The wrapped Gamma distribution and wrapped sums and linear combinations of independent Gamma and Laplace distributions

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Abstract

In this paper we first obtain an expression for the probability density function of the wrapped or circular Gamma distribution and then we show how it may be seen, both for integer and non-integer shape parameter, as a mixture of truncated Gamma distributions. Some other properties of the wrapped Gamma distribution are studied and it is shown how this distribution and namely mixtures of these distributions may be a much useful tool in modelling directional data in biology and meteorology. Based on the results obtained, namely the ones concerning mixtures, and on some properties of the distributions of the sum of independent Gamma random variables, the wrapped versions of the distributions of such sums, for both integer and non-integer shape parameters are derived. Also the wrapped sum of independent generalized Laplace distributions. Among the particular cases of the distributions introduced there are symmetrical, slightly skewed and highly skewed wrapped distributions as well as the recently introduced wrapped Exponential and Laplace distributions.

Keywords: mixtures, truncated Gamma distributions, circular data, circular distributions, wrapped symmetrical and skew distributions.

1 Introduction

The wrapped Exponential and Laplace distributions have been recently introduced and studied by Jammalamadaka and Kozubowski (2003, 2004) and their adequacy to model directional data in biology was shown. However, the wrapped or circular Gamma distribution and associated distributions have never been adequately studied, although they may be even better candidates to model directional data in several areas of research such as biology and meteorology.

Although wrapping the Exponential distribution is equivalent to only truncate it at 2π , wrapping the Gamma distribution places other interesting problems and opens interesting views. In this paper, expressions for both the pdf (probability density function) and the cdf (cumulative distribution function) of the wrapped Gamma distribution are obtained under much manageable forms. A couple of alternative representations, based on mixtures, are obtained. These representations show that for both integer and non-integer values of the shape parameter the wrapped Gamma distribution may be seen as a mixture of Gamma distributions truncated at 2π and they are thus much useful in obtaining manageable expressions for the cdf of the

wrapped Gamma distribution. These mixtures will be finite for integer values of the shape parameter and infinite when the shape parameter assumes a non-integer value.

To establish the notation and nomenclature used, we will say that the r.v. (random variable) X has a Gamma distribution with shape parameter r (> 0) and rate parameter λ (> 0) if its pdf may be written as

$$f_X(x) = \frac{\lambda^r}{\Gamma(r)} e^{-\lambda x} x^{r-1}, \quad (x > 0)$$

We will represent this fact by

 $X \sim \Gamma(r, \lambda)$,

and we will use $\Gamma(r, \lambda)$ to denote such distribution. If r is an integer we will call the Gamma distribution an integer Gamma distribution.

We will also use

$$\gamma(r;\lambda y) = \int_0^y \lambda^r e^{-\lambda x} x^{r-1} dx$$
(1)

to denote the incomplete Gamma function and the notation

$$\gamma^*(r;\lambda y) = \frac{\gamma(r;\lambda y)}{\Gamma(r)}.$$
(2)

With this notation, the cdf of the r.v. $X \sim \Gamma(r, \lambda)$ is given by

$$F_X(x) = P(X \le x) = \gamma^*(r; \lambda x) = \frac{\gamma(r; \lambda x)}{\Gamma(r)}$$

and the pdf of a Gamma distribution with shape parameter r and rate parameter λ , truncated at b > 0, written as

$$f(x) = \frac{\frac{\lambda'}{\Gamma(r)} e^{-\lambda x} x^{r-1}}{\gamma^*(r; \lambda b)} = \frac{\lambda^r}{\gamma(r; \lambda b)} e^{-\lambda x} x^{r-1}.$$

Hereon, we will denote the fact that the r.v. X has a Gamma distribution with shape parameter r and rate parameter λ , truncated at b > 0, by

$$X \sim \Gamma(r, \lambda; [0, b)),$$

and we will use $\Gamma(r, \lambda; [0, b))$ to denote such a distribution.

Yet, to shorten the notation, we will use: i) \mathbb{I} to denote the set of positive integers and $\mathbb{I}N_0$ to denote the set of non-negative integers; ii) $\mathbb{I}R$ to denote the set of all reals, with $\mathbb{I}R^+$ for the positive, $\mathbb{I}R_0^+$ for the non-negative, and $\mathbb{I}R_0^-$ for the non-positive reals.

2 The wrapped Gamma distribution

2.1 A first representation for the pdf and some properties of the distribution

Let

$$\Phi(z, s, a) = \sum_{k=0}^{\infty} \frac{z^k}{(a+k)^s},$$
(3)

where any term with a + k = 0 is excluded. The function $\Phi(z, s, a)$ is usually called Lerch's transcendental function and it is a generalization of both Riemann's generalized Zeta function and the Polylogarithm function. The generalized Riemann's Zeta function is defined as

$$\xi(s,a) \, = \, \sum_{k=0}^\infty \frac{1}{(a+k)^s} \, = \, \Phi(1,s,a)$$

where any term with a + k = 0 is also excluded, while the Polylogarithm function is defined as

$$PLog(z,s) = \sum_{k=1}^{\infty} \frac{z^k}{k^s}$$

$$= \frac{1}{\Gamma(s)} \int_0^1 \frac{z \, (-\log t)^{s-1}}{1-tz} \, dt = \Phi(z,s,0) \,.$$
(4)

Let $X \sim \Gamma(r, \lambda)$ and let

$$\theta \equiv \theta(X) = X \mod 2\pi$$
.

Then θ is a wrapped (around the circle) or circular Gamma r.v., that for $t \in [0, 2\pi)$ has pdf

$$f_{\theta}(t) = \sum_{k=0}^{\infty} f_X(t+2k\pi)$$

$$= \frac{\lambda^r}{\Gamma(r)} \sum_{k=0}^{\infty} e^{-\lambda(t+2k\pi)} (t+2k\pi)^{r-1}$$

$$= \frac{\lambda^r}{\Gamma(r)} e^{-\lambda t} (2\pi)^{r-1} \sum_{k=0}^{\infty} e^{-2\lambda\pi k} \left(\frac{t}{2\pi}+k\right)^{r-1}$$

$$= \frac{\lambda^r}{\Gamma(r)} e^{-\lambda t} (2\pi)^{r-1} \Phi\left(e^{-2\lambda\pi}, 1-r, \frac{t}{2\pi}\right).$$
(5)

We will denote the fact that the r.v. θ has a wrapped Gamma distribution with shape parameter r and rate parameter λ , with the above pdf, by

$$\theta \sim W\Gamma(r,\lambda)$$
.

The representation for the pdf of the wrapped Gamma distribution in (5) is rather convenient since Lerch's function is already well implemented in a number of software packages like Mathematica (from Wolfram Research). However, it may present some problems in terms of the determination of a manageable representation for the cdf.

Alternative representations of the pdf of the wrapped Gamma distribution may be obtained directly from the series expression in (5), under the form of mixtures. These representations will shed some more light on the intrinsic structure of the wrapped Gamma distribution and will also enable us to obtain simple representations for the cdf. Details are shown in the next subsection, where it is also shown that if the shape parameter r assumes integer values then negative values of the rate parameter λ are allowed, producing mirrored distributions (around the origin) of the ones with positive rate parameter.

Concerning the arbitrariness of the choice of the origin we should indeed rather write the pdf of the $W\Gamma(r, \lambda)$ distribution as

$$f_{\theta}(t) = \frac{\lambda^r}{\Gamma(r)} e^{-\lambda(t-a)} (2\pi)^{r-1} \Phi\left(e^{-2\lambda\pi}, 1-r, \frac{t-a}{2\pi}\right)$$

for $t - a \in [0, 2\pi)$ or $t \in [a, a + 2\pi)$, being a the origin. However, for the sake of simplicity, we will use throughout this manuscript a = 0, thus using the notation in (5).

