})$, for $c = 1.1$ and $\lambda = 1$.

u	b	exponential claims $t^{\#}$	exponential claims $V_d(u, b, t^{\#})$	Pareto claims $t^{\#}$	Pareto claims $V_d(u, b, t^{\#})$
10	10	9.3364	3.5310	5.2182	2.7003
20	20	47.7545	9.9923	20.6545	6.8102
30	50	373.9454	22.1306	75.6455	2.4628
40	50	671.3545	48.1897	135.4909	14.4870
50	50	734.5727	60.8803	258.9365	35.8758

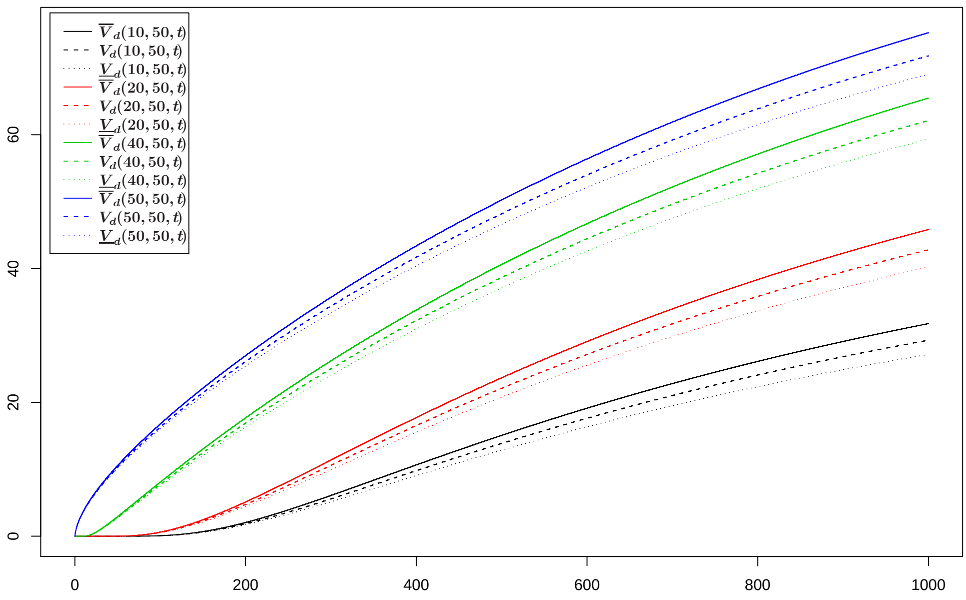


Figure 1: Approximations to and bounds, for $V(u, 50, t)$, exponential claims.

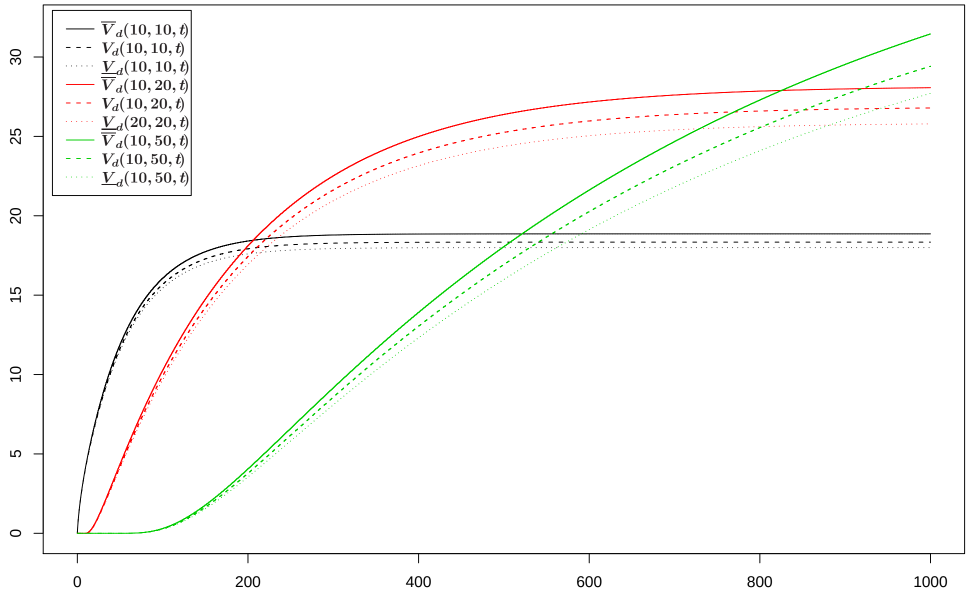


Figure 2: Approximations to and bounds for $V(10, b, t)$, Pareto claims.

this happens later as b increases.

Concerning the expected values $L(u, b, t)$ and $\varphi(u, b, t)$, their approximations are shown in Tables 7 and 8. These numerical values were obtained using the Markov chain approach and recursions described in Section 3, giving the same numbers with at least four decimal places. It seems that the values $\varphi(u, b, t)$ are, in general, small and have a lower weight in the calculation of $L(u, b, t)$ than the (positive) initial surplus u. We note that in our case the expected individual claim size is 1. Different parameters and claim distributions could give a different result. Also, from the tables it seems that $\varphi(u, b, t)$ is much bigger for Pareto claims than for exponential for the same values of (u, b, t). Another interesting feature that comes from comparing these tables is that $\varphi(u, b, t)$ decreases with u, for fixed b and for all $t \in (0, 1000]$ when claims are exponentially distributed, whereas in the Pareto case $\varphi(u, b, t)$ is not always monotone with respect to u, for all values of t, as we show in Figure 3.

Figure 4 shows graphically approximations of $L(10, b, t)$ for Pareto(3,2) claim amounts and b =10, 20 and 50. There are similarities with the lines described in Figure 2 although now the curves of $L(10, b, t)$, for $b > 10$, are decreasing during some time interval due to the lack of dividend payouts. From this graph we can also deduce the time point at which the shareholders

Table 7: Approximations to $\varphi(u, b, t)$ and $L(u, b, t)$, for $\delta = 0.001$, $c = 1.1$, $\lambda = 1$ and exponential(1) claims.