We should note that as either $\lambda \to 0$ or $r \to \infty$, the $W\Gamma(r, \lambda)$ distribution tends to the wrapped uniform distribution, with density

$$f(t) = \frac{1}{2\pi}, \qquad t \in [0, 2\pi)$$

This fact may be analysed in Figures 2 and 3 and it is derived from the facts that

r

$$\lim_{t \to +\infty} \frac{\Phi\left(e^{-2\lambda\pi}, 1-r, \frac{t}{2\pi}\right)}{e^{\lambda t} \Gamma(r) \left(2\lambda\pi\right)^{-r}} = 1$$
(6)

and

$$\lim_{\lambda \to 0^+} \frac{\Phi\left(e^{-2\lambda\pi}, 1-r, \frac{t}{2\pi}\right)}{\Gamma(r) \left(2\lambda\pi\right)^{-r}} = 1, \qquad (7)$$

so that

$$\lim_{r \to +\infty} f_{\theta}(t) = \lim_{r \to +\infty} \frac{\lambda^r}{\Gamma(r)} e^{-\lambda t} (2\pi)^{r-1} \Phi\left(e^{-2\lambda\pi}, 1-r, \frac{t}{2\pi}\right) = \frac{1}{2\pi}$$

and also

$$\lim_{\lambda \to 0^+} f_{\theta}(t) = \lim_{\lambda \to 0^+} \frac{\lambda^r}{\Gamma(r)} e^{-\lambda t} (2\pi)^{r-1} \Phi\left(e^{-2\lambda\pi}, 1-r, \frac{t}{2\pi}\right) = \frac{1}{2\pi}$$

The rate of convergence of the limit in (6) is higher for smaller values of λ , while the rate of convergence of the limit in (7) is somehow erratic for different values of r. See Figure 1.



The graphical aspects of the $W\Gamma(r, \lambda)$ distribution are much variable, somehow even more than for its linear counterpart. Contrary to what happens with the linear Gamma distribution, with the $W\Gamma(r, \lambda)$ distribution the lower point or antimode of the pdf is not always placed at the origin (see Figures 2, 3 and 4).

The mode is, for sufficiently large λ and r > 1 (the largest r is, the largest λ has to be), placed a little below $t = \frac{r-1}{\lambda}$. For $0 < r \le 1$ it is placed exactly on the right of t = 0 and the antimode on the left of $t = 2\pi$. For $1 < r \le 2$ the antimode is placed on the right of t = 0. For r = 2 the mode is at $t = \frac{1}{\lambda} - \frac{2\pi}{e^{2\lambda\pi}-1}$, while for r = 3 the value of t where the mode is placed already has a very complicated expression,

$$t = \frac{1}{\lambda} + \frac{2\lambda\pi \left(e^{2\lambda\pi} - 1\right) - \sqrt{\sum_{i=0}^{4} \binom{4}{i} (-1)^{i} e^{2\lambda\pi i} + 4\lambda^{2}\pi^{2} \sum_{i=0}^{2} (-1)^{i+1} \binom{2}{i} e^{2\lambda\pi i}}{\lambda \sum_{i=0}^{2} (-1)^{i+1} \binom{2}{i} e^{2\lambda\pi i}}$$

with the antimode at

$$t = \frac{1}{\lambda} + \frac{2\lambda\pi \left(e^{2\lambda\pi} - 1\right) + \sqrt{\sum_{i=0}^{4} \binom{4}{i} (-1)^{i} e^{2\lambda\pi i} + 4\lambda^{2}\pi^{2} \sum_{i=0}^{2} (-1)^{i+1} \binom{2}{i} e^{2\lambda\pi i}}{\lambda \sum_{i=0}^{2} (-1)^{i+1} \binom{2}{i} e^{2\lambda\pi i}} \,.$$

The mode and the antimode are anyway, for any value of r and λ , at the values of t for which $\frac{d}{dt} f_{\theta}(t)$ does not exist or $\frac{d}{dt} f_{\theta}(t) = 0$.



Figure 2. – Three dimensional views of wrapped Gamma densities for $\lambda = 2.8$ and increasing values of r. All plots have the origin placed at the same spot. In each plot are marked the origin (O), the mode (M), the antimode (m) and the mean direction, this one marked with a continuous line (for a definition of the mean direction see section 3). (All plots on the same row have the same vertical scale but plots in different rows have different vertical scales).

Since

$$\frac{d}{dt}\Phi\left(e^{-2\lambda\pi}, 1-r, \frac{t}{2\pi}\right) = \frac{1}{2\pi}\Phi\left(e^{-2\lambda\pi}, 2-r, \frac{t}{2\pi}\right)$$

as it is easy to derive from the definition of Lerch's Phi function, the abscissas for the mode and the antimode are the values of $t \in [0, 2\pi)$ such that $\frac{d}{dt}f_{\theta}(t)$ does not exist or

$$\frac{d}{dt}f_{\theta}(t) = 0 \iff \Phi\left(e^{-2\lambda\pi}, 1-r, \frac{t}{2\pi}\right) = \frac{r-1}{2\lambda\pi}\Phi\left(e^{-2\lambda\pi}, 2-r, \frac{t}{2\pi}\right).$$

For values of $r \leq 1$ the mode is at the point t = 0, where $\frac{d}{dt}f(t)$ is not defined.



Figure 3. – Three-dimensional views of wrapped Gamma densities for r = 6.1 and increasing values of λ . All plots have the origin placed at the same spot. In each plot are marked the origin (O), the mode (M), the antimode (m) and the mean direction, this one marked with a continuous line (for a definition of the mean direction see section 3). All plots have the same vertical scale.

As it happens with the wrapped Exponential distribution, also the integer wrapped Gamma distribution is defined for $\lambda < 0$, with the pdf for $\lambda < 0$ yielding a symmetrical aspect (relative to the origin) of the one it does have for positive λ . More precisely, we have the following Theorem.

Theorem 1: For positive integer r, if

$$\theta_1 \sim W\Gamma(r,\lambda)$$
 and $\theta_2 \sim W\Gamma(r,-\lambda)$

then

$$f_{\theta_1}(t) = f_{\theta_2}(2\pi - t)$$
 or $f_{\theta_2}(t) = f_{\theta_1}(2\pi - t)$

that is, for positive integer r, wrapped Gama distributions with symmetric rate parameters have pdf's that are mirrored about the origin.

Proof: We may note that for positive integer r we may write the pdf of the linear $\Gamma(r, \lambda)$ distribution as

$$f(x;r,\lambda) = \frac{\lambda^r}{\Gamma(r)} e^{-\lambda x} x^{r-1}$$
$$= -\frac{(-\lambda)^r}{\Gamma(r)} e^{-(-\lambda)(-x)} (-x)^{r-1}$$
$$= -f(-x;r,-\lambda)$$

what shows that for positive integer r, the symmetric of $f(x; r, \lambda)$ with $\lambda(> 0)$ replaced by $-\lambda$, is a pdf on \mathbb{R}^- , mirrored from $f(x; r, \lambda)$.

Using this result we may easily show that for positive integer r, we have for the $W\Gamma(r, \lambda)$ distribution (with $\lambda > 0$),

$$f_{\theta}(2\pi - t; r, -\lambda) = f_{\theta}(t; r, \lambda), \qquad (t \in [0, 2\pi))$$

since for $t \in [0, 2\pi)$,

$$\begin{split} f_{\theta}(2\pi - t; r, -\lambda) &= \sum_{k=-1}^{-\infty} -f(2\pi - t + 2k\pi; r, -\lambda) = \sum_{k=-1}^{-\infty} -f(-t + 2\pi(k+1); r, -\lambda) \\ &= \sum_{k=0}^{-\infty} -f(-t + 2\pi k; r, -\lambda) = \sum_{k=0}^{\infty} -f(-t - 2\pi k; r, -\lambda) \\ &= \sum_{k=0}^{\infty} f(t + 2\pi k; r, \lambda) = f_{\theta}(t; r, \lambda) \,, \end{split}$$

where the equality from the row before the last to the last row is only valid for $r \in \mathbb{N}$.

The definition of Lerch's Phi function in most packages, as it happens with the software Mathematica, usually has this duality built in, so that if we implement the pdf of the $W\Gamma(r, \lambda)$ distribution through the use of Lerch's Phi function, and if then we take, for $r \in \mathbb{N}$ and $\lambda > 0$, $f_{\theta}(2\pi - t; r, -\lambda)$ we obtain the same values as for $f_{\theta}(t; r, \lambda)$.

Also for wrapped or circular distributions the usual properties of mixtures hold. Namely:

i) if $f_i(t)$ (i = 0, 1, ...) are pdf's of wrapped distributions and if $\sum_{i=0}^n p_i = 1$, with $0 < p_i < 1$ and where n may be either finite or infinite, then

$$f(t) = \sum_{i=0}^{n} p_i f_i(t)$$

is the pdf of a wrapped distribution;

ii) if $\Phi_i(p)$ (i = 0, 1, ...) are the c.f.'s (characteristic functions) of the wrapped distributions with pdf's $f_i(t)$ in i), then

$$\Phi(p) = \sum_{i=0}^{n} p_i \, \Phi_i(p)$$

is the c.f. of the wrapped mixture distribution with pdf f(t) in i) and thus the cdf's and moments also mix.



marked the origin (O) and the mean direction (M) (for a definition of the mean direction see section 3). Plots on the same row have the same vertical scale but plots on different rows may have different vertical scales.

Mixtures of wrapped Gamma distributions are extremely flexible and are a powerful modeling tool for circular data, adequate to model both symmetric and asymmetric circular data with more than one mode, since a mixture of k wrapped Gamma distributions may have from 1 to k modes. Some plots of densities of mixtures of wrapped Gamma distributions are displayed in Figures 5, 6 and 7.

Linearized plots of the pdf's may here be useful to better analyse the location and number of modes, although the circular plots may be also quite elucidative.

In Figures 5 and 6 we may see how simple changes in the weight may dramatically affect the aspect of the density of the mixture of two wrapped Gamma distributions, while in Figures 7 and 8 we may see how mixtures of wrapped Gamma distributions may exhibit multimodal densities.



2.2 Alternative representations for the pdf of the wrapped Gamma distribution

vertical scale.

The following two theorems establish two results which show how the wrapped Gamma distribution may be seen as a mixture of truncated linear Gamma distributions both for integer and non-integer r.

Theorem 2: For non-integer r the $W\Gamma(r, \lambda)$ distribution is a mixture with weights

$$p_{i} = \gamma^{*}(i+1;2\lambda\pi) (-1)^{i} \frac{\lambda^{r-i-1}}{\Gamma(r)} \frac{\Gamma(1-r+i)}{\Gamma(1-r)} PLog\left(e^{-2\lambda\pi}, i+1-r\right), \quad i = 0, 1, \dots$$
(8)

of infinitely many $\Gamma(i + 1, \lambda; [0, 2\pi))$ distributions (i = 0, 1, ...) with a further $\Gamma(r, \lambda; [0, 2\pi))$ distribution with weight

$$p^* = \gamma^*(r, 2\lambda\pi) \,. \tag{9}$$

Proof: Since for $t \in [0, 2\pi)$ and $k \ge 1$ we have $\left|\frac{t}{2k\pi}\right| < 1$, we may write, for $c \notin \mathbb{N}_0$,

$$\left(1+\frac{t}{2k\pi}\right)^c = {}_1F_0\left(-c;-\frac{t}{2k\pi}\right) = \sum_{i=0}^{\infty} \frac{\Gamma(-c+i)}{\Gamma(-c)} \frac{(-1)^i}{i!} \left(\frac{t}{2k\pi}\right)^i.$$

Then, given the uniform convergence of the series involved, we may write, directly from the definition of $f_{\theta}(t)$ and from (5),

$$\begin{split} f_{\theta}(t) &= \frac{\lambda^{r}}{\Gamma(r)} e^{-\lambda t} \sum_{k=0}^{\infty} e^{-2\pi\lambda k} \left(t + 2k\pi\right)^{r-1} \\ &= \frac{\lambda^{r}}{\Gamma(r)} e^{-\lambda t} t^{r-1} \\ &+ \frac{\lambda^{r}}{\Gamma(r)} e^{-\lambda t} \sum_{k=1}^{\infty} e^{-2\pi\lambda k} \left(\frac{t}{2k\pi} + 1\right)^{r-1} (2k\pi)^{r-1} \\ &= \frac{\lambda^{r}}{\Gamma(r)} e^{-\lambda t} t^{r-1} \frac{\Gamma(r)}{\gamma(r; 2\lambda\pi)} \frac{\gamma(r; 2\lambda\pi)}{\Gamma(r)} \\ &+ \frac{\lambda^{r}}{\Gamma(r)} e^{-\lambda t} \sum_{k=1}^{\infty} e^{-2\pi\lambda k} (2k\pi)^{r-1} \sum_{i=0}^{\infty} (-1)^{i} \left(\frac{t}{2k\pi}\right)^{i} \frac{\Gamma(1-r+i)}{\Gamma(1-r)} \frac{1}{i!} \\ &= \frac{\lambda^{r}}{\gamma(r; 2\lambda\pi)} e^{-\lambda t} t^{r-1} \frac{\Gamma(r, 2\lambda\pi)}{\Gamma(r)} + \sum_{i=0}^{\infty} \frac{\lambda^{i+1}}{\gamma(i+1; 2\lambda\pi)} e^{-\lambda t} t^{i} \\ &\qquad \frac{\gamma(i+1; 2\lambda\pi)}{\Gamma(i+1)} (-1)^{i} \frac{\lambda^{r-i-1}}{\Gamma(r)} \frac{\Gamma(1-r+i)}{\Gamma(1-r)} \sum_{k=1}^{\infty} e^{-2\pi\lambda k} (2k\pi)^{r-1-i} \\ &= \underbrace{\frac{\lambda^{r}}{\gamma(r; 2\lambda\pi)} e^{-\lambda t} t^{r-1}}_{pdf \text{ of } \Gamma(r, \lambda; [0, 2\pi))} \underbrace{\gamma^{*}(r; 2\lambda\pi)}_{p^{*}} + \sum_{i=0}^{\infty} \frac{\lambda^{i+1}}{\gamma(i+1; 2\lambda\pi)} e^{-\lambda t} t^{i} \\ &\qquad \underbrace{\gamma^{*}(i+1; 2\lambda\pi) (-1)^{i} \frac{\lambda^{r-i-1}}{\Gamma(r)} \frac{\Gamma(1-r+i)}{\Gamma(1-r)} PLog\left(e^{-2\lambda\pi}, i+1-r\right)}_{p_{i}} \dots \\ \end{matrix}$$

We should note that indeed for the weights p_i in (8), and p^* in (9),

$$\sum_{i=0}^{\infty} p_i = 1 - p^*, \quad \text{or} \quad p^* + \sum_{i=0}^{\infty} p_i = 1.$$

For negative integer s the Lerch and Polylogarithm functions in (3) and (4) have simpler finite representations, which are useful to develop. Such representations will be used to obtain a finite mixture representation for the $W\Gamma(r, \lambda)$ distribution for $r \in \mathbb{N}$.

We will define, for positive integers n and i, with 1 < i < n,

$$t_{n,i} = (n-i+1) t_{n-1,i-1} + i t_{n-1,i}$$

with

$$t_{n,n} = t_{n,1} = 1$$

and

$$t_{n,i} = 0 \quad \text{for} \quad i > n \text{ or } i < 0.$$

We may note that as a consequence we have, as it is really desirable,

$$t_{n,0} = \begin{cases} 1, & n = 0\\ 0, & n > 0 \end{cases} \quad \text{and} \quad t_{n,i} = 0 \text{ for } n < 0.$$

Values of $t_{n,i}$ for n = 1, ..., 10 (and $1 \le i \le n$) are tabulated in Appendix A.