	$t = 50$	$t = 100$	$t = 200$	$t = 500$	$t = 700$	$t = 1000$
$\varphi_d(0, 10, t)$	0.8869	0.9368	0.9725	0.9832	0.9833	0.9833
$L_d(0, 10, t)$	0.0909	0.8308	1.3602	1.5188	1.5207	1.5208
$\varphi_d(10, 10, t)$	0.4310	0.6865	0.8692	0.9240	0.9247	0.9247
$L_d(10, 10, t)$	-0.4056	3.3839	6.0946	6.9069	6.9167	6.9172
$\varphi_d(0, 20, t)$	0.8682	0.8874	0.9084	0.9403	0.9489	0.9544
$L_d(0, 20, t)$	-0.7309	-0.1137	1.0062	2.7473	3.2156	3.5139
$\varphi_d(10, 20, t)$	0.1821	0.2802	0.4171	0.6274	0.6840	0.7200
$L_d(10, 20, t)$	-7.3864	-2.7563	4.6680	16.1479	19.2354	21.2020
$\varphi_d(20, 20, t)$	0.0527	0.1515	0.3128	0.5624	0.6296	0.6724
$L_d(20, 20, t)$	-9.7567	-3.9001	4.9404	18.5706	22.2364	24.5713
$\varphi_d(0, 50, t)$	0.8681	0.8852	0.8948	0.9000	0.9010	0.9021
$L_d(0, 50, t)$	-0.8681	-0.8835	-0.7176	0.9249	1.9498	3.1367
$\varphi_d(10, 50, t)$	0.1793	0.2509	0.3006	0.3317	0.3388	0.3464
$L_d(10, 50, t)$	-10.1793	-10.1861	-8.4378	3.5363	10.6860	18.9453
$\varphi_d(20, 50, t)$	0.0238	0.0570	0.0925	0.1234	0.1326	0.1427
$L_d(20, 50, t)$	-20.0179	-19.4965	-15.3438	1.9799	11.5993	22.6641
$\varphi_d(30, 50, t)$	0.0021	0.0102	0.0258	0.0495	0.0592	0.0703
$L_d(30, 50, t)$	-29.7113	-27.3845	-20.3238	0.0016	10.5925	22.7283
$\varphi_d(40, 50, t)$	0.0001	0.0015	0.0071	0.0257	0.0356	0.0470
$L_d(40, 50, t)$	-37.1993	-32.2619	-23.0926	-1.3545	9.5778	22.0775
$\varphi_d(50, 50, t)$	0.0000	0.0004	0.0037	0.0209	0.0308	0.0423
$L_d(50, 50, t)$	-39.7038	-33.7074	-23.9120	-1.8255	9.1793	21.7547

Table 8: Approximations to $\varphi(u, b, t)$ and $L(u, b, t)$, for $\delta = 0.001$, $c = 1.1$, $\lambda = 1$ and Pareto(3,2) claims.

	$t=50$	$t=100$	$t=200$	$t=500$	$t=700$	$t=1000$
$\varphi_d(0, 10, t)$	1.5288	1.7346	1.8488	1.8700	1.8701	1.8701
$L_d(0, 10, t)$	0.1070	0.9819	1.4673	1.5579	1.5582	1.5582
$\varphi_d(10, 10, t)$	1.8057	2.5754	3.0025	3.0821	3.0824	3.0824
$L_d(10, 10, t)$	-0.1712	3.1016	4.9174	5.2561	5.2574	5.2574
$\varphi_d(0, 20, t)$	1.4394	1.5762	1.7437	1.9174	1.9424	1.9517
$L_d(0, 20, t)$	-1.0875	-0.0948	1.3083	2.7671	2.9767	3.0545
$\varphi_d(10, 20, t)$	1.0998	1.7228	2.5241	3.3558	3.4753	3.5197
$L_d(10, 20, t)$	-6.8982	-1.8150	4.9105	11.8950	12.8985	13.2709
$\varphi_d(20, 20, t)$	0.7745	1.4903	2.4283	3.4023	3.5423	3.5942
$L_d(20, 20, t)$	-8.5958	-2.4994	5.3798	13.5591	14.7342	15.1703
$\varphi_d(0, 50, t)$	1.4364	1.5383	1.6169	1.7103	1.7482	1.7868
$L_d(0, 50, t)$	-1.4364	-1.5233	-1.0685	1.3261	2.4789	3.6535
$\varphi_d(10, 50, t)$	1.0368	1.4246	1.7762	2.2493	2.4462	2.6465
$L_d(10, 50, t)$	-11.0368	-11.1472	-7.9979	4.6796	10.6690	16.7698
$\varphi_d(20, 50, t)$	0.4777	0.8036	1.1844	1.8108	2.0803	2.3544
$L_d(20, 50, t)$	-20.4470	-19.3934	-13.3859	4.4251	12.6300	20.9849
$\varphi_d(30, 50, t)$	0.2331	0.4536	0.7974	1.4909	1.7980	2.1105
$L_d(30, 50, t)$	-29.5246	-25.9300	-17.2611	3.4679	12.8232	22.3468
$\varphi_d(40, 50, t)$	0.1323	0.2884	0.5995	1.3199	1.6441	1.9742
$L_d(40, 50, t)$	-35.9073	-29.8260	-19.4966	2.6505	12.5324	22.5905
$\varphi_d(50, 50, t)$	0.1031	0.2413	0.5424	1.2696	1.5984	1.9332
$L_d(50, 50, t)$	-37.9196	-31.0079	-20.1971	2.3340	12.3561	22.5565

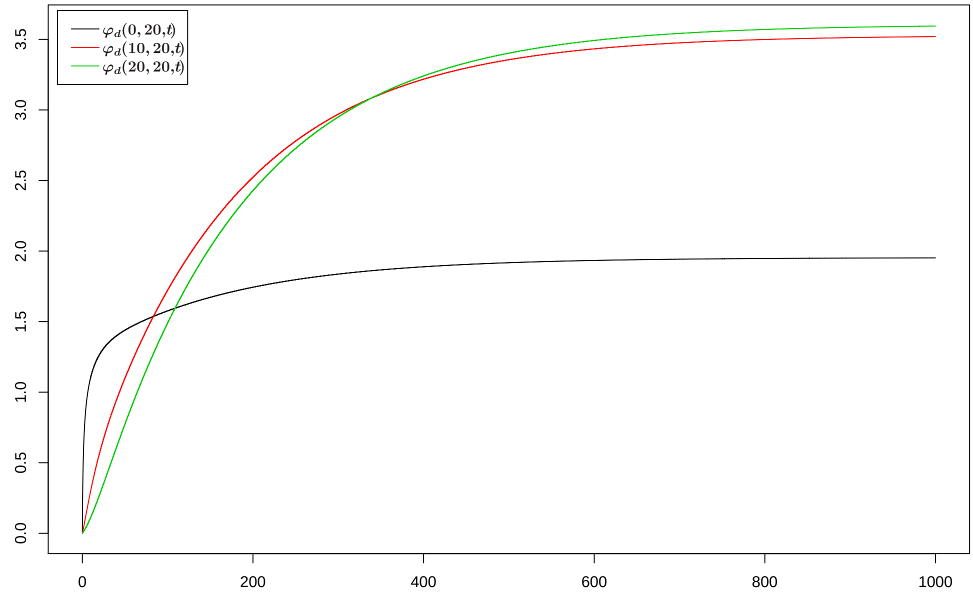


Figure 3: Approximations to $\varphi(u, 20, t)$, Pareto claims.

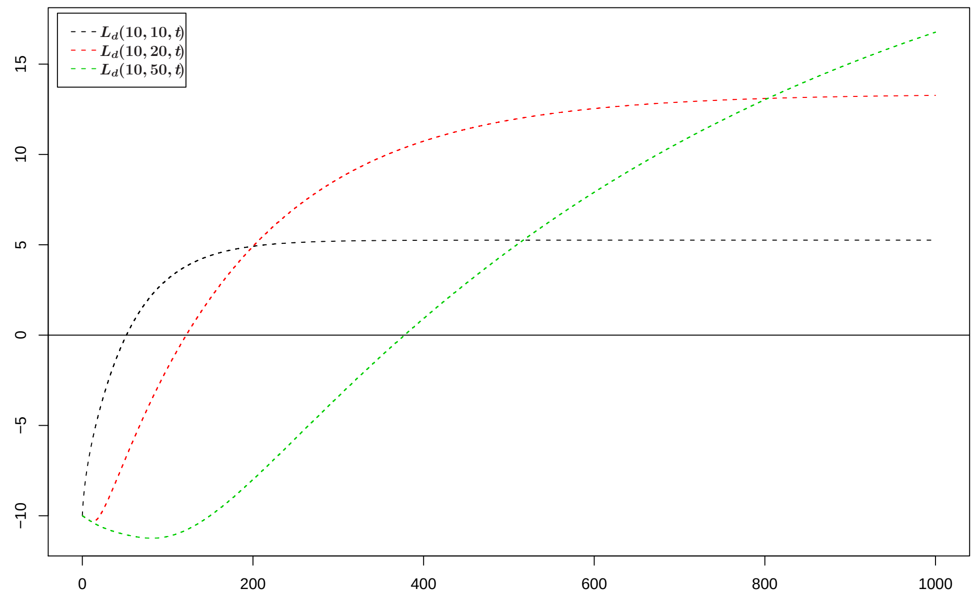


Figure 4: Approximations to $L(10, b, t)$, Pareto claims.

can expect a positive net income. This happens much later as b increases.