Figure 6. – Three-dimensional views of the densities of the mixture of three wrapped Gamma distributions, $3/7 * W\Gamma(2.8, 13.5) + 2/7 * W\Gamma(2.8, 4.5) + 2/7 * W\Gamma(2.8, r)$, for different values of r. All plots have the origin placed at the same spot and all plots have the same vertical scale.

Then we may write, for positive integer n, and $z \in \mathbb{R}$

$$\Phi^*(z,n) = \sum_{k=0}^{\infty} z^k k^n = \sum_{k=1}^{\infty} z^k k^n = \Phi(z,-n,0) = PLog(z,-n)$$

$$= \frac{\sum_{i=1}^n t_{n,i} z^{-i}}{(z^{-1}-1)^{n+1}} = \frac{\sum_{i=1}^n t_{n,i} z^{n-i+1}}{(1-z)^{n+1}}$$
(11)

and for n = 0

$$\Phi^*(z,0) = \sum_{k=0}^{\infty} z^k k^0 = 1 + \sum_{k=1}^{\infty} z^k k^0 = 1 + \Phi(z,0,0) = 1 + PLog(z,0)$$
$$= \frac{z^{-1}}{z^{-1} - 1} = \frac{1}{1 - z},$$

so that we may write, for $n \in \mathbb{I}N_0$,

$$\Phi^*(z,n) = \sum_{k=0}^{\infty} z^k k^n = \frac{t_{n,0} + \sum_{i=1}^n t_{n,i} z^{n+1-i}}{(1-z)^{n+1}}.$$
(12)



Figure 7. – Three dimensional views of the density of a mixture of three wrapped Gamma distributions: $1/3 * W\Gamma(-2.3,3) + 1/3 * W\Gamma(2,7) + 1/3 * W\Gamma(2.3,3)$ and linear plot of the same density. In each plot are marked the origin (O) and the mean direction (M).



plot of the same density. In each plot are marked the origin (O) and the mean direction (M).

Theorem 3: For $r \in \mathbb{N}$ the $W\Gamma(r, \lambda)$ distribution is a mixture with weights

$$p_h = \gamma^* (h+1; 2\lambda\pi) \,\frac{(2\pi\lambda)^{r-h-1}}{(r-h-1)!} \,\Phi^* \left(e^{-2\lambda\pi}, r-h-1 \right) \,, \quad h = 0, \dots, r-1 \tag{13}$$

of r distributions $\Gamma(h+1,\lambda,[0,2\pi))$ distributions, with $h=0,\ldots,r-1$.

Proof: For $r \in \mathbb{N}$ we may write

$$\begin{aligned} f_{\theta}(t) &= \frac{\lambda^{r}}{\Gamma(r)} e^{-\lambda t} \sum_{k=0}^{\infty} e^{-2\lambda \pi k} (t+2\pi k)^{r-1} \\ &= \frac{\lambda^{r}}{\Gamma(r)} e^{-\lambda t} \sum_{k=0}^{\infty} e^{-2\lambda \pi k} \sum_{h=0}^{r-1} \binom{r-1}{h} t^{h} (2\pi k)^{r-1-h} \\ &= \sum_{h=0}^{r-1} \frac{\lambda^{h+1}}{\Gamma(h+1)} e^{-\lambda t} t^{h} \frac{\lambda^{r-h-1}}{(r-h-1)!} (2\pi)^{r-1-h} \sum_{k=0}^{\infty} e^{-2\lambda \pi k} k^{r-1-h} \\ &= \sum_{h=0}^{r-1} \frac{\lambda^{h+1}}{\gamma(h+1;2\lambda \pi)} e^{-\lambda t} t^{h} \frac{\lambda^{r}(h+1;2\lambda \pi)}{(r-h-1)!} \frac{(2\pi \lambda)^{r-h-1}}{(r-h-1)!} \Phi^{*} \left(e^{-2\lambda \pi}, r-h-1\right)}{p_{h}} \end{aligned}$$
(14)

where $\sum_{h=0}^{r-1} p_h = 1$, $\Phi^*(\cdot, \cdot)$ is given by (12) above, and where, for non-negative integer h,

$$\gamma^*(h+1;2\lambda\pi) = \frac{\gamma(h+1;2\lambda\pi)}{\Gamma(h+1)} = 1 - e^{-2\lambda\pi} \sum_{j=0}^h \frac{(2\lambda\pi)^j}{j!} . \bullet$$

For the particular case r = 1 we have the wrapped Exponential distribution of Jammalamadaka and Kozubowski (2003, 2004).

We should note that the mixture expression obtained for non-integer r still holds for integer r.

The mixture representations of the $W\Gamma(r, \lambda)$ distribution are useful for a number of reasons. Besides showing us an interesting result, they allow us to easily derive expressions for the cdf, both for integer and non-integer r. Using the incomplete Gamma functions defined in (1) and (2) and the mixture representation in (10) we may write the cdf of the $W\Gamma(r, \lambda)$ distribution for non-integer r as

$$F_{\theta}(t) = p^* \frac{\gamma(r; \lambda t)}{\gamma(r; 2\pi\lambda)} + \sum_{i=0}^{\infty} p_i \frac{\gamma(i+1; \lambda t)}{\gamma(i+1; 2\pi\lambda)}$$
(15)

for p_i (i = 0, 1, ...) given by (8) and p^* given by (9), while for $r \in \mathbb{N}$ we may write, from (14), the cdf of the $W\Gamma(r, \lambda)$ distribution as

$$F_{\theta}(t) = \sum_{h=0}^{r-1} p_h \frac{\gamma(h+1;\lambda t)}{\gamma(h+1;2\pi\lambda)}$$
(16)

for p_h (h = 0, ..., r - 1) given in (13).

3 Trigonometric moments and related parameters

The c.f. (characteristic function) of the $W\Gamma(r,\lambda)$ distribution is

$$\phi(p) = \lambda^r (\lambda - ip)^{-r}, \qquad p = \pm 1, \pm 2, \dots,$$

where $i = (-1)^{1/2}$.

Using the fact that for $a, b, r \in \mathbb{R}^+$ we may write

$$(a - ib)^{-r} = (a^2 + b^2)^{-r/2} e^{ir \arctan(b/a)}$$

we have for $\lambda > 0$,

$$\phi(p) = \lambda^r \left(\lambda^2 + p^2\right)^{-r/2} e^{ir \arctan(p/\lambda)}$$
$$= \rho_p e^{i\mu_p}$$

with

 $\rho_p = \lambda^r \left(\lambda^2 + p^2\right)^{-r/2}$

and

 $\mu_p = r \arctan(p/\lambda),$

where $\arctan(\cdot)$ is the usual inverse function of the tangent function, with counterdomain $] - \pi/2, \pi/2[$. This way, the non-central trigonometric moments are

$$\alpha_p = \rho_p \cos(\mu_p) = \lambda^r \left(\lambda^2 + p^2\right)^{-r/2} \cos\left(r \arctan(p/\lambda)\right)$$
$$\beta_p = \rho_p \sin(\mu_p) = \lambda^r \left(\lambda^2 + p^2\right)^{-r/2} \sin\left(r \arctan(p/\lambda)\right)$$

with

$$\phi(p) = \alpha_p + i \beta_p, \quad p = \pm 1, \pm 2, \dots,$$

and the central trigonometric moments are given by

$$\overline{\alpha}_p = \rho_p \cos(\mu_p - p\mu_1) = \lambda^r \left(\lambda^2 + p^2\right)^{-r/2} \cos\left(r \arctan(p/\lambda) - pr \arctan(1/\lambda)\right)$$
$$\overline{\beta}_p = \rho_p \sin(\mu_p - p\mu_1) = \lambda^r \left(\lambda^2 + p^2\right)^{-r/2} \sin\left(r \arctan(p/\lambda - pr \arctan(1/\lambda))\right).$$

The resultant length and mean direction are thus respectively

$$\rho = \rho_1 = \lambda^r \left(\lambda^2 + 1\right)^{-r/2}$$

and

$$\mu_1 = r \arctan(1/\lambda)$$
.

The circular variance is

$$V_0 = 1 - \rho = 1 - \lambda^r \left(\lambda^2 + 1\right)^{-r/2}$$

and the circular standard deviation is

$$\sigma_0 = \sqrt{-2\log \rho} = \sqrt{-r \log \left(1 + \frac{1}{\lambda^2}\right)},$$

with the skewness and kurtosis not having any particularly interesting expressions, being just defined through the usual expressions, respectively

$$\gamma_1^0 = \overline{\beta}_2 / V_0^{3/2}$$

and

$$\gamma_2^0 = \frac{\overline{\alpha}_2 - \rho^4}{(1-\rho)^2}.$$

For $r \in \mathbb{N}$ and $\lambda < 0$ we have to use the fact that for a < 0, and $b, r \in \mathbb{R}^+$, we may write

$$(a-ib)^{-r} = (a^2+b^2)^{-r/2} e^{ir(\pi+\arctan(b/a))}$$

so that in this case

$$\phi(p) = \lambda^r \left(\lambda^2 + p^2\right)^{-r/2} e^{ir(\pi + \arctan(p/\lambda))}$$

with ρ_p being the same as for $\lambda > 0$ and

$$\mu_p = r \left(\pi + \arctan(p/\lambda) \right)$$

with the concomitant changes in the mean direction and all other moments and parameters depending on μ_p .

All these results confirm, for r = 1, the results from Jammalamadaka and Kozubowski (2003, 2004), except for the mean direction with $\lambda < 0$, where there seems to be a small mistake in Jammalamadaka and Kozubowski (2003, 2004).

4 The wrapped sum and linear combination of Gamma distributions and the wrapped Generalized Integer Gamma distribution

It may be shown that wrapped sums of independent Gamma distributions are particular types of mixtures of wrapped Gamma distributions.

In fact both for integer and non-integer shape parameters, the sum of independent Gamma distributions is a mixture of Gamma distributions. Thus its wrapped version is a mixture of wrapped Gamma distributions. The same applies to the wrapped version of linear combinations of independent Gamma distributions with all positive coefficients.

Let

$$X_i \sim \Gamma(r_i, \lambda_i), \qquad i = 1, \dots, n$$

be n independent Gamma r.v.'s, and consider the r.v.

$$Z = \sum_{i=1}^{n} a_i X_i, \quad a_i \in \mathbb{R}^+, \ \forall i.$$

Then we may see the r.v. Z as the sum of n independent Gamma r.v.'s

$$Y_i = a_i X_i \sim \Gamma(r_i, \mu_i)$$

where $\mu_i = \lambda_i / a_i \ (> 0)$.

In the general case, Moschopoulos (1985) has shown that the distribution of Z is an infinite mixture of $\Gamma(r+k,\mu)$ $(k=0,1,\ldots)$ distributions, where

$$r = \sum_{i=1}^{n} r_i \quad \text{and} \quad \mu = \max_{1 \le i \le n} \mu_i \tag{17}$$

with weights

$$w_k = \delta_k \prod_{i=1}^n \left(\frac{\mu_i}{\mu}\right)^{r_i}$$
 (k = 0, 1, ...) (18)

where $\delta_0 = 1$ and for $k = 1, 2, \ldots$

$$\delta_k = \frac{1}{k} \sum_{j=1}^k j \gamma_j \,\delta_{k-j} \tag{19}$$

with

$$\gamma_j = \frac{1}{j} \sum_{i=1}^n r_i \left(1 - \frac{\mu_i}{\mu} \right)^j, \quad j = 1, 2, \dots$$
 (20)

Thus the wrapped distribution of Z, that is, the distribution of the r.v.

$$\theta(Z) = Z \mod 2\pi$$

is an infinite mixture, with weights w_k (k = 0, 1, ...) of $W\Gamma(r + k, \mu)$ distributions, each one of which is itself an infinite mixture of truncated Gamma distributions at 2π , unless $r \in \mathbb{N}$, in which case each of the $W\Gamma(r + k, \mu)$ (k = 0, 1, ...) distributions is a finite mixture of r + ktruncated Gamma distributions. In the general case, that is, for $r_i \in \mathbb{R}^+$ (i = 1, ..., n), we may write the pdf of $\theta(Z)$ as

$$f_{\theta(Z)}(t) = \sum_{k=0}^{\infty} w_k \, \frac{\mu^{r+k}}{\Gamma(r+k)} \, e^{-\mu t} \, (2\pi)^{r+k-1} \, \Phi\left(e^{-2\mu\pi}, 1-r-k, \frac{t}{2\pi}\right) \tag{21}$$

with w_k given by (18) through (20) above, and μ and r given by (17).

The corresponding cdf is then easy to derive from (21) above, using the incomplete Gamma function and the mixture structure of the distribution. For non-integer r, using (15) as a reference, it may be written as

$$F_{\theta(Z)}(t) = \sum_{k=0}^{\infty} w_k \left(p_k^* \frac{\gamma(r+k;\mu t)}{\gamma(r+k;2\pi\mu)} + \sum_{i=0}^{\infty} p_{ik} \frac{\gamma(i+1;\mu t)}{\gamma(i+1;2\pi\mu)} \right)$$

with w_k , r and μ defined as in (21) above and, from (9), with

$$p_k^* = \gamma^*(r+k; 2\mu\pi)$$

and, from (8), for i, k = 0, 1, ...,

$$p_{ik} = \gamma^*(i+1;2\mu\pi) \left(-1\right)^i \frac{\mu^{r+k-i-1}}{\Gamma(r+k)} \frac{\Gamma(1-r-k+i)}{\Gamma(1-r-k)} \left(2\pi\right)^{r+k-1-i} PLog\left(e^{-2\mu\pi}, i+1-r-k\right),$$

while for integer r, taking (16) as a reference, we may write

$$F_{\theta(Z)}(t) = \sum_{k=0}^{\infty} w_k \sum_{h=0}^{r+k-1} p_{hk} \frac{\gamma(h+1;\mu t)}{\gamma(h+1;2\pi\mu)}$$

with, from (13), for $h = 0, \ldots, r + k - 1$ and $k = 0, 1, \ldots$, and taking $\Phi^*(\cdot, \cdot)$ defined as in (11),

$$p_{hk} = \gamma^*(h+1; 2\mu\pi) \frac{(2\pi\mu)^{r+k-h-1}}{(r+k-h-1)!} \Phi^*\left(e^{-2\mu\pi}, r+k-h-1\right) \,.$$

However, if all the r_i 's are integer, the distribution of $\theta(Z)$ may be shown to have a finite representation, since in this case the distribution of Z is what Coelho (1998) called a GIG (Generalized Integer Gamma) distribution of depth n, with pdf

$$f_Y(y) = K \sum_{j=1}^n P_j(y) e^{-\mu_j y}$$

and cdf

$$F_Y(y) = 1 - K \sum_{j=1}^n P_j^*(y) e^{-\mu_j y}$$

where

$$K = \prod_{j=1}^{n} \mu_j^{r_j} , \qquad P_j(y) = \sum_{k=1}^{r_j} c_{jk} y^{k-1}$$

and

$$P_j^*(y) = \sum_{k=1}^{r_j} c_{jk} (k-1)! \sum_{i=0}^{k-1} \frac{y^i}{i! \, \mu_i^{k-i}}$$

with

$$c_{j,r_j} = \frac{1}{(r_j - 1)!} \prod_{\substack{i=1 \ i \neq j}}^n (\mu_i - \mu_j)^{-r_i}, \qquad j = 1, \dots, n,$$

and

$$c_{j,r_j-k} = \frac{1}{k} \sum_{i=1}^{k} \frac{(r_j - k + i - 1)!}{(r_j - k - 1)!} R(i, j, n) c_{j,r_j-(k-i)}, \quad (k = 1, \dots, r_j - 1)$$
$$(j = 1, \dots, n)$$

where

$$R(i, j, n) = \sum_{\substack{k=1\\k\neq j}}^{n} r_k (\mu_j - \mu_k)^{-i} \quad (i = 1, \dots, r_j - 1),$$

and where we assumed all μ_i (i = 1, ..., n) to be different. In case $2 \le m \le n$ of them are equal, we have to add the corresponding shape parameters, and then consider the n - m + 1 different rate parameters and corresponding shape parameters to obtain a GIG distribution of depth n - m + 1.