5.2 For the modified risk model II

We now present numerical results that we obtained using the algorithm described in Subsection 4.2. In Tables 9 and 10 we show the approximate values $V_d(u,b,t)$, $\varphi_d(u,b,t)$ and $M_d(u,b,t)$ considering our modified risk model II. The first two approximations are now much higher than the ones presented in the last subsection considering the same governing parameters. Also it seems, from our examples, for a fixed value of the initial surplus, that $M(u,b,t)$ decreases as b increases and, for a fixed value of the dividend barrier, $M(u,b,t)$ increases as u increases. This conclusion agrees with the results shown in Figure 4 of Dickson and Waters (2004). Concerning the transition probabilities in n steps, $p_{\omega,j}^{(n)}$, for the risk model I we have $p_{\omega,0}^{(n)} = 1$ after some time point. This is no longer true for the present risk model. In fact the probabilities $p_{\omega,j}^{(n)}$ will converge, each one, to some positive value. From our experience $p_{\omega,0}^{(n)}$ is, in general, very small.

In Figure 5 we compare the approximations $L_d(10,b,t)$ and $M_d(10,b,t)$, for $b = 10, 20, 50$ and Pareto(3,2) claim amounts in the range $0 < t \leq 1000$. During some time the first expected value is above the second but some time later, after becoming positive, these expected values switch positions since the values $L_d(u,b,t)$ seem to converge to some value. From our examples we cannot make any conclusions about the convergence of the values $M_d(u,b,t)$.

5.3 For the modified risk model III

Numerical results concerning this risk model were calculated for different governing parameters which are listed in Table 11 including the corresponding tables. These results were calculated within a time range $0 < t \leq 40$.

From these numerical examples we make the following remarks:

1. $\psi(u,b,t)$ and $V(u,b,t)$ decrease as b increases.

2. We can see that $\psi(u,b,t)$ is higher for the Pareto than for the exponential claim amounts. However this is not always true for an initial surplus of zero due to its unusual features. Also, these probabilities are small, excepting the case $u = 0$.

3. Comparing the values in Tables 12 and 13 we note that the bounds for $V(u,10,t)$, with $u > 10$, are not much different, which is probably due to the small probabilities of ruin.

4. In general, for $u > b$, the bounds for $V(u,b,t)$ obtained for Pareto claim amounts are smaller than for exponential distributed claims.

Table 9: Approximations to $V(u,b,t)$, $\varphi(u,b,t)$ and $M(u,b,t)$, for $\delta = 0.001$, $c = 1.1$, $\lambda = 1$ and exponential(1) claims.

	$t=50$	$t=100$	$t=200$	$t=500$	$t=700$	$t=1000$
$V_d(0,10,t)$	4.3187	11.5977	25.1774	58.6438	75.9835	96.2819
$\varphi_d(0,10,t)$	5.6790	8.3710	13.3405	25.5870	31.9322	39.3600
$M_d(0,10,t)$	-1.3604	3.2267	11.8369	33.0569	44.0513	56.9219
$V_d(10,10,t)$	10.5561	17.8872	31.4673	64.9337	82.2734	102.5718
$\varphi_d(10,10,t)$	2.0879	4.7596	9.7289	21.9754	28.3206	35.7485
$M_d(10,10,t)$	-1.5318	3.1276	11.7384	32.9583	43.9528	56.8233
$V_d(0,20,t)$	0.4504	3.9454	13.4801	38.3373	51.2330	66.3292
$\varphi_d(0,20,t)$	5.4194	6.8128	8.4698	12.1488	14.0501	16.2758
$M_d(0,20,t)$	-4.9690	-2.8674	5.0103	26.1884	37.1829	50.0535
$V_d(10,20,t)$	2.8068	7.9099	17.9160	42.8001	55.6958	70.7920
$\varphi_d(10,20,t)$	0.9044	1.8022	3.3190	6.9901	8.8914	11.1170
$M_d(10,20,t)$	-8.0976	-3.8923	4.5969	25.8099	36.8044	49.6749
$V_d(20,20,t)$	10.2964	16.3290	26.5966	51.4956	64.3913	79.4875
$\varphi_d(20,20,t)$	0.2141	0.8425	2.2816	5.9483	7.8495	10.0752
$M_d(20,20,t)$	-9.9176	-4.5135	4.3150	25.5473	36.5418	49.4123
$V_d(0,50,t)$	0.0000	0.0052	0.9143	14.5992	25.0040	37.8305
$\varphi_d(0,50,t)$	5.4191	6.7768	7.9649	8.8883	9.0541	9.1937
$M_d(0,50,t)$	-5.4191	-6.7716	-7.0507	5.7109	15.9499	28.6368
$V_d(10,50,t)$	0.0000	0.0649	1.9600	17.4316	28.0122	40.8814
$\varphi_d(10,50,t)$	0.8957	1.6628	2.4839	3.2151	3.3660	3.5020
$M_d(10,50,t)$	-10.8957	-11.5979	-10.5239	4.2165	14.6461	27.3794
$V_d(20,50,t)$	0.0059	0.5605	4.7598	23.0538	33.8955	46.8282
$\varphi_d(20,50,t)$	0.0992	0.3300	0.7045	1.1620	1.2909	1.4215
$M_d(20,50,t)$	-20.0933	-19.7695	-15.9446	1.8918	12.6046	25.4067
$V_d(30,50,t)$	0.2907	2.6258	9.7031	30.2964	41.3342	54.3146
$\varphi_d(30,50,t)$	0.0075	0.0524	0.1804	0.4444	0.5568	0.6834
$M_d(30,50,t)$	-29.7168	-27.4267	-20.4773	-0.1479	10.7774	23.6312
$V_d(40,50,t)$	2.8009	7.7396	16.9146	38.7410	49.8770	62.8812
$\varphi_d(40,50,t)$	0.0004	0.0068	0.0452	0.2175	0.3216	0.4462
$M_d(40,50,t)$	-37.1995	-32.2672	-23.1306	-1.4764	9.5554	22.4350
$V_d(50,50,t)$	10.2962	16.2930	26.0918	48.2350	59.3953	72.4054
$\varphi_d(50,50,t)$	0.0000	0.0015	0.0218	0.1721	0.2742	0.3983
$M_d(50,50,t)$	-39.7039	-33.7085	-23.9301	-1.9371	9.1211	22.0071

Table 10: Approximations to $V(u,b,t)$, $\varphi(u,b,t)$ and $M(u,b,t)$, for $\delta = 0.001$, $c = 1.1$, $\lambda = 1$ and Pareto(3,2) claims.

	$t=50$	$t=100$	$t=200$	$t=500$	$t=700$	$t=1000$
$V_d(0,10,t)$	6.3276	16.0321	34.0638	78.5010	101.5249	128.4774
$\varphi_d(0,10,t)$	8.0279	13.1069	22.5280	45.7453	57.7747	71.8566
$M_d(0,10,t)$	-1.7003	2.9252	11.5358	32.7557	43.7502	56.6207
$V_d(10,10,t)$	12.9049	22.6231	40.6549	85.0921	108.1159	135.0684
$\varphi_d(10,10,t)$	4.7056	9.7805	19.2016	42.4188	54.4482	68.5302
$M_d(10,10,t)$	-1.8006	2.8427	11.4533	32.6732	43.6677	56.5382
$V_d(0,20,t)$	1.0464	6.6739	19.4034	51.2183	67.7036	87.0019
$\varphi_d(0,20,t)$	7.3184	10.0330	14.3874	24.9858	30.4767	36.9045
$M_d(0,20,t)$	-6.2720	-3.3591	5.0160	26.2324	37.2269	50.0974
$V_d(10,20,t)$	4.2668	11.0699	23.9603	55.7775	72.2628	91.5611
$\varphi_d(10,20,t)$	2.5707	4.9326	9.2391	19.8368	25.3277	31.7554
$M_d(10,20,t)$	-8.3038	-3.8627	4.7212	25.9407	36.9352	49.8057
$V_d(20,20,t)$	12.1954	19.5493	32.5143	64.3326	80.8179	100.1162
$\varphi_d(20,20,t)$	1.4978	3.6962	7.9804	18.5778	24.0686	30.4964
$M_d(20,20,t)$	-9.3023	-4.1470	4.5339	25.7548	36.7493	49.6198
$V_d(0,50,t)$	0.0000	0.0427	2.5769	21.9643	34.0039	48.2767
$\varphi_d(0,50,t)$	7.3065	9.7172	12.2331	15.3732	16.6237	18.0544
$M_d(0,50,t)$	-7.3065	-9.6746	-9.6562	6.5911	17.3802	30.2223
$V_d(10,50,t)$	0.0001	0.2799	4.2650	24.7693	36.8559	51.1352
$\varphi_d(10,50,t)$	2.4611	4.1875	6.2464	9.1711	10.4129	11.8424
$M_d(10,50,t)$	-12.4610	-13.9077	-11.9813	5.5981	16.4429	29.2928
$V_d(20,50,t)$	0.0307	1.4108	7.9505	30.0574	42.2106	56.4990
$\varphi_d(20,50,t)$	0.8875	1.8593	3.3102	5.9316	7.1610	8.5888
$M_d(20,50,t)$	-20.8568	-20.4486	-15.3597	4.1258	15.0495	27.9103
$V_d(30,50,t)$	0.7085	4.5237	13.5898	37.0060	49.2127	63.5086
$\varphi_d(30,50,t)$	0.3652	0.8747	1.8671	4.2454	5.4648	6.8911
$M_d(30,50,t)$	-29.6567	-26.3510	-18.2773	2.7606	13.7480	26.6175
$V_d(40,50,t)$	4.2250	10.4625	21.1252	45.2787	57.5154	71.8154
$\varphi_d(40,50,t)$	0.1834	0.4794	1.2271	3.4700	4.6838	6.1094
$M_d(40,50,t)$	-35.9584	-30.0169	-20.1018	1.8088	12.8316	25.7060
$V_d(50,50,t)$	12.1835	19.2335	30.3600	54.7199	66.9650	81.2661
$\varphi_d(50,50,t)$	0.1346	0.3747	1.0550	3.2602	4.4725	5.8979
$M_d(50,50,t)$	-37.9511	-31.1413	-20.6951	1.4597	12.4925	25.3683

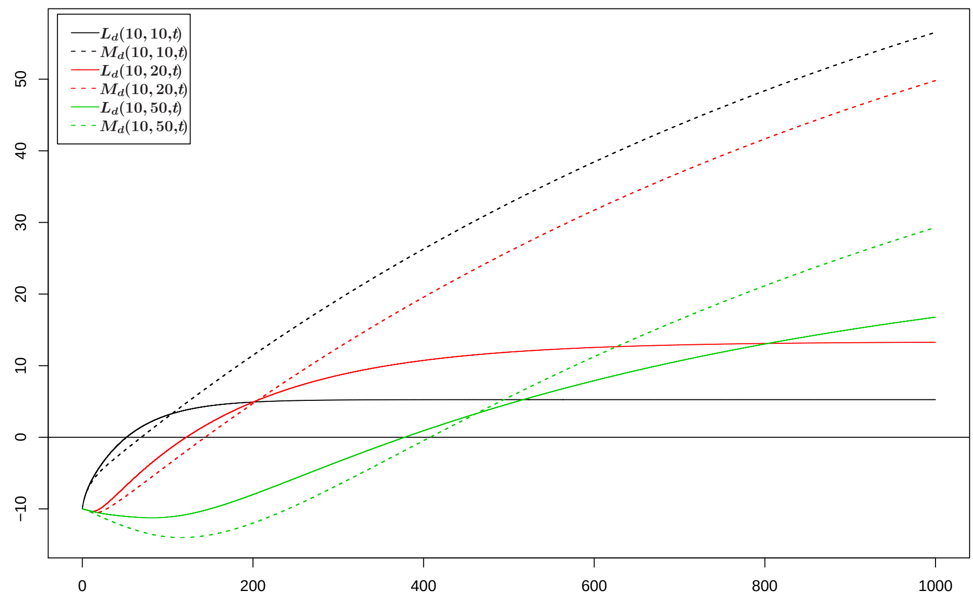


Figure 5: Approximations to $L(10, b, t)$ and $M(10, b, t)$, Pareto claims.

Table 11: Parameter values for the numerical results in the modified risk model III.

Premium rate	Dividend rate	Interest force	Tables
1.3	0.2	0.01	12–13
1.2	0.15	0.001	14–15
2.1	1.0	0.001	16–17

5. From Tables 14 to 17 we observe that the bounds for the expected value $V(u, b, t)$ are very much alike when $u = b$, when $u - b = 10$ and when $b - u = 10$, excepting the case of an initial surplus equal to zero. However we should expect to have bigger differences when t becomes larger.

In Figure 6 we compare graphically the bounds for $V(u, 10, t)$, for three different levels of the initial surplus (10, 20 and 40), and the corresponding ruin probabilities. We can see that the bounds produced for the expected discounted payments are indeed very close and as u increases these bounds tends to be tighter, a feature that we also found for other combinations of u and b.

A comparison between the modified risk models I and III, in terms of the bounds for $V(u, b, t)$, is depicted in Figure 7. The premium rate is, for each surplus risk process, 1.1 and 2.1, respectively. Although the dividend rate for the first surplus is larger than the dividend rate for the second surplus ($\hat{c} = 1$), we conclude, for such a case, that $V(u, b, t)$ shall be much bigger for Model III than for Model I. A great part of this difference must be due to the corresponding small ruin probabilities obtained in Model III. Note that in Model I the surplus process does not go above b (and therefore ruin is certain) whereas for Model III the surplus can go above b although with a slope 0.1.