The important point is that we may indeed look at the GIG distribution of Z as a mixture of integer Gamma distributions, since we may write the pdf of Z as

$$f_Z(z) = K \sum_{j=1}^n \sum_{k=1}^{r_j} c_{jk} z^{k-1} e^{-\mu_j z}$$

=
$$\sum_{j=1}^n \sum_{k=1}^{r_j} \underbrace{\frac{K}{\mu_j^k} \Gamma(k) c_{jk}}_{p_{kj}^*} \frac{\mu_j^k}{\Gamma(k)} e^{-\mu_j z} z^{k-1}$$

what shows that it is a mixture of $\sum_{j=1}^{n} r_j$ integer $\Gamma(k, \mu_j)$ distributions with weights

$$p_{kj}^* = \frac{K}{\mu_j^k} \Gamma(k) c_{jk}, \qquad k = 1, \dots, r_j; \ j = 1, \dots, n$$

with $\sum_{j=1}^{n} \sum_{k=1}^{r_j} p_{kj}^* = 1.$



Figure 9. – Three-dimensional views of densities of wrapped GIG distributions of depth 2 with shape parameters $r_1 = 2$, $r_2 = 3$ and rate parameters $\lambda_1 = 1.5$ and different values of λ_2 . All plots have the origin placed at the same spot and all plots have the same vertical scale.

As such, the wrapped GIG distribution of depth n is a mixture with weights p_{kj}^* of $\sum_{j=1}^n r_j$ integer $W\Gamma(k, \mu_j)$ distributions $(k = 1, \ldots, r_j; j = 1, \ldots, n)$, its pdf being given by

$$f_{\theta(Z)}(t) = \sum_{j=1}^{n} \sum_{k=1}^{r_j} p_{kj}^* \frac{\mu_j^k}{\Gamma(k)} e^{-\mu_j t} (2\pi)^{k-1} \Phi\left(e^{-2\mu_j \pi}, 1-k, \frac{t}{2\pi}\right)$$
$$= K \sum_{j=1}^{n} \sum_{k=1}^{r_j} c_{jk} e^{-\mu_j t} (2\pi)^{k-1} \Phi\left(e^{-2\mu_j \pi}, 1-k, \frac{t}{2\pi}\right)$$

or

$$f_{\theta(Z)}(t) = K \sum_{j=1}^{n} \sum_{k=1}^{r_j} \frac{\Gamma(k)}{\mu_j^k} c_{jk} \sum_{h=0}^{k-1} p_h \frac{\mu_j^{h+1}}{\Gamma(h+1)} \frac{e^{-\mu_j t} t^h}{\gamma^*(h+1; 2\mu_j \pi)}$$

and the cdf by

$$F_{\theta(Z)}(t) = K \sum_{j=1}^{n} \sum_{k=1}^{r_j} \frac{\Gamma(k)}{\mu_j^k} c_{jk} \sum_{h=0}^{k-1} p_h \frac{\gamma(h+1;\mu_j t)}{\gamma(h+1;2\mu_j \pi)},$$

where p_h (h = 0, ..., k - 1) is given by (13), with λ replaced by μ_i and r replaced by k.

We may then also easily consider mixtures of wrapped GIG distributions and of wrapped sums of Gammas.

On Figure 9 we may see the effect of the change in one of the rate parameters on the shape of the density of wrapped GIG distributions of depth 2, while on Figure 10 we may see the effect of the change in one of the shape parameters.



5 The wrapped sum of independent generalized Laplace distributions or the wrapped difference of two sums of independent Exponentials

One important particular type of mixture of wrapped Gamma distributions is the mixture of two wrapped Exponential distributions, that is, two wrapped integer Gamma distributions both with shape parameter equal to 1 and weights equal to 1/2 and symmetric rate parameters. Such a distribution is a wrapped Laplace distribution (see Jammalamadaka and Kozubowski (2003,

2004)). Indeed one way of looking at the linear Laplace distribution is to see it as the difference of two independent Exponential distributions with the same rate parameter, what also turns out to be a mixture with weights 1/2 of two Exponential distributions with symmetrical rate parameters, one with support \mathbb{R}_0^+ and the other with support \mathbb{R}_0^- .

Indeed we may even define the generalized linear Laplace distribution with rate parameters λ_1 and λ_2 , as the distribution with pdf

$$f(x) = \begin{cases} \frac{\lambda_1}{\lambda_1 + \lambda_2} \lambda_2 e^{\lambda_2 x}, & x \le 0\\ \frac{\lambda_2}{\lambda_1 + \lambda_2} \lambda_1 e^{-\lambda_1 x}, & x \ge 0 \end{cases}$$
$$= \frac{\lambda_1}{\lambda_1 + \lambda_2} \lambda_2 e^{\lambda_2 x} I_{]-\infty,0]}(x) + \frac{\lambda_2}{\lambda_1 + \lambda_2} \lambda_1 e^{-\lambda_1 x} I_{[0,+\infty[}(x))$$

where

$$I_{[a,b[}(x) = \begin{cases} 1 & \text{if} \quad x \in [a,b[\\ 0 & \text{if} \quad x \notin [a,b[\end{cases} \end{cases}$$

is the indicator function for the set [a, b]. This shows that the generalized Laplace distribution may be seen as a mixture with weights $\frac{\lambda_2}{\lambda_1+\lambda_2}$ and $\frac{\lambda_1}{\lambda_1+\lambda_2}$ respectively of an Exponential distribution with rate parameter λ_1 with support \mathbb{R}_0^+ and a 'negative' Exponential distribution with rate parameter λ_2 with support \mathbb{R}_0^- . For $\lambda_1 = \lambda_2$ this distribution yields the usual Laplace distribution.

When wrapping the generalized Laplace distribution the mixture structure and the weights are kept, with the wrapped version of the generalized Laplace distribution with pdf

$$f(t) = \frac{\lambda_2}{\lambda_1 + \lambda_2} \frac{\lambda_1 e^{-\lambda_1 t}}{1 - e^{-2\lambda_1 \pi}} - \frac{\lambda_1}{\lambda_1 + \lambda_2} \frac{\lambda_2 e^{\lambda_2 t}}{1 - e^{2\lambda_2 \pi}}$$
$$= \frac{\lambda_2}{\lambda_1 + \lambda_2} \frac{\lambda_1 e^{-\lambda_1 t}}{1 - e^{-2\lambda_1 \pi}} + \frac{\lambda_1}{\lambda_1 + \lambda_2} \frac{(-\lambda_2) e^{\lambda_2 t}}{1 - e^{2\lambda_2 \pi}}$$

a mixture with weights $\frac{\lambda_2}{\lambda_1+\lambda_2}$ and $\frac{\lambda_1}{\lambda_1+\lambda_2}$ respectively of a wrapped Exponential with rate λ_1 and another wrapped Exponential distribution with rate $-\lambda_2$.