Acknowledgements

The author is very much in debt to Professor Howard Waters for his comments and suggestions and for reading this manuscript.

Table 12: Bounds for $V(u,10,t)$ and for $\psi(u,10,t)$, for $\delta = 0.01$, $c = 1.3$, $\hat{c} = 0.2$, $\lambda = 1$ and exponential(1) claims

	$t=1$	$t=5$	$t=10$	$t=20$	$t=30$	$t=40$
$\overline{V}_d(0,10,t)$	0.0000	0.0000	0.0048	0.1559	0.4018	0.6607
$\underline{V}_d(0,10,t)$	0.0000	0.0000	0.0044	0.1480	0.3822	0.6287
$\overline{\psi}_d(0,10,t)$	0.4408	0.6614	0.7132	0.7460	0.7591	0.7667
$\underline{\psi}_d(0,10,t)$	0.4373	0.6565	0.7077	0.7398	0.7523	0.7595
$\overline{V}_d(10,10,t)$	0.1564	0.6813	1.3249	2.5856	3.7841	4.9004
$\underline{V}_d(10,10,t)$	0.1547	0.6726	1.3055	2.5415	3.7142	4.8053
$\overline{\psi}_d(10,10,t)$	0.0003	0.0071	0.0215	0.0492	0.0707	0.0874
$\underline{\psi}_d(10,10,t)$	0.0002	0.0067	0.0202	0.0458	0.0653	0.0803
$\overline{V}_d(20,10,t)$	0.1990	0.9738	1.8925	3.5743	5.0761	6.4248
$\underline{V}_d(20,10,t)$	0.1989	0.9735	1.8911	3.5677	5.0613	6.4001
$\overline{\psi}_d(20,10,t)$	0.0001	0.0001	0.0004	0.0024	0.0059	0.0102
$\underline{\psi}_d(20,10,t)$	0.0000	0.0000	0.0002	0.0020	0.0050	0.0086
$\overline{V}_d(30,10,t)$	0.1991	0.9755	1.9033	3.6242	5.1789	6.5826
$\underline{V}_d(30,10,t)$	0.1989	0.9753	1.9030	3.6236	5.1773	6.5794
$\overline{\psi}_d(30,10,t)$	0.0001	0.0001	0.0001	0.0001	0.0003	0.0008
$\underline{\psi}_d(30,10,t)$	0.0000	0.0000	0.0000	0.0000	0.0002	0.0005
$\overline{V}_d(40,10,t)$	0.1991	0.9755	1.9034	3.6256	5.1837	6.5934
$\underline{V}_d(40,10,t)$	0.1989	0.9753	1.9031	3.6252	5.1832	6.5926
$\overline{\psi}_d(40,10,t)$	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001
$\underline{\psi}_d(40,10,t)$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$\overline{V}_d(50,10,t)$	0.1991	0.9755	1.9034	3.6256	5.1839	6.5939
$\underline{V}_d(50,10,t)$	0.1989	0.9753	1.9031	3.6252	5.1834	6.5933
$\overline{\psi}_d(50,10,t)$	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001
$\underline{\psi}_d(50,10,t)$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Table 13: Bounds for $V(u,10,t)$ and for $\psi(u,10,t)$, for $\delta = 0.01$, $c = 1.3$, $\hat{c} = 0.2$, $\lambda = 1$ and Pareto(3,2) claims

	$t=1$	$t=5$	$t=10$	$t=20$	$t=30$	$t=40$
$\overline{V}_d(0,10,t)$	0.0000	0.0000	0.0087	0.2359	0.5479	0.8504
$\underline{V}_d(0,10,t)$	0.0000	0.0000	0.0080	0.2253	0.5253	0.8163
$\overline{\psi}_d(0,10,t)$	0.3987	0.6130	0.6735	0.7178	0.7389	0.7526
$\underline{\psi}_d(0,10,t)$	0.3946	0.6078	0.6680	0.7118	0.7324	0.7457
$\overline{V}_d(10,10,t)$	0.1622	0.7179	1.3794	2.6345	3.7871	4.8301
$\underline{V}_d(10,10,t)$	0.1606	0.7103	1.3634	2.5999	3.7334	4.7577
$\overline{\psi}_d(10,10,t)$	0.0054	0.0325	0.0661	0.1196	0.1589	0.1892
$\underline{\psi}_d(10,10,t)$	0.0053	0.0318	0.0644	0.1158	0.1531	0.1816
$\overline{V}_d(20,10,t)$	0.1986	0.9639	1.8599	3.4825	4.9188	6.1989
$\underline{V}_d(20,10,t)$	0.1985	0.9633	1.8580	3.4750	4.9030	6.1733
$\overline{\psi}_d(20,10,t)$	0.0009	0.0052	0.0121	0.0284	0.0453	0.0614
$\underline{\psi}_d(20,10,t)$	0.0008	0.0050	0.0117	0.0273	0.0431	0.0579
$\overline{V}_d(30,10,t)$	0.1990	0.9736	1.8956	3.5951	5.1192	6.4869
$\underline{V}_d(30,10,t)$	0.1989	0.9734	1.8951	3.5935	5.1151	6.4795
$\overline{\psi}_d(30,10,t)$	0.0003	0.0015	0.0035	0.0083	0.0142	0.0207
$\underline{\psi}_d(30,10,t)$	0.0002	0.0014	0.0033	0.0080	0.0135	0.0194
$\overline{V}_d(40,10,t)$	0.1990	0.9749	1.9010	3.6161	5.1632	6.5583
$\underline{V}_d(40,10,t)$	0.1989	0.9747	1.9007	3.6155	5.1618	6.5558
$\overline{\psi}_d(40,10,t)$	0.0002	0.0007	0.0014	0.0032	0.0054	0.0080
$\underline{\psi}_d(40,10,t)$	0.0001	0.0006	0.0013	0.0030	0.0051	0.0075
$\overline{V}_d(50,10,t)$	0.1990	0.9752	1.9023	3.6216	5.1753	6.5792
$\underline{V}_d(50,10,t)$	0.1989	0.9751	1.9021	3.6212	5.1745	6.5779
$\overline{\psi}_d(50,10,t)$	0.0001	0.0004	0.0007	0.0016	0.0025	0.0037
$\underline{\psi}_d(50,10,t)$	0.0000	0.0003	0.0006	0.0014	0.0024	0.0034

Table 14: Bounds for $V(u,b,t)$ and for $\psi(u,b,t)$, for $\delta = 0.001$, $c = 1.2$, $\hat{c} = 0.15$, $\lambda = 1$ and exponential(1) claims

	$t=1$	$t=5$	$t=10$	$t=20$	$t=30$	$t=40$
$\overline{V}_d(0,10,t)$	0.0000	0.0000	0.0011	0.0783	0.2339	0.4189
$\underline{V}_d(0,10,t)$	0.0000	0.0000	0.0009	0.0737	0.2205	0.3949
$\overline{\psi}_d(0,10,t)$	0.4528	0.6912	0.7504	0.7904	0.8074	0.8177
$\underline{\psi}_d(0,10,t)$	0.4493	0.6865	0.7451	0.7845	0.8010	0.8108
$\overline{V}_d(10,10,t)$	0.1167	0.5039	0.9853	1.9739	2.9889	4.0163
$\underline{V}_d(10,10,t)$	0.1154	0.4971	0.9694	1.9351	2.9226	3.9193
$\overline{\psi}_d(10,10,t)$	0.0004	0.0084	0.0274	0.0677	0.1014	0.1289
$\underline{\psi}_d(10,10,t)$	0.0002	0.0079	0.0258	0.0633	0.0943	0.1192
$\overline{V}_d(20,10,t)$	0.1500	0.7467	1.4821	2.9135	4.3000	5.6518
$\underline{V}_d(20,10,t)$	0.1499	0.7464	1.4810	2.9069	4.2832	5.6210
$\overline{\psi}_d(20,10,t)$	0.0000	0.0001	0.0005	0.0035	0.0094	0.0171
$\underline{\psi}_d(20,10,t)$	0.0000	0.0000	0.0003	0.0029	0.0080	0.0146
$\overline{V}_d(10,20,t)$	0.0000	0.0000	0.0012	0.1276	0.4769	0.9917
$\underline{V}_d(10,20,t)$	0.0000	0.0000	0.0010	0.1203	0.4513	0.9398
$\overline{\psi}_d(10,20,t)$	0.0003	0.0078	0.0247	0.0579	0.0827	0.1007
$\underline{\psi}_d(10,20,t)$	0.0002	0.0074	0.0235	0.0546	0.0777	0.0942
$\overline{V}_d(20,20,t)$	0.1167	0.5039	0.9853	1.9744	2.9946	4.0390
$\underline{V}_d(20,20,t)$	0.1154	0.4971	0.9694	1.9355	2.9283	3.9420
$\overline{\psi}_d(20,20,t)$	0.0000	0.0001	0.0004	0.0023	0.0057	0.0098
$\underline{\psi}_d(20,20,t)$	0.0000	0.0000	0.0002	0.0020	0.0050	0.0086
$\overline{V}_d(30,20,t)$	0.1500	0.7467	1.4821	2.9135	4.3001	5.6526
$\underline{V}_d(30,20,t)$	0.1499	0.7464	1.4810	2.9069	4.2833	5.6218
$\overline{\psi}_d(30,20,t)$	0.0000	0.0000	0.0001	0.0001	0.0004	0.0009
$\underline{\psi}_d(30,20,t)$	0.0000	0.0000	0.0000	0.0000	0.0002	0.0006
$\overline{V}_d(40,50,t)$	0.0000	0.0000	0.0012	0.1276	0.4769	0.9923
$\underline{V}_d(40,50,t)$	0.0000	0.0000	0.0010	0.1203	0.4513	0.9403
$\overline{\psi}_d(40,50,t)$	0.0000	0.0000	0.0000	0.0000	0.0001	0.0001
$\underline{\psi}_d(40,50,t)$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$\overline{V}_d(50,50,t)$	0.1167	0.5039	0.9853	1.9744	2.9946	4.0390
$\underline{V}_d(50,50,t)$	0.1154	0.4971	0.9694	1.9355	2.9283	3.9420
$\overline{\psi}_d(50,50,t)$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$\underline{\psi}_d(50,50,t)$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Table 15: Bounds for $V(u,b,t)$ and for $\psi(u,b,t)$, for $\delta = 0.001$, $c = 1.2$, $\hat{c} = 0.15$, $\lambda = 1$ and Pareto(3,2) claims

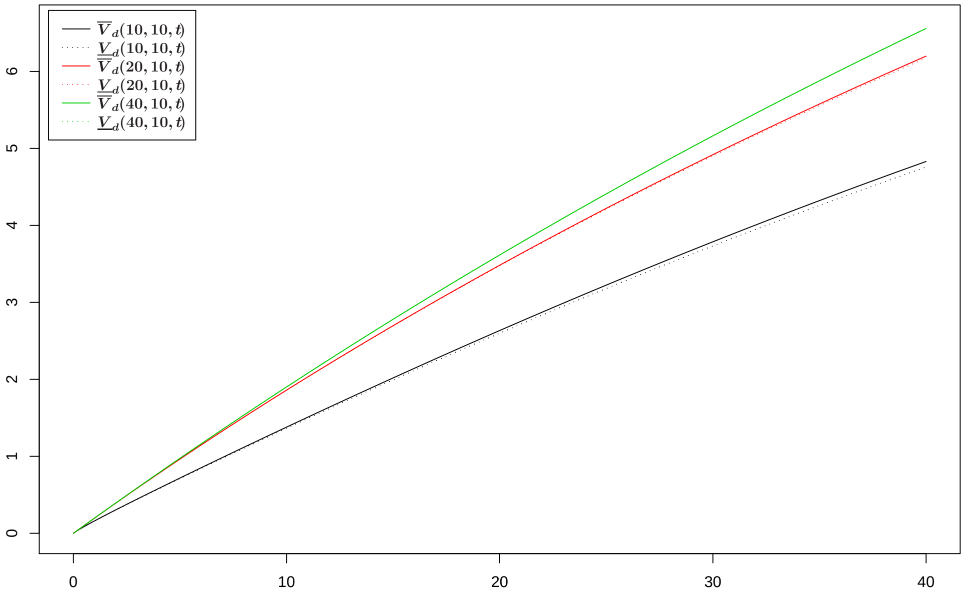
	$t=1$	$t=5$	$t=10$	$t=20$	$t=30$	$t=40$
$\overline{V}_d(0,10,t)$	0.0000	0.0000	0.0020	0.1307	0.3500	0.5882
$\underline{V}_d(0,10,t)$	0.0000	0.0000	0.0017	0.1238	0.3330	0.5603
$\overline{\psi}_d(0,10,t)$	0.4110	0.6409	0.7078	0.7578	0.7815	0.7969
$\underline{\psi}_d(0,10,t)$	0.4069	0.6357	0.7023	0.7518	0.7752	0.7902
$\overline{V}_d(10,10,t)$	0.1212	0.5353	1.0389	2.0459	3.0485	4.0354
$\underline{V}_d(10,10,t)$	0.1199	0.5292	1.0256	2.0149	2.9972	3.9617
$\overline{\psi}_d(10,10,t)$	0.0055	0.0344	0.0726	0.1370	0.1861	0.2248
$\underline{\psi}_d(10,10,t)$	0.0053	0.0337	0.0708	0.1326	0.1794	0.2159
$\overline{V}_d(20,10,t)$	0.1496	0.7387	1.4548	2.8321	4.1546	5.4348
$\underline{V}_d(20,10,t)$	0.1495	0.7383	1.4533	2.8251	4.1382	5.4060
$\overline{\psi}_d(20,10,t)$	0.0009	0.0053	0.0129	0.0320	0.0531	0.0740
$\underline{\psi}_d(20,10,t)$	0.0008	0.0051	0.0125	0.0307	0.0504	0.0697
$\overline{V}_d(10,20,t)$	0.0000	0.0000	0.0022	0.2055	0.6677	1.2744
$\underline{V}_d(10,20,t)$	0.0000	0.0000	0.0019	0.1952	0.6385	1.2220
$\overline{\psi}_d(10,20,t)$	0.0054	0.0327	0.0672	0.1214	0.1596	0.1879
$\underline{\psi}_d(10,20,t)$	0.0052	0.0322	0.0658	0.1183	0.1550	0.1821
$\overline{V}_d(20,20,t)$	0.1212	0.5353	1.0389	2.0478	3.0661	4.0928
$\underline{V}_d(20,20,t)$	0.1199	0.5292	1.0256	2.0168	3.0143	4.0182
$\overline{\psi}_d(20,20,t)$	0.0009	0.0052	0.0122	0.0287	0.0456	0.0616
$\underline{\psi}_d(20,20,t)$	0.0008	0.0050	0.0119	0.0277	0.0438	0.0589
$\overline{V}_d(30,20,t)$	0.1496	0.7387	1.4548	2.8324	4.1571	5.4446
$\underline{V}_d(30,20,t)$	0.1495	0.7383	1.4533	2.8253	4.1407	5.4158
$\overline{\psi}_d(30,20,t)$	0.0003	0.0016	0.0036	0.0088	0.0152	0.0223
$\underline{\psi}_d(30,20,t)$	0.0002	0.0015	0.0034	0.0084	0.0145	0.0211
$\overline{V}_d(40,50,t)$	0.0000	0.0000	0.0022	0.2055	0.6679	1.2774
$\underline{V}_d(40,50,t)$	0.0000	0.0000	0.0019	0.1952	0.6387	1.2248
$\overline{\psi}_d(40,50,t)$	0.0002	0.0007	0.0014	0.0030	0.0049	0.0069
$\underline{\psi}_d(40,50,t)$	0.0001	0.0006	0.0013	0.0029	0.0047	0.0066
$\overline{V}_d(50,50,t)$	0.1212	0.5353	1.0389	2.0478	3.0661	4.0933
$\underline{V}_d(50,50,t)$	0.1199	0.5292	1.0256	2.0168	3.0144	4.0186
$\overline{\psi}_d(50,50,t)$	0.0001	0.0004	0.0007	0.0015	0.0025	0.0035
$\underline{\psi}_d(50,50,t)$	0.0000	0.0003	0.0006	0.0014	0.0023	0.0033