The linear generalized Laplace distribution is quite simple and flexible since it may exhibit both symmetrical and asymmetrical forms and its wrapped version was studied by Jammalamadaka and Kozubowski (2003, 2004). However, these distributions have the limitation of having a sharp top always placed at the origin. It would be interesting to be able to develop somehow similar distributions, able to exhibit both symmetrical and asymmetrical forms but which could have either sharp or smooth tops, not necessarily placed always at the origin, namely when the distribution would assume an asymmetric form.

The answer to such a goal is the development of the wrapped sum of independent generalized Laplace distributions. However, since another way to look at the linear Laplace distributions is also as the difference of two independent Exponential distributions, exactly the Exponential distributions that appear in the mixture structure, we may alternatively think of the wrapped difference of two independent sums of independent Exponential distributions, which if each of these two sums is the sum of an equal number of Exponential distributions, may then be seen as the sum of independent generalized Laplace distributions. Also in the case where the two sums of independent Exponential distributions are sums of an equal number of distributions all the formulation becomes a bit simpler, so this was our choice.

Let Z be a linear combination of n independent Exponential distributions, with all positive coefficients, that is, let

$$Z = \sum_{i=1}^{n} a_i X_i$$

where $a_i > 0$ (i = 1, ..., n) and $X_i \sim Exp(\lambda_i)$, so that

$$Z = \sum_{i=1}^{n} Y_i$$

where $Y_i \sim Exp(\mu_i)$ with $\mu_i = \lambda_i/a_i > 0$. Then the distribution of Z is just a GIG distribution of depth n (see the previous section), with all shape parameters equal to 1 and rate parameters μ_i (i = 1, ..., n), which has pdf

$$f_Z(z) = K \sum_{j=1}^n c_j e^{-\mu_j z}$$

where

$$K = \prod_{j=1}^{n} \mu_j$$
 and $c_j = \prod_{\substack{k=1 \ k \neq j}}^{n} \frac{1}{\mu_j - \mu_k}$ $(j = 1, \dots, n)$.

We will denote the fact that Z has this distribution by

$$Z \sim SE(\mu_j, j \in \{1, \ldots, n\}).$$

Let then

$$Z_1 \sim SE(\mu_j, j \in \{1, \dots, n\})$$
 and $Z_2 \sim SE(\nu_j, j \in \{1, \dots, n\})$

be two independent r.v.^s and let

 $W = Z_1 - Z_2.$

The pdf's of Z_1 and Z_2 may then be respectively written as

$$f_{Z_1}(z_1) = K_1 \sum_{j=1}^n c_j e^{-\mu_j z_1}$$
 and $f_{Z_2}(z_2) = K_2 \sum_{j=1}^n d_j e^{-\nu_j z_2}$

where

$$K_1 = \prod_{j=1}^n \mu_j , \quad K_2 = \prod_{j=1}^n \nu_j$$

and, for j = 1, ..., p,

$$c_j = \prod_{\substack{k=1 \ k \neq j}}^n \frac{1}{\mu_j - \mu_k} , \quad d_j = \prod_{\substack{k=1 \ k \neq j}}^n \frac{1}{\nu_j - \nu_k}$$

so that the p.d.f. of W will be given by

$$f_W(w) = \int_{\max(w,0)}^{+\infty} K_1 K_2 \left(\sum_{j=1}^n c_j \, e^{-\mu_j \, z_1} \right) \left(\sum_{j=1}^n d_j \, e^{-\nu_j (z_1 - w)} \right) dz_1$$

= $K_1 K_2 \sum_{j=1}^n \sum_{k=1}^n e^{\nu_k w} c_j \, d_k \int_{\max(z_1,0)}^{+\infty} e^{-(\mu_j + \nu_k) z_1} \, dz_1$

or,

$$\begin{split} f_W(w) &= \begin{cases} K_1 K_2 \sum_{j=1}^n H_{1j} c_j e^{-\mu_j w} & w \ge 0\\ K_1 K_2 \sum_{j=1}^n H_{2j} d_j e^{\nu_j w} & w \le 0 \end{cases}\\ &= K_1 K_2 \sum_{j=1}^n H_{2j} d_j e^{\nu_j w} I_{]-\infty,0]}(w) + K_1 K_2 \sum_{j=1}^n H_{1j} c_j e^{-\mu_j w} I_{[0,+\infty[}(w)) \end{cases} \end{split}$$

where

$$H_{1j} = \sum_{h=1}^{n} \frac{d_h}{\mu_j + \nu_h}$$
 and $H_{2j} = \sum_{h=1}^{n} \frac{c_h}{\mu_h + \nu_j}$

Therefore, the wrapped version of this sum of independent generalized Laplace distributions or difference of two sums of independent Exponentials, that is, the distribution of

$$\theta(W) = W \mod 2\pi$$

is a mixture of 2n wrapped Exponentials, n of which are $W\Gamma(\mu_j, 1)$ distributions, with weights $K_1K_2 H_{1j} c_j$ (j = 1, ..., n), and the other n are $W\Gamma(-\nu_j, 1)$ distributions, with weights $K_1K_2 H_{2j} d_j$ (j = 1, ..., n), with pdf

$$f_{\theta(W)}(t) = K_1 K_2 \sum_{j=1}^n \left(H_{1j} c_j \frac{\mu_j e^{-\mu_j t}}{1 - e^{-2\mu_j \pi}} + H_{2j} d_j \frac{-\nu_j e^{\nu_j t}}{1 - e^{2\nu_j \pi}} \right)$$

and cdf

$$F_{\theta(W)}(t) = K_1 K_2 \sum_{j=1}^n \left(H_{1j} c_j \frac{1 - e^{-2\mu_j t}}{1 - e^{-2\mu_j \pi}} + H_{2j} d_j \frac{1 - e^{2\nu_j t}}{1 - e^{2\nu_j \pi}} \right).$$

We will denote this distribution by $WSL(\mu_1, \ldots, \mu_n; \nu_1, \ldots, \nu_n)$. For n = 1 this distribution reduces to the wrapped generalized Laplace distribution.

This distribution has a symmetric shape if the set of rate parameters μ_j and the set of rate parameters ν_j (j = 1, ..., n) are the same, the order of the rate parameters μ_j and ν_j (j = 1, ..., n) in each set being irrelevant. We should also note that for the distribution above all the μ_j are supposed to be different as well as all the ν_j , since this is what is required in the GIG distribution, from which this distribution was derived. Two wrapped sums of independent generalized Laplace distributions are symmetrical, relative to the origin if the two sets of parameters μ_j and ν_J are interchanged. Also, this distribution tends to the uniform distribution if at least one of the μ_j 's and one of the ν_j 's tend to zero. Both the sum of generalized Laplace distributions and the GIG or the sum of independent Gammas are unimodal, both in their linear and wrapped forms. However, mixtures of these distributions are a powerful tool in modelling circular data, since a criterious choice of parameters may easily yield multimodal distributions. The trigonometric moments for these distribution, and related parameters, may

be easily obtained from the mixture structure and the results for the single wrapped Gamma distribution.

In Figures 11 and 12 we may see some plots of densities of wrapped sums of generalized Laplace distributions.



Figure 11. – Three-dimensional views of densities of $WSL(2.9, 3.3; 4.5, \nu_2)$ for different values of ν_2 . All plots have the origin placed at the same spot and all plots have the same vertical scale.



6 The wrapped non-central Gamma distribution

We will say that the r.v. X has a non-central Gamma distribution with shape parameter r, rate parameter λ and non-centrality parameter δ if the r.v. X has pdf

$$f_X(x) = \frac{\lambda^r}{\Gamma(r)} e^{-\lambda x} x^{r-1} e^{-\delta/2} {}_0 F_1(r; \lambda x \delta/2)$$

=
$$\sum_{i=0}^{\infty} \frac{e^{-\delta/2} (\delta/2)^i}{i!} \frac{\lambda^{r+i}}{\Gamma(r+i)} e^{-\lambda x} x^{r+i-1},$$

that is, if the distribution of X is a mixture with Poisson weights with parameter δ of infinitely many $\Gamma(r+i,\lambda)$ distributions $(i=0,1,\ldots)$.