Table 16: Bounds for $V(u,b,t)$ and for $\psi(u,b,t)$, for $\delta = 0.001$, $c = 2.1$, $\hat{c} = 1.0$, $\lambda = 1$ and exponential(1) claims

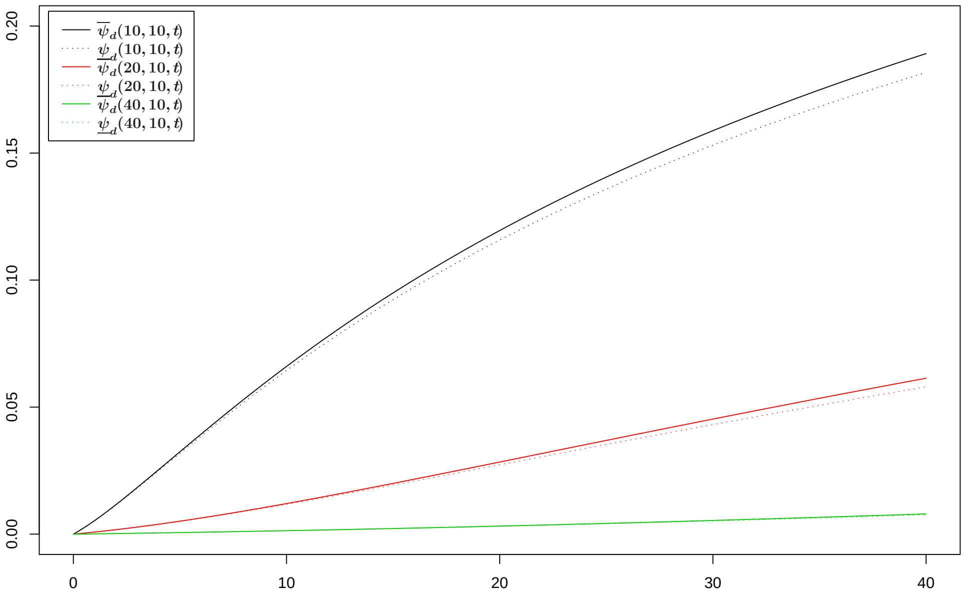
	$t=1$	$t=5$	$t=10$	$t=20$	$t=30$	$t=40$
$\overline{V}_d(0,10,t)$	0.0000	0.0045	1.1740	5.4497	10.1029	14.8326
$\underline{V}_d(0,10,t)$	0.0000	0.0042	1.1435	5.3399	9.9165	14.5716
$\overline{\psi}_d(0,10,t)$	0.3587	0.4659	0.4767	0.4805	0.4822	0.4834
$\underline{\psi}_d(0,10,t)$	0.3556	0.4615	0.4720	0.4756	0.4771	0.4782
$\overline{V}_d(10,10,t)$	0.8166	3.9632	8.1752	17.0035	26.0209	35.0840
$\underline{V}_d(10,10,t)$	0.8106	3.9307	8.1079	16.8692	25.8238	34.8271
$\overline{\psi}_d(10,10,t)$	0.0003	0.0027	0.0053	0.0089	0.0113	0.0131
$\underline{\psi}_d(10,10,t)$	0.0002	0.0024	0.0049	0.0082	0.0103	0.0119
$\overline{V}_d(0,20,t)$	0.0000	0.0000	0.0004	1.7581	5.8821	10.4832
$\underline{V}_d(0,20,t)$	0.0000	0.0000	0.0003	1.6969	5.7317	10.2539
$\overline{\psi}_d(0,20,t)$	0.3587	0.4659	0.4765	0.4785	0.4786	0.4786
$\underline{\psi}_d(0,20,t)$	0.3556	0.4615	0.4717	0.4737	0.4738	0.4738
$\overline{V}_d(10,20,t)$	0.0000	0.0046	1.6041	9.0403	17.7452	26.7105
$\underline{V}_d(10,20,t)$	0.0000	0.0043	1.5656	8.9038	17.5345	26.4373
$\overline{\psi}_d(10,20,t)$	0.0002	0.0016	0.0023	0.0026	0.0027	0.0027
$\underline{\psi}_d(10,20,t)$	0.0001	0.0014	0.0021	0.0024	0.0024	0.0024
$\overline{V}_d(20,20,t)$	0.8166	3.9632	8.1760	17.0261	26.0985	35.2409
$\underline{V}_d(20,20,t)$	0.8106	3.9307	8.1086	16.8926	25.9052	34.9930
$\overline{\psi}_d(20,20,t)$	0.0000	0.0001	0.0001	0.0001	0.0001	0.0001
$\underline{\psi}_d(20,20,t)$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Table 17: Bounds for $V(u,b,t)$ and for $\psi(u,b,t)$, for $\delta = 0.001$, $c = 2.1$, $\hat{c} = 1.0$, $\lambda = 1$ and Pareto(3,2) claims

	$t=1$	$t=5$	$t=10$	$t=20$	$t=30$	$t=40$
$\overline{V}_d(0,10,t)$	0.0000	0.0060	1.4363	5.8698	10.4438	14.9724
$\underline{V}_d(0,10,t)$	0.0000	0.0055	1.4029	5.7669	10.2731	14.7331
$\overline{\psi}_d(0,10,t)$	0.3195	0.4397	0.4650	0.4879	0.5026	0.5138
$\underline{\psi}_d(0,10,t)$	0.3161	0.4354	0.4605	0.4829	0.4973	0.5081
$\overline{V}_d(10,10,t)$	0.8437	4.0490	8.1861	16.6055	24.9745	33.2152
$\underline{V}_d(10,10,t)$	0.8379	4.0220	8.1321	16.4974	24.8101	32.9913
$\overline{\psi}_d(10,10,t)$	0.0052	0.0258	0.0462	0.0762	0.0983	0.1160
$\underline{\psi}_d(10,10,t)$	0.0051	0.0254	0.0453	0.0744	0.0956	0.1125
$\overline{V}_d(0,20,t)$	0.0000	0.0000	0.0008	2.1810	6.3799	10.9021
$\underline{V}_d(0,20,t)$	0.0000	0.0000	0.0006	2.1158	6.2384	10.6901
$\overline{\psi}_d(0,20,t)$	0.3195	0.4397	0.4629	0.4742	0.4788	0.4822
$\underline{\psi}_d(0,20,t)$	0.3161	0.4354	0.4583	0.4695	0.4740	0.4773
$\overline{V}_d(10,20,t)$	0.0000	0.0060	1.8959	9.2079	17.4462	25.8777
$\underline{V}_d(10,20,t)$	0.0000	0.0056	1.8565	9.0922	17.2689	25.6445
$\overline{\psi}_d(10,20,t)$	0.0047	0.0187	0.0277	0.0374	0.0441	0.0494
$\underline{\psi}_d(10,20,t)$	0.0046	0.0184	0.0273	0.0368	0.0432	0.0484
$\overline{V}_d(20,20,t)$	0.8437	4.0490	8.1932	16.7720	25.5228	34.3162
$\underline{V}_d(20,20,t)$	0.8379	4.0220	8.1391	16.6639	25.3627	34.1061
$\overline{\psi}_d(20,20,t)$	0.0009	0.0043	0.0084	0.0152	0.0208	0.0256
$\underline{\psi}_d(20,20,t)$	0.0007	0.0042	0.0082	0.0148	0.0203	0.0249



(a) Bounds for $V(u, 10, t)$.



(b) Bounds for $\psi(u, 10, t)$.

Figure 6: Comparison of bounds for $V(u, 10, t)$ and $\psi(u, 10, t)$, for $u = 10$, 20, 40, $c = 1.3$, $\hat{c} = 0.2$, $\delta = 0.01$ and Pareto claims.

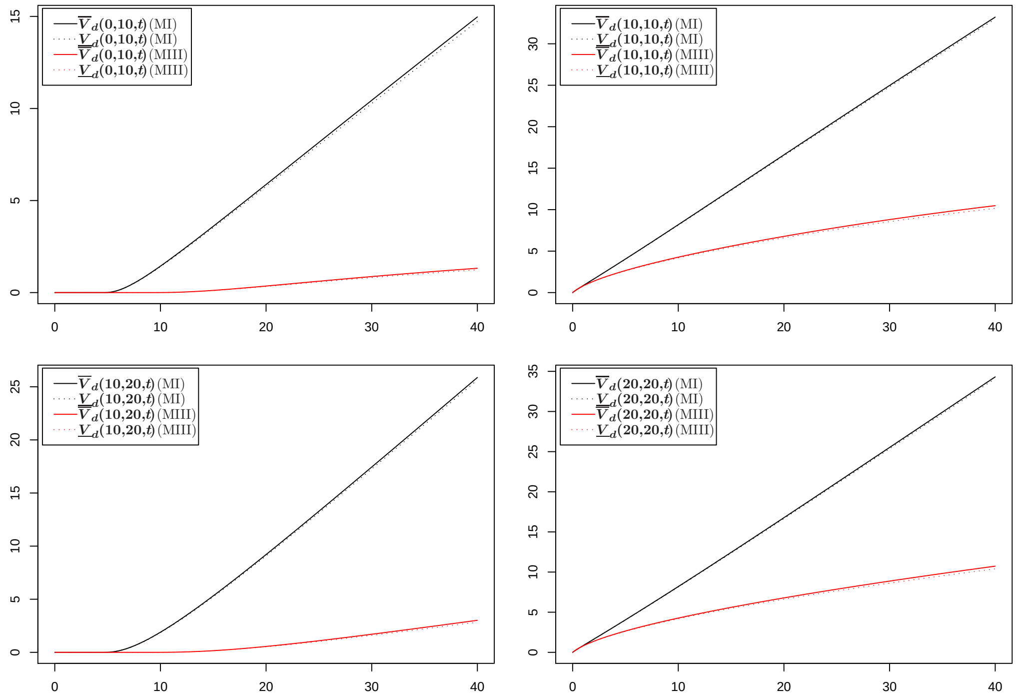


Figure 7: Model I *vs* Model III: a comparison of bounds for $V(u, b, t)$, for $\delta = 0.001$ and Pareto claims.

References

Albrecher, H. and Hartinger, J. (2007). A risk model with multilayer dividend strategy. *North American Actuarial Journal*, 11(2):43–64.

Badescu, A., Drekic, S., and Landriault, D. (2007). Analysis of a threshold dividend strategy for a map risk model. *Scandinavian Actuarial Journal*, (4):227 – 247.

Cardoso, R. M. R. and Egídio dos Reis, A. D. (2002). Recursive calculation of time to ruin distributions. *Insurance: Mathematics and Economics*, 30(2):219–230.

Cardoso, R. M. R. and Waters, H. R. (2003). Recursive calculation of finite time ruin probabilities under interest force. *Insurance: Mathematics and Economics*, 33(3):659–676.

Cardoso, R. M. R. and Waters, H. R. (2005). Calculation of finite time ruin probabilities for some risk models. *Insurance: Mathematics and Economics*, 37(2):197–215.

De Vylder, F. and Goovaerts, M. J. (1988). Recursive calculation of finite-time ruin probabilities. *Insurance: Mathematics and Economics*, 7(1):1–7.

Dickson, D. C. M. and Drekic, S. (2006). Optimal dividends under a ruin constraint. *Annals of Actuarial Science*, 1:291–306.

Dickson, D. C. M. and Waters, H. R. (1991). Recursive calculation of survival probabilities. *Astin Bulletin*, 21(2):199–221.

Dickson, D. C. M. and Waters, H. R. (2004). Some optimal dividends problems. *Astin Bulletin*, 34(1):49–74.

Gerber, H., Lin, X., and Yang, H. (2006a). A note on the dividends-penalty identity and the optimal dividend barrier. *Astin Bulletin*, 36(2).

Gerber, H., Shiu, E., and Smith, N. (2006b). Maximizing dividends without bankruptcy. *Astin Bulletin*, 36(1):5–23.

Gerber, H. U. and Shiu, E. S. (2006). On optimal dividend strategies in the compound poisson model. *North American Actuarial Journal*, 10(2):76–93.

Leung, K. S., Kwok, Y. K., and Leung, S. Y. (2008). Finite-time dividend-ruin models. *Insurance: Mathematics and Economics*, 42(1):154–162.

Lin, X. and Pavlova, K. P. (2006). The compound Poisson risk model with a threshold dividend strategy. *Insurance: Mathematics and Economics*, 38(1):57–80.

Lin, X. S. and Sendova, K. P. (2007). The compound poisson risk model with multiple thresholds. *Insurance: Mathematics and Economics*. In press.