This way, the wrapped form of the non-central Gamma distribution will be a mixture with the same Poisson weights of infinitely many $W\Gamma(r+i,\lambda)$ distributions (i = 0, 1, ...), with pdf

$$f_{\theta}(t) = e^{-(\lambda t + \delta/2)} \sum_{i=0}^{\infty} \frac{(\delta/2)^{i}}{i!} \frac{\lambda^{r+i}}{\Gamma(r+i)} (2\pi)^{r+i-1} \Phi\left(e^{-2\lambda\pi}, 1 - r - i, \frac{t}{2\pi}\right)$$

where

$$\theta \equiv \theta(X) = X \mod 2\pi$$

and where, for non-integer r we may write

$$\begin{aligned} f_{\theta}(t) &= \sum_{j=0}^{\infty} \frac{e^{-\delta/2} \, (\delta/2)^{j}}{j!} \left(p_{j}^{*} \frac{\lambda^{r+j}}{\gamma(r+j;2\lambda\pi)} \, e^{-\lambda t} \, t^{r+j-1} + \sum_{i=0}^{\infty} p_{i,j} \frac{\lambda^{i+1}}{\gamma(i+1;2\lambda\pi)} \, e^{-\lambda t} \, t^{i} \right) \\ &= e^{-\delta/2+\lambda t} \, \sum_{j=0}^{\infty} \frac{(\delta/2)^{j}}{j!} \left(\frac{t^{r+j-1}}{\Gamma(r+j)} + \sum_{i=0}^{\infty} \frac{t^{i}(-1)^{i}}{\Gamma(i+1)} \, \frac{(2\pi\lambda)^{r+j-h-1}}{(r+j-h-1)!} \, \Phi^{*}\!\left(e^{-2\lambda\pi}, r+j-h-1\right) \right) \end{aligned}$$

where

$$p_j^* = \gamma^* (r+j, 2\lambda\pi) \tag{22}$$

and

$$p_{i,j} = \gamma^* (i+1, 2\lambda\pi) (-1)^i \frac{\lambda^{r+j-i-1}}{\Gamma(r)} \frac{\Gamma(1-r-j+i)}{\Gamma(1-r-j)} PLog\left(e^{-2\lambda\pi}, i+1-r-j\right)$$
(23)

while for $r \in \mathbb{N}$,

$$f_{\theta}(t) = \sum_{j=0}^{\infty} \frac{e^{-\delta/2} (\delta/2)^j}{j!} \sum_{h=0}^{r-1} p_{h,j}^* \frac{\lambda^{h+1}}{\gamma(h+1,2\lambda\pi)} e^{-\lambda t} t^h$$

= $e^{-(\delta/2+\lambda t)} \sum_{j=0}^{\infty} \frac{(\delta/2)^j}{j!} \sum_{h=0}^{r-1} \frac{\lambda^{h+1}}{\Gamma(h+1)} \frac{(2\lambda\pi)^{r+j-h-1}}{(r+j-h-1)!} \Phi^* \left(e^{-2\lambda\pi}, r+j-h-1 \right)$

where

$$p_{h,j}^* = \gamma^*(h+1, 2\lambda\pi) \,\frac{(2\lambda\pi)^{r+j-h-1}}{(r+j-h-1)!} \,\Phi^*\!\left(e^{-2\lambda\pi}, r+j-h-1\right) \,. \tag{24}$$

The corresponding cdf's are given by

$$F_{\theta}(t) = \sum_{j=0}^{\infty} \frac{e^{-\delta/2} (\delta/2)^j}{j!} \left(p_j^* \frac{\gamma(r+j,\lambda t)}{\gamma(r+j,2\lambda\pi)} + \sum_{i=0}^{\infty} p_{i,j} \frac{\gamma(i+1,\lambda t)}{\gamma(i+1,2\lambda\pi)} \right)$$

for non-integer r, with $p_{i,j}$ (i = 0, 1, ...) given by (23) and p_j^* (j = 0, 1, ...) given by (22), and

$$F_{\theta}(t) = \sum_{j=0}^{\infty} \frac{e^{-\delta/2} (\delta/2)^j}{j!} \sum_{h=0}^{r-1} p_{h,j}^* \frac{\gamma(h+1,\lambda t)}{\gamma(h+1,2\lambda \pi)}$$

for $r \in \mathbb{N}$, where $p_{h,j}^*$ (h = 0, ..., r - 1; j = 0, 1, ...) are given by (24).

We may note that, interestingly enough, this latter non-central distribution depends on r only through the weights $p_{h,j}^*$, while for non-integer r the non-central distribution has all of its components but the first depending on r only through the weights $p_{i,j}$, with the first component being a function of r both through the weight p_i^* and also through the distribution part itself.

We should also note that when $\delta \to \infty$ the wrapped non-central Gamma distribution tends to the wrapped Uniform distribution, since then the components in the mixture with higher values for the shape parameter will have higher weights in the mixture, and as we saw in section 2, when the shape parameter tends to infinity the wrapped Gamma distribution tends to the Uniform distribution. In Figure 13 we may see a few three-dimensional plots of densities of wrapped non-central Gamma distributions, where this fact may be analyzed.



Figure 13. – Three-dimensional views of densities of non-central wrapped Gamma densities with shape parameter r = 3, rate parameter $\lambda = 1.9$ and different values of the non-centrality parameter δ . All plots have the origin placed at the same spot and all plots have the same vertical scale.

7 Final remarks

Once unveiled the mixture structure of the wrapped Gamma distribution it became easy to obtain expressions for its cdf and also once considered the mixture structure of the other linear distributions considered, as the distribution of linear combinations of independent Gamma distributions, the distribution of the sum of independent generalized Laplace distributions and the non-central Gamma distribution, it was easy to obtain expressions for both the pdf and cdf of their wrapped versions.

The great advantage of the wrapped sum of generalized Laplace distributions is that these distributions may exhibit both symmetrical and fairly asymmetrical shapes with either a sharp or smooth top. The sharp top only occurs for n = 1, that is for the single generalized Laplace distribution. Although these distributions are always unimodal, we may use mixtures of them to obtain very flexible multimodal distributions.

The wrapped sum or linear combination of independent Gamma distributions, the wrapped GIG distribution and the wrapped distribution of the sum of independent generalized Laplace distributions are all very flexible distributions. However, they are all unimodal. Of course when one needs to handle multimodal circular data we may think about using mixtures of these distributions. However, then they may become a bit hard to handle from an estimation point of view.

Mere mixtures of wrapped Gamma distributions seem to be quite a powerful tool, since they may have from one to as many modes as the number of Gamma distributions in the mixture. However, in cases where we need some degree of symmetry, the wrapped sum of generalized Laplace distributions will be a distribution to consider. Of course one may use mixtures of different distributions, as for example a mixture of a wrapped sum of generalized Laplace distributions with a wrapped GIG distribution.

Modules programmed in Mathematica[©] (from Wolfram Research, Inc.) to compute and plot the pdf's and cdf's in this paper are available from the author.

n = 1									1								
2								1		1							
3							1		4		1						
4						1		11		11		1					
5					1		26		66		26		1				
6				1		57		302		302		57		1			
7			1		120		1191		2416		1191		120	1			
8		1		247		4293		15619		15619		4293		247	1		
9		1	50	2	14608	8	88234	1	156190)	88234	Ł	14608	3 5	02	1	
10	1	10	13	47840) 4	45519	2 1	131035	4 1	31035	4 4	45519	2 4	47840	1013	3	1

Appendix A

